RESIDUAL SPECTRUM OF GL_{2n} DISTINGUISHED BY THE SYMPLECTIC GROUP

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Abstract. We determine which automorphic representations of the discrete spectrum of GL_{2n} are distinguished by the symplectic group. This concludes a project initiated by Jacquet and Rallis.

1. Introduction

Let F be a number field and let $\mathbb{A}$ be the ring of adèles of F. For any positive integer r we denote by G_r the group GL_r viewed as an algebraic group over F. We fix an integer n and denote $G = G_{2n}$. Let K be the standard maximal compact of $G(\mathbb{A})$. For any algebraic group Q defined over F, denote $Q(\mathbb{A})^1 = \cap_\chi \ker|\chi|$ where χ ranges over the algebraic characters of Q. There is a direct sum decomposition

$$L^2(G(F)\backslash G(\mathbb{A})^1) = L^2_{disc}(G) \oplus L^2_{cont}(G)$$

to a discrete and a continuous part. The discrete part $L^2_{disc}(G)$ decomposes into a direct sum of irreducible representations. By an irreducible, discrete spectrum representation of $G(\mathbb{A})^1$ we mean an irreducible summand of $L^2_{disc}(G)$. In this work we determine the irreducible, discrete spectrum representation of $G(\mathbb{A})^1$, that have a non-vanishing symplectic period. This completes the work of Jacquet and Rallis in [JR92b].

Mœglin and Waldspurger obtained the following classification for the discrete spectrum [MW89]. For $\lambda \in \mathbb{C}$, let ν^λ denote the character of $G_r(\mathbb{A})$ defined by $g \mapsto |\det g|^\lambda$, where $|\,| = |\,|_{\mathbb{A}}$ is the standard choice of absolute value on $\mathbb{A}^\times$. Let σ be a cuspidal automorphic representation of $G_r(\mathbb{A})$. For $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{C}^s$ we denote

$$\pi[\lambda] = \pi_\sigma[\lambda] = \nu^{\lambda_1}\sigma \otimes \cdots \otimes \nu^{\lambda_s}\sigma.$$

It is an irreducible, cuspidal, automorphic representation of $M(\mathbb{A})$, where M is the standard Levi subgroup of G_{rs} of type $(r, \dots, r)$. Let $P = MU$ be the standard parabolic of G_{rs} with Levi M and unipotent radical U. We denote by $I(\pi, \lambda)$ the representation induced from the representation $\pi[\lambda] \otimes 1_{U(\mathbb{A})}$ of $P(\mathbb{A})$ to $G_{rs}(\mathbb{A})$ using normalized parabolic induction. Let

$$\Lambda = \Lambda_s = \left(\frac{s-1}{2}, \frac{s-3}{2}, \dots, \frac{1-s}{2}\right) \in \mathbb{C}^s.$$

The representation $I(\pi, \Lambda)$ has a unique irreducible quotient, which we denote by $L(\sigma, \Lambda)$.

Theorem 1 (Mœglin-Waldspurger)**.** *Let Π be an irreducible discrete spectrum representation of $G(\mathbb{A})^1$, then there is a decomposition $2n = rs$ and an irreducible, cuspidal automorphic representation σ of $G_r(\mathbb{A})$ such that*

$$\Pi \simeq L(\sigma, \Lambda).$$

The pair (r, σ) is determined uniquely.

In what follows we refer to the body of the text for precise definitions. Fix a decomposition $2n = rs$ and a cuspidal automorphic representation σ of $G_r(\mathbb{A})$. The representation $L(\sigma, \Lambda)$ is spanned by multi-residues of Eisenstein series, as already explained in [Jac84]. Let $\pi = \pi[0] = \sigma \otimes \cdots \otimes \sigma$ be the associated representation of $M(\mathbb{A})$. By restricting functions to K we may identify the spaces of the representations $I(\pi, \lambda)$, $\lambda \in \mathbb{C}^s$ with the space of $I(\pi, 0)$. For $\varphi \in I(\pi, \lambda)$ there is associated an Eisenstein series $E(g, \varphi, \lambda)$, $\lambda \in \mathbb{C}^s$. We denote by $E_{-1}(\varphi)$ the multi-residue of the Eisenstein series $E(\varphi, \lambda)$ at $\lambda = \Lambda$. The space of $L(\sigma, \Lambda)$ is spanned by $\{E_{-1}(\varphi),\ \varphi \in I(\pi, \Lambda)\}$ and $\varphi \mapsto E_{-1}(\varphi)$ defines an intertwining operator from $I(\pi, \Lambda)$ onto $L(\sigma, \Lambda)$. Let w_r be the $r \times r$ permutation matrix with unit anti-diagonal, and let

$$\epsilon_{2r} = \begin{pmatrix} & w_r \\ -w_r & \end{pmatrix}.$$

Let $H_{2r} = Sp_{2r}$ be the symplectic group with respect to the skew-symmetric form defined by ϵ_{2r}. It is a subgroup of G_{2r}. We will denote $H = H_{2n}$ and $\epsilon = \epsilon_{2n}$. For any permutation w on the blocks of M we denote by $M(w, \lambda)$ the standard intertwining operator from $I(\pi, \lambda)$ to $I(\pi, w\lambda)$. Let $M_{-1}(w)$ denote its multi-residue at Λ and let $j = j_M$ be the linear functional

$$j(\varphi) = \int_{K_H} \int_{M_H(F)\backslash M_H(\mathbb{A})^1} \varphi(mk)\,dm\,dk$$

where $K_H = K \cap H(\mathbb{A})$ and $M_H = M \cap H$. When s is even we denote $s = 2k$ and define a permutation w' that will play a central role in this work:

$$w'(2i-1) = i,\ w'(2i) = 2k + 1 - i,\ i = 1, \dots, k. \tag{1}$$

In [Off], we computed the symplectic period of $E_{-1}(\varphi)$ as follows.

Theorem 2. *For all $\varphi \in I(\pi,\Lambda)$ we have*

$$\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh = \begin{cases} v_{P_H} j(M_{-1}(w')\varphi) & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd} \end{cases} \tag{2}$$

where v_P is a certain volume defined in [Off].

In particular, the period integral is convergent, for automorphic forms in the discrete spectrum, so the following makes sense.

Definition 1. An irreducible discrete representation Π of $G(\mathbb{A})^1$ is called *H-distinguished* if there is an automorphic form φ in the space of Π such that the period integral

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi(h)dh$$

is not zero.

In this work we determine the discrete automorphic representations of $G(\mathbb{A})$ which are H-distinguished.

Theorem 3. *Let $\Pi = L(\sigma,\Lambda_s)$ be an irreducible discrete spectrum representation of $G(\mathbb{A})^1$, then Π is H-distinguished if and only if s is even.*

The theorem was proved by Jacquet and Rallis [JR92b] for the cases $s = 1$ and $s = 2$. In light of Theorem 2, to prove Theorem 3 it is enough to show that for s even the linear form $l_H = j \circ M_{-1}(w')$ is not identically zero on $I(\pi,\Lambda)$. The form l_H decomposes into local factors, i.e the representation π and hence also the representation $I(\pi,\Lambda)$ decompose into a restricted tensor product of local factors, and up to a non-zero global constant, we may write $l_H = \otimes_\nu l_{v,H_v}$ where the tensor product is over all places v of F, $l_{v,H_v} = j_v \circ M_v(w')$, is an H_v-invariant form on the induced representation $I(\pi_v,\Lambda)$, j_v is the local analogue of j and $M_v(w')$ is the normalized local intertwining operator. We may therefore reduce the global non-vanishing problem to the local problem of the non-vanishing of l_{v,H_v} at all places of F. The reduction to the local problem is explained in §2. We then treat the non-archimedean local problem in §3 and the archimedean local problem in §4.

Remark 1. Unlike in the Jacquet-Rallis case, when $s > 2$ the residual Eisenstein series is not a functorial lift from a representation which appears (even weakly) in $L^2(G(F)\backslash G(\mathbb{A})^1)$. It will be interesting to understand its affect on the relative trace formula of [JR92a].

As an application of the formula (2), we obtain an explicit expression for the symplectic period of an automorphic form in an unramified situation. Let $n = kr$ and let σ be a cuspidal automorphic representation of $GL_r(\mathbb{A})$ which is everywhere unramified. The measure on $G(\mathbb{A})$ is given by the Iwasawa decomposition $G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})K$.

Theorem 4. *Let ϕ_0 be the K-invariant, L^2-normalized automorphic form in $L(\sigma, \Lambda_{2k})$. With Haar measure on $M(\mathbb{A})$ properly normalized we have,*

$$\left| \int_{H(F)\backslash H(\mathbb{A})} \phi_0(h) dh \right|^2 =$$
$$\frac{L(2, \sigma \times \tilde{\sigma}) L(4, \sigma \times \tilde{\sigma}) \cdots L(2k, \sigma \times \tilde{\sigma})}{Res_{s=1} L(s, \sigma \times \tilde{\sigma}) L(3, \sigma \times \tilde{\sigma}) \cdots L(2k-1, \sigma \times \tilde{\sigma})}.$$

Proof. Let v_0 be the spherical element in the space of $\sigma^{\otimes 2k}$ of norm one, and let $\varphi_0 \in I(\pi, \Lambda)$ be the K-invariant section, normalized so that φ_0 takes the value v_0 on K. We then have, $j(\varphi_0) = 1$. Clearly

$$\phi_0 = \frac{E_{-1}(\varphi_0)}{\|E_{-1}(\varphi_0)\|_2}$$

where $\| \cdot \|_2$ denotes the L^2-norm. It follows from Langlands inner product formula ([Art80], Lemma 4.2) and his local computation in [Lan71], that up to a certain volume depending on M

$$\|E_{-1}(\varphi_0)\|_2^2 = \frac{\prod_{i=2}^{2k} L(i, \sigma \times \tilde{\sigma})}{[\mathrm{Res}_{s=1} L(s, \sigma \times \tilde{\sigma})]^{2k-1}}.$$

Note that

$$\{(i,j) | 1 \le i < j \le 2k,\, w'(i) > w'(j)\} = \{(2i, j) | 1 \le 2i < j \le 2k\}.$$

It therefore also follows that

$$M_{-1}(w')\varphi_0 = \frac{[\mathrm{Res}_{s=1} L(s, \sigma \times \tilde{\sigma})]^{k-1}}{\prod_{i=1}^{k-1} L(2i+1, \sigma \times \tilde{\sigma})} \varphi_0.$$

Combining all this with (2) we obtain the theorem. □

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2. Reduction to a local problem

2.1. **Eisenstein series, intertwining operators and multi-residues.** In this section F is a number field. For an algebraic group Q defined over F, let $\mathfrak{a}_Q^* = X^*(Q)\otimes\mathbb{R}$, where $X^*(Q)$ is the the lattice of rational characters of Q, and let $\mathfrak{a}_Q$ denote the dual vector space. Let δ_Q denote the modulus function on $Q(\mathbb{A})$. When $Q = LV$ is a parabolic subgroup of G with its Levi decomposition, let ρ_Q denote half the sum of the positive roots of Q. We define a height function $H_L : G(\mathbb{A}) \to \mathfrak{a}_Q$. It is the left $V(\mathbb{A})$-invariant, right K-invariant function on $G(\mathbb{A})$ such that for $l \in L(\mathbb{A})$

$$e^{\langle \chi, H_L(l)\rangle} = |\chi|(l) = \prod_v |\chi_v(l_v)|_v$$

for all $\chi \in X^*(L)$. Here, χ_v is the extension of χ to the completion F_v of F at v, and the product is over all places v of F. Let $2n = 2kr$ and let $P = MU$ be the standard parabolic subgroup of G of type $(r,\dots,r)$. Let σ be a cuspidal automorphic form of $G_r(\mathbb{A})$. For every place v of F we denote by σ_v the local component of σ at v so that σ is isomorphic to the restricted tensor product $\otimes'_v\sigma_v$. For σ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}} \simeq \mathbb{C}^{2k}$ we associated in the introduction, the automorphic representation $\pi[\lambda] = \nu^{\lambda_1}\sigma\otimes\cdots\otimes\nu^{\lambda_{2k}}\sigma$ of $M(\mathbb{A})$ and the induced representation $I(\pi,\lambda)$ of $G(\mathbb{A})$. For $\varphi \in I(\pi, 0)$ we define the Eisenstein series $E(\varphi,\lambda)$ as the meromorphic continuation of

$$E(g,\varphi,\lambda) = \sum_{\delta\in P\backslash G} \varphi(\delta g) e^{\langle\lambda, H_M(\delta g)\rangle}$$

to $\lambda \in \mathbb{C}^{2k}$. The series converges absolutely if $\mathrm{Re}(\lambda_i - \lambda_{i+1} - 1) > 0$ for all i, and defines an automorphic form on $G(\mathbb{A})$. The function $E_{-1}(\varphi)$ is defined by

$$E_{-1}(g,\varphi) = \lim_{\lambda\to\Lambda}\left[E(g,\varphi,\lambda)\prod_{i=1}^{2k-1}(\lambda_i - \lambda_{i+1} - 1)\right].$$

The limit exists, and $E_{-1}(\varphi)$ is an L^2-automorphic form on $G(\mathbb{A})^1$, which we refer to as the multi-residue of the Eisenstein series $E(\varphi,\lambda)$.

We recall some properties of intertwining operators that we will need, which appear in [Jac84] and [MW89]. For a permutation w in $2k$ variables and $\lambda \in \mathbb{C}^{2k}$, the standard un-normalized intertwining operator $M(w,\lambda) : I(\pi,\lambda) \to I(\pi, w\lambda)$ is meromorphic. Up to a scalar factor it is decomposable into local intertwining operators, i.e. there is a scalar valued meromorphic function $m(w,\lambda)$ so that if we set

$$M(w,\lambda) = m(w,\lambda)R(w,\lambda)$$

then the normalized intertwining operators $R(w,\lambda)$ are decomposable. We denote the decomposition into local intertwining operators by

$$R(w,\lambda) = \otimes_v' R_v(w,\lambda).$$

The multi-residue $M_{-1}(w)$ of $M(w,\lambda)$ is

$$M_{-1}(w) = \lim_{\lambda\to\Lambda}\left[M(w,\lambda)\prod_{\{1\le i\le 2k-1\,|\,w(i)>w(i+1)\}}(\lambda_i-\lambda_{i+1}-1)\right].$$

The operator $R(w,\lambda)$ is holomorphic at $\lambda=\Lambda$ and

$$M_{-1}(w) = m_{-1}(w)R(w,\Lambda)$$

where

$$m_{-1}(w) = \lim_{\lambda\to\Lambda}\left[m(w,\lambda)\prod_{\{1\le i\le 2k-1\,|\,w(i)>w(i+1)\}}(\lambda_i-\lambda_{i+1}-1)\right].$$

We see that $M_{-1}(w)$ is a decomposable intertwining operator, up to the scalar factor $m_{-1}(w)$. It is also known that $m_{-1}(w)\neq 0$.

2.2. **Factorization of the period.** Our task in this work is to show that the linear functional $l_H = j\circ M_{-1}(w')$ is not identically zero on $I(\pi,\Lambda)$. As already mentioned in the introduction, we will use a factorization of l_H and will obtain its non-vanishing from the non-vanishing at all places of its local factors. From §2.1 it follows that up to a non-zero scalar, the intertwining operator $M_{-1}(w')$ decomposes into local factors $\otimes_v R_v(w',\Lambda)$. Since our focus is only on non-vanishing, this global scalar will play no role in what follows. We stress, however, that the local intertwining periods $R_v(w',\Lambda)$ are the normalized ones. For $\varphi\in I(\pi,\lambda)$ we defined

$$j(\varphi) = \int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}\varphi(mk)dm\,dk.$$

The integral over K_H is global and decomposes into a product of local integrals, each over the maximal compact $K_{v,H_v}=K_v\cap H_v$ of H_v. The inner integral is the unique (up to a scalar) $M_H(\mathbb{A})$-invariant form on π. Denote it by l_{M_H}. This is observed as follows. The group $M_H = M\cap H$ consists of elements of the form $\mathrm{diag}(g_1,\dots,g_k,\tilde g_k,\dots,\tilde g_1)$ where $g_i\in G_r$ and for $g\in G_r$ we denote $\tilde g = w_r{}^tg^{-1}w_r$. The contragradiant representation of σ may be realized in the space of σ with action $g\mapsto$

$\sigma(\tilde{g})$. With this realization the inner integral is the (unique up to a scalar) pairing between $\sigma^{\otimes k}$ and its contragradiant. Therefore l_{M_H} decomposes into local factors $\otimes_v l_{M_{v,H_v}}$ where $l_{M_{v,H_v}}$ is the $M_{v,H_v} = (M_v \cap H_v)$-invariant form on π_v and thus $j = \otimes_\nu j_v$ is decomposable. The form $l_{v,H_v} = j_v \circ R_v(w', \Lambda)$ on $I(\pi_v, \Lambda)$ is given by

$$\varphi_v \mapsto \int_{K_{v,H_v}} l_{M_{v,H_v}}(R_v(w', \Lambda)\varphi_v(k))dk.$$

If v is a place where σ_v is unramified, then so are the representations $I(\pi_v, \lambda)$. Let v_0 be the spherical vector in the space of $\sigma_v^{\otimes 2k}$ normalized to have norm one. The spherical section in $I(\pi_v, \lambda)$ normalized to take the value v_0 at the identity will be referred to as the normalized spherical section. In this case $R_v(w', \Lambda)$ maps the normalized spherical section of $I(\pi_v, \Lambda)$ to the normalized spherical section of $I(\pi_v, w'\Lambda)$. The functional j_v applied to the normalized spherical section is $l_{M_{v,H_v}} v_0$ which is 1 by our choice of v_0. Thus at almost all places, l_{v,H_v} is 1 on the normalized spherical section and l_H differs from $\otimes_v l_{v,H_v}$ by a non-zero scalar. To show that l_H is not identically zero on $I(\pi, \Lambda)$ it is therefore enough to show that l_{v,H_v} is not identically zero on $I(\pi_v, \Lambda)$ for each place v. The local components of a cuspidal automorphic representation σ of $G_r(\mathbb{A})$ are unitary and generic. It remains to show the following.

Theorem 5. *For any place v of F let σ_v be an irreducible, unitary, generic representation of $G_r(F_v)$ and let $\pi_v = \sigma_v^{\otimes 2k}$. The form l_{v,H_v} is not identically zero on $I(\pi_v, \Lambda)$.*

The rest of this manuscript treats this local problem. The archimedean and non-archimedean places are treated separately using different methods. We will switch to a local setting and drop the index v from our notation. Thus for an algebraic group Q we will, by abuse of notation, also denote the local group $Q(F)$ by Q. We will denote by $M(w)$ the standard intertwining operator from $I(\pi, \Lambda)$ to $I(\pi, w\Lambda)$ and we need not worry about weather or not it is the normalized one.

3. The local problem (non-archimedean)

In this section F will denote a non-archimedean local field.

The main tool we use to prove Theorem 5 in the non-archimedean case is the geometric lemma of Bernstein-Zelevinsky. Applying it, requires an analysis of the double coset space $P\backslash G/H$ which we present in §3.1. In §3.2 we provide the representation theoretic background. The prove is then given in §3.3.

3.1. **Double cosets.** We denote by W the Weyl group of G. Let $P = MU$ be a standard parabolic of G of type $(r_1, \dots, r_t)$. In [Off], we studied the double coset space $P\backslash G/H$. In this section, we sum up the results and make them more explicit. We will denote by W_M the Weyl group of M viewed as a subgroup of W and by ${}_MW_M$ the set of left and right M-reduced elements in W. There is a one to one correspondence

$$ {}_MW_M \simeq W_M\backslash W/W_M. $$

We will denote by w_M the longest element of W_M and set $w_0 = w_G$.

Definition 2. An element $\xi \in W$ is called a twisted involution if $(\xi w_0)^2 = 1$. A twisted involution $\xi \in {}_MW_M$ is called M-admissible if

$$ M = \xi w_0 M w_0 \xi^{-1}. $$

We denote by $\mathfrak{I}_M$ the set of M-admissible twisted involutions.

Denote

$$ W(H) = \{ww_0w^{-1}w_0 | w \in W\} = \{w \in W | ww_0 \text{ is conjugate to } w_0\}. $$

For $\xi \in {}_MW_M \cap W(H)$ let $\eta \in W$ be such that $\eta w_0 \eta^{-1} = \xi w_0$. The map

$$ P\eta H \longleftrightarrow \xi $$

is a well defined bijection

$$ P\backslash G/H \simeq {}_MW_M \cap W(H). \tag{3} $$

We wish to make a convenient choice of representatives $\{\eta\}$ for the double cosets. For that task, it will be useful to first analyze further the set ${}_MW_M \cap W(H)$. We set up some notation that will help us view certain permutations as elements of W. For any two real numbers $a \le b$ such that $b - a \in \mathbb{Z}$ denote by $[a, b]$ the set $\{a, a+1, \dots, b\}$. Let $\mathbf{r} = (r_1, \dots, r_t)$ be a partition of r and τ a permutation of the set $[1, t]$. We define a permutation matrix in G_r, which we denote by $w_{\mathbf{r}}(\tau)$. In $r_i \times r_{\tau(j)}$-block form, it is given by

$$ w_{\mathbf{r}}(\tau) = (A_{i,j}) $$

where

$$ A_{i,j} = \begin{cases} I_{r_i} & \tau(j) = i \\ 0_{r_i \times r_{\tau(j)}} & \tau(j) \neq i \end{cases}. $$

If M is the standard Levi of G_r of type $\mathbf{r}$, we also denote

$$ w_M(\tau) = w_{\mathbf{r}}(\tau). $$

We say that τ is an M-admissible or $\mathbf{r}$-admissible permutation on the blocks of M if

$$ \mathbf{r} = \tau\mathbf{r} = (r_{\tau^{-1}(1)}, \dots, r_{\tau^{-1}(t)}) $$

and whenever $\tau(i) = i$, r_i is even. We note that for permutations τ_1, τ_2 of $[1, t]$ we have

$$w_{\mathbf{r}}(\tau_1\tau_2) = w_{\mathbf{r}}(\tau_1)w_{\tau_1^{-1}\mathbf{r}}(\tau_2). \tag{4}$$

3.1.1. *The $\xi \leftrightarrow \tau$ correspondence.* An involution of $[1, t]$, is a permutation τ of the set $[1, t]$ such that $\tau^2 = 1$. Denote by T_M the set of all M-admissible involutions of $[1, t]$. There is a bijection

$$\mathfrak{I}_M \cap W(H) \simeq T_M.$$

Elements $\xi \in \mathfrak{I}_M \cap W(H)$ and $\tau \in T_M$ correspond, and we write $\xi \leftrightarrow \tau$ if

$$w_M\xi w_0 = w_M(\tau).$$

We now wish to generalize this description to non-admissible twisted involutions. We define a subset Γ_M of pairs (M', τ), where $M' \subset M$ is a Levi subgroup of G and τ is an M'-admissible involution on the blocks of M' as follows. Let the type of M' be given by $(\gamma_1, \ldots, \gamma_t)$ where

$$\gamma_l = (m_1^l, \ldots, m_{q_l}^l) \tag{5}$$

is a partition of r_l for all $l \in [1, t]$. Thus τ is an involution of the set

$$\{(j, l)|j \in [1, q_l], \, l \in [1, t]\}.$$

Such a pair (M', τ) is in Γ_M if for all $j < j_1 \in [1, q_l]$ we have $l' > l_1'$ where $\tau(j, l) = (j', l')$ and $\tau(j_1, l) = (j_1', l_1')$. The definition of Γ_M is set so that there is a bijection

$${}_MW_M \cap W(H) \simeq \Gamma_M \tag{6}$$

given by $\xi \leftrightarrow (M', \tau)$ where

$$M' = M \cap \xi w_0 M w_0 \xi^{-1} \tag{7}$$

and

$$w_{M'}\xi w_0 = w_{M'}(\tau).$$

We will often omit M' from the notation and refer to the bijection (6) as the $\xi \leftrightarrow \tau$ correspondence.

3.1.2. *The $\tau \leftrightarrow \eta$ correspondence.* Here we make special choices of representatives η for the double cosets $P\backslash G/H$. For any positive integer s, we define the permutation

$$\sigma_s(i) = \begin{cases} 2i - 1 & 1 \le i \le s \\ 2(2s + 1 - i) & s + 1 \le i \le 2s \end{cases}$$

of the set $[1, 2s]$. For $d \le \frac{t}{2}$ let τ_d be the involution on $[1, t]$ of the form

$$\tau_d = (1, t)(2, t - 1) \cdots (d, t + 1 - d).$$

If τ_d is M-admissible, then the type of M has the form

$$(r_1, \dots, r_t) = (r_1, \dots, r_d, 2k_1, \dots, 2k_{t-2d}, r_d, \dots, r_1).$$

Let $\mathbf{k} = (k_1, k_1, \dots, k_{t-2d}, k_{t-2d})$ and define $\eta_d \in G_{2n}$ to be

$$\eta_d = \mathrm{diag}(I_{r_1+\cdots+r_d}, w_{\mathbf{k}}(\sigma_{t-2d}), I_{r_1+\cdots+r_d}). \tag{8}$$

If $\xi_d \in {}_M W_M \cap W(H)$ is such that $\xi_d \leftrightarrow \tau_d$, then $\eta_d w_0 \eta_d^{-1} = \xi_d w_0$. Therefore, the double coset $P\eta_d H$ corresponds to ξ_d in the bijection (3). If τ is any M-admissible involution of $[1,t]$, we may write $\tau = (i_1, j_1) \cdots (i_d, j_d)(l_1) \cdots (l_{t-2d})$ for some $d \le \frac{t}{2}$. Writing τ in this form, let us assume that $i_k < j_k$ for all $k \in [1,d]$, that $i_1 < \cdots < i_d$ and that $l_1 < \cdots < l_{t-2d}$. Let σ_τ be the permutation of $[1,t]$ defined by

$$\sigma_\tau(k) = \begin{cases} i_k & k \in [1,d] \\ l_{k-d} & k \in [d+1, t-d] \\ j_{t+1-k} & k \in [t+1-d, t] \end{cases}.$$

Note then that

$$\tau = \sigma_\tau \tau_d \sigma_\tau^{-1}$$

and that τ_d is $\sigma_\tau^{-1}\mathbf{r}$-admissible. Therefore η_d is defined as in (8). Let

$$\eta = w_{\mathbf{r}}(\sigma_\tau)\eta_d. \tag{9}$$

Then, if $\xi \in \mathfrak{I}_M \cap W(H)$ is such that $\xi \leftrightarrow \tau$ then $\eta w_0 \eta^{-1} = \xi w_0$. This gives our special choice of double coset representatives, corresponding to admissible twisted involutions. We write in this case $\tau \leftrightarrow \eta$. We further make a choice of η for every $\xi \in {}_M W_M \cap W(H)$. Let $(M', \tau) \in \Gamma_M$ be such that $\xi \leftrightarrow \tau$. Then there is a unique η defined as above, with respect to (M', τ).

Whenever referring to $\xi \in {}_M W_M \cap W(H)$ or to $(M', \tau) \in \Gamma_M$ we will always refer to the corresponding triple

$$\xi \leftrightarrow \tau \leftrightarrow \eta \tag{10}$$

given by the above correspondence. This special choice of representatives $\{\eta\}$ for the double cosets was designed for the proof of Proposition 1 in §3.1.3.

3.1.3. *Exponents.* Fix $(M', \tau) \in \Gamma_M$, with a corresponding triple (10). For any subgroup X of G we define

$$X_\xi = X \cap \eta H \eta^{-1}.$$

Keeping notation as in (5), the group M'_ξ is the subgroup of M' consisting of matrices of the form

$$\mathrm{diag}(a_1^1, \dots, a_{q_1}^1, a_1^2, \dots, a_{q_t}^t)$$

where

$$a_j^l = w_{m_j^l}{}^t(a_{j'}^{l'})^{-1} w_{m_j^l} \in G_{m_j^l}$$

whenever $\tau(j,l) = (j',l') \neq (j,l)$ and

$$a_j^l \in H_{m_j^l}$$

whenever $\tau(j,l) = (j,l)$. Let

$$U' = M \cap \xi w_0 U w_0 \xi^{-1}$$

then, $P' = M'U'$ is the standard parabolic of M and $Q' = M'U'U$ is the standard parabolic of G with Levi component M'. We have

$$P_\xi = Q'_\xi = M'_\xi R$$

where R is the unipotent radical of P_ξ. If proj_M denotes the projection from P_ξ to M, then

$$\ker(\mathrm{proj}_M) = U_\xi$$

and

$$\mathrm{proj}_M(R) = U'.$$

Thus proj_M defines a bijection

$$U_\xi \backslash R \simeq U'.$$

We may identify $\mathfrak{a}_{M'}^*$ with the space $\mathbb{R}^{q_1+\cdots+q_t}$. We will write an element $x \in \mathbb{R}^{q_1+\cdots+q_t}$ in coordinate form, as $x = (x_j^l)$ with $j \in [1, q_l]$ and $l \in [1,t]$. We may view $\mathfrak{a}_{M'_\xi}^*$ as the subspace of all $x = (x_j^l)$ such that $x_j^l = -x_{j'}^{l'}$ if $\tau(j,l) = (j',l') \neq (j,l)$ and $x_j^l = 0$ if $\tau(j,l) = (j,l)$ where $\xi \leftrightarrow \tau$. Note that $\xi \in \mathfrak{I}_{M'}$. As explained in ([Off], §3.4), there is then an element $\rho_\xi \in (\mathfrak{a}_{M'_\xi}^*)$ such that for all $m \in M'_\xi(\mathbb{A})$

$$\delta_{Q'_\xi}(m) = e^{\langle 2\rho_\xi, H_{M'}(m)\rangle}.$$

Define

$$\alpha_j^l(\tau) = \begin{cases} -\frac{1}{2} & (j,l) < \tau(j,l) \\ 0 & (j,l) = \tau(j,l) \\ \frac{1}{2} & (j,l) > \tau(j,l) \end{cases}.$$

The following proposition complements the analysis of double cosets in [Off].

Proposition 1. *For every triple (10) we may write $\rho_{Q'} - 2\rho_\xi \in \mathbb{R}^{q_1+\cdots+q_t}$ in coordinate form as*

$$(\rho_{Q'} - 2\rho_\xi)_j^l = \alpha_j^l(\tau).$$

Proof. Here is were we need the explicit choice (9) of η associated to the triple (10). The decomposition (9) defining η is such that $\eta = w\eta_1$, where ξ_1 is a minimal admissible twisted involution with respect to M' and $w \in W^0(\xi, \xi_1)$. Both minimal twisted involutions, and the subset $W^0(\xi, \xi_1)$ of W are defined in ([Off], §3.1). We may therefore apply ([Off], Proposition 3.9) to reduce the statement to the case that ξ is minimal. Following the explicit description of minimal twisted involutions in ([Off], §3.5) it is easy to compute ρ_ξ explicitly. We leave the computation for the reader. $\square$

Example. We illustrate the $\xi \leftrightarrow \tau \leftrightarrow \eta$ correspondence by an example. Let M be the Levi subgroup of type $(2, 2, 4, 2)$ of GL_{10} and let $\tau = (2\ 4)$ be an involution on the blocks of M. Note that $\tau \in T_M$. We set

$$\xi = (1\ 3\ 5\ 7\ 9)(2\ 4\ 6\ 8\ 10) \text{ and } \eta = (1\ 3)(2\ 4\ 5\ 6\ 7\ 8).$$

Thus $\xi \in \mathfrak{I}_M \cap W(H)$,

$$\xi w_0 = (1\ 2)(3\ 10)(4\ 9)(5\ 8)(6\ 7),\ \eta w_0 \eta^{-1} = \xi w_0$$

and $\xi \leftrightarrow \tau \leftrightarrow \eta$ is a corresponding triple. Assume further that M' is the Levi subgroup of type $(2, 2, 2, 2, 2)$ and let $\tau' = (1\ 4)(2\ 3)$ be a permutation on the blocks of M'. Thus $(M', \tau') \in \Gamma_M$. We set

$$\xi' = (1\ 9\ 7\ 5\ 3)(2\ 10\ 8\ 6\ 4) = \xi^{-1} \text{ and } \eta' = (5\ 9\ 7)(6\ 10\ 8).$$

Thus $\xi' \in {}_M W_M \cap W(H)$ is not admissible,

$$\xi' w_0 = (1\ 8)(2\ 7)(3\ 6)(4\ 5)(9\ 10),\ \eta' w_0 \eta'^{-1} = \xi' w_0$$

and $\xi' \leftrightarrow \tau' \leftrightarrow \eta'$ is another corresponding triple.

3.2. **Representations of GL_n over a $\wp$-adic field.** In this subsection F is a non-archimedean local field. We will use notation from [BZ77] that has now become standard in the literature, sometimes without further comment. Let $r = r_1 + \cdots + r_l$ and let π_i be an admissible representation of G_{r_i}. We denote by $\pi_1 \times \cdots \times \pi_l$ the representation of G_r parabolically induced from $\pi_1 \otimes \cdots \otimes \pi_l$. Denote by $M_{\mathbf{r}}$ the Levi subgroup of G_r of type $\mathbf{r} = (r_1, \ldots, r_l)$. We will use the functors $i_{G_r, M_{\mathbf{r}}}$ and $r_{M_{\mathbf{r}}, G_r}$ as defined in ([BZ77], §2.3). Thus $i_{G_r, M_{\mathbf{r}}}(\pi_1 \otimes \cdots \otimes \pi_l) = \pi_1 \times \cdots \times \pi_l$ and $r_{M_{\mathbf{r}}, G_r}$ is the normalized Jacquet functor. Let C denote the set of all irreducible super-cuspidal representations of G_r for all $r \geq 1$, and let C^u denote the subset of unitary representations in C. For $\rho \in C$ we set $\deg \rho = r$ when ρ is a representation of G_r. For a positive integer k and $\rho \in C$, the set $[\rho, \nu^{k-1}\rho] = \{\nu^i \rho \mid i = 0, \ldots, k-1\}$ is called a segment in C. Denote by $S(C)$ the set of all segments in C. Following Zelevinsky we denote by $\langle [\rho, \nu^{k-1}\rho] \rangle^t$ the unique irreducible sub-quotient of $\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho$.

The representations $\langle[\rho, \nu^{k-1}\rho]\rangle^t$ are exactly the irreducible, essentially square integrable representations [Zel80].

3.2.1. *Jacquet functor of a square integrable representation.* Let $\Delta = [\rho, \nu^{k-1}\rho] \in S(C)$ be a segment and let $r = k \deg \rho$. The following lemma follows by a simple induction from ([Zel80], §9.5).

Lemma 1. *Let M be a standard Levi of G_r of type $(m_1, \dots, m_q)$. Then,*

$$r_{M,G_r}(\langle\Delta\rangle^t) = 0$$

unless m_j is divisible by $\deg \rho$ *for all $j \in [1, q]$. When this is the case, denote $m_j = k_j \deg \rho$ then*

$$r_{M,G_r}(\langle\Delta\rangle^t) = \langle\Delta_1\rangle^t \otimes \cdots \otimes \langle\Delta_q\rangle^t$$

where

$$\Delta_j = [\nu^{k_{j+1}+\cdots+k_s}\rho, \nu^{k_j+\cdots+k_s-1}\rho].$$

We will need to use two applications of the geometric lemma of Bernstein-Zelevinsky given in [BZ77]. Next we describe them, in a form suitable for our needs.

3.2.2. *First application of the geometric lemma.* We describe in 'coordinate form' as presented in ([Zel80], §1.6) a particular case of ([BZ77], Lemma 2.12). We further make an assumption that is sufficient to our use. Let M and L be Levi subgroups of G_r of types $(n_1, \dots, n_s)$ and $(m_1, \dots, m_q)$ respectively. The geometric lemma, provides a filtration for the functor

$$r_{L,G_r} \circ i_{G_r,M} : Alg(M) \to Alg(L).$$

Let $B = (b_{i,j})$ be an $s \times q$ matrix that satisfies:

$$\text{(11)} \qquad \begin{cases} \text{All } b_{i,j} \text{ are non-negative integers;} \\ \sum_{j=1}^q b_{i,j} = n_i \text{ for all } i \in [1, s]; \\ \sum_{i=1}^s b_{i,j} = m_j \text{ for all } j \in [1, q]. \end{cases}$$

Let $\rho = \rho_1 \otimes \cdots \otimes \rho_s$ be an irreducible representation of M. Denote $\beta_i = (b_{i,1}, \dots, b_{i,q})$ and $\gamma_j = (b_{1,j}, \dots, b_{s,j})$. The assumption we make is that $r_{M_{\beta_i},G_{n_i}}(\rho_i)$ is either zero or irreducible for all $i \in [1, s]$ and all matrices B as in (11). When this is the case, we describe composition factors of the representation $r_{L,G_r} \circ i_{G_r,M}(\rho)$ indexed by the set of all matrices B as in (11). Denote

$$r_{M_{\beta_i},G_{n_i}}(\rho_i) = \rho_{i,1} \otimes \cdots \otimes \rho_{i,q}$$

where $\rho_{i,j}$ is an irreducible representation of $G_{b_{i,j}}$ and set

$$\pi_j(\rho, B) = \rho_{1,j} \times \cdots \times \rho_{s,j}$$

and

$$F_B(\rho) = \pi_1(\rho, B) \otimes \cdots \otimes \pi_q(\rho, B).$$

Proposition 2. *With the above notation and assumption, the representation $r_{L,G} \circ i_{G,M}(\rho)$ is glued from the representations $F_B(\rho)$ for all matrices B that satisfy (11).*

3.2.3. *Second application of the geometric lemma.* Here we describe a particular case of ([BZ77], Theorem 5.2). Let $P = MU$ be a parabolic subgroup of G. We describe a decomposition series for the functor

$$\mathrm{res}_{|H} \circ i_{G,M} : Alg(M) \to Alg(H),$$

where $\mathrm{res}_{|H} : Alg(G) \to Alg(H)$ is the restriction of representations of G to representations of H. The decomposition series is indexed by the double coset space $P\backslash G/H$. To apply the geometric lemma to our needs, we will need our analysis of the double coset space. In particular, the fact that $P\backslash G/H$ is finite is necessary. We will use the notation introduced in §3.1. Thus, in particular, we fix the choice of representatives $\{\eta\}$ to the double cosets as explained in §3.1, and for each η we associate the corresponding triple (10). Let $\pi \in Alg(M)$ and let V be the space of the representation $i_{G,M}(\pi)$. According to ([BZ76], §1.5), there is an ordering $\eta_1, \dots, \eta_m$ of the double coset representatives, so that

$$Y_i = \bigcup_{j=1}^{i} P\eta_j H$$

is open in G for all i. The space V is a space of functions with values in the space of π. Let

$$V_i = \{f \in V \,|\, \mathrm{supp}(f) \subset Y_i\}$$

The sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$$

is a filtration of V by H-invariant subspaces. Denote

$$X_i = X_{\eta_i} = \frac{V_i}{V_{i-1}}.$$

The spaces X_η are the decomposition factors we wish to describe. Fix a representative η. Define

$$H^P_\eta = H \cap \eta^{-1} P \eta = \eta^{-1} P_\xi \eta.$$

Let Y be a subgroup of a group X. For a representation ρ of Y and $x \in X$ we denote by ρ^x the representation of $x^{-1}Yx$, on the space of ρ, given by $\rho^x(z) = \rho(xzx^{-1})$, $z \in x^{-1}Yx$.

Proposition 3. *We have*

$$X_\eta \simeq \mathrm{ind}_{H_\eta^P}^H([\delta_P^{1/2}\pi_{|P_\xi}]^\eta)$$

where by $\mathrm{ind}_{H_\eta^P}^H$ *we mean, the un-normalized induction with compact support.*

Remark 2. In the form that the geometric lemma is stated in [BZ77], it only applies under the assumption that all double cosets are associated to admissible twisted involutions. This assumption, only appears there for an aesthetic reason – in order to remain in the context of *parabolic* induction and Jacquet functor. The proof in [BZ77] is valid in our case, once we allow in the answer non-parabolic induction such as $\mathrm{ind}_{H_\eta^P}^H$.

Corollary 1. *For every* η *we have*

$$\mathrm{Hom}_H(X_\eta, 1) \simeq \mathrm{Hom}_{M'_\xi}(r_{M',M}(\pi))[\rho_{Q'} - 2\rho_\xi], 1). \tag{12}$$

Proof. Applying Frobenius reciprocity ([BZ76], §2.29) to the identity obtained in the proposition, we get

$$\mathrm{Hom}_H(X_\eta, 1) = \mathrm{Hom}_{H_\eta^P}([\delta_P^{1/2}\delta_{P_\xi}^{-1}\pi_{|P_\xi}]^\eta, 1).$$

Recall that $\delta_P^{1/2}\delta_{P_\xi}^{-1}\pi$ is a representation of M, or as we view it here a representation of P trivial on U and that $\mathrm{proj}_M(P_\xi) = M'_\xi U'$. Note that for $p = mu \in P_\xi$ with $m \in M'_\xi$ and $u \in R$, such that $\mathrm{proj}_M(u) = u' \in U'$, we have $(\delta_P^{1/2}\delta_{P_\xi}^{-1}\pi)(p) = (\delta_P^{1/2}\delta_{P_\xi}^{-1}(m))\pi(mu')$. Thus, we have

$$\mathrm{Hom}_{H_\eta^P}([\delta_P^{1/2}\delta_{P_\xi}^{-1}\pi_{|P_\xi}]^\eta, 1) = \mathrm{Hom}_{M'_\xi U'}(\pi[\rho_P - 2\rho_\xi], 1).$$

Factoring through U' in the right hand side and keeping in mind the normalization of the Jacquet functor and that $\rho_{Q'} = \rho_P + \rho_{Q'}^P$, we obtain the corollary. □

3.2.4. *Irreducible, generic, unitary representations of* GL_r *after Zelevinsky and Tadić.* The irreducible unitary representations of $G_r(F)$ were classified in [Tad86]. Together with the classification of generic representations in [Zel80], it is easy to point out which are the irreducible, unitary, generic representations. Denote by B_{nd}^u the set

$$\Big\{\langle\Delta\rangle^t,\ \nu^\beta\langle\Delta\rangle^t \times \nu^{-\beta}\langle\Delta\rangle^t |\ \Delta = [\nu^{\frac{1-k}{2}}\rho, \nu^{\frac{k-1}{2}}\rho],\ \rho \in C^u, k \in \mathbb{Z}_{>0}, \beta \in (0, 1/2)\Big\}.$$

Theorem 6. *Let* σ *be an irreducible, unitary, generic representation of* G_r*. Then, there exist* $\sigma_1, \ldots, \sigma_m \in B_{nd}^u$ *such that*

$$\sigma = \sigma_1 \times \cdots \times \sigma_m.$$

Furthermore, any representation of the form $\sigma_1 \times \cdots \times \sigma_m$ is irreducible, unitary and generic, and determines uniquely the multi-set $(\sigma_1, \ldots, \sigma_m)$.

3.3. **The non-vanishing.** We fix a decomposition $2n = 2kr$. Let σ be an irreducible, unitary, generic representation of G_r and let $\pi = \sigma^{\otimes 2k}$ be a representation of the standard Levi M of G of type $(r, \ldots, r)$. The local linear form $l_H = j \circ M(w')$ on $I(\pi, \Lambda) = i_{G,M}(\pi[\Lambda])$ was defined in §2.2. Recall that the permutation w' was defined by (1). We have

$$\{i < 2k | w'(i) > w'(i+1)\} = \{2i | i = 1, \ldots, k-1\}.$$

Set

$$w' = w'_1 w'_2,$$

where $w'_2 = (23)(45)\cdots(2k-2,\, 2k-1)$. The intertwining operator $M(w')$ decomposes up to a non-zero constant into $R(w'_1, w'_2\Lambda) \circ M(w'_2)$. The operator $R(w'_1, w'_2\Lambda)$ is an isomorphism and l_H is given by the decomposition

$$i_{G,M}(\pi[\Lambda]) \xrightarrow{M(w'_2)} i_{G,M}(\pi[w'_2\Lambda]) \underset{\sim}{\xrightarrow{R(w'_1, w'_2\Lambda)}} i_{G,M}(\pi[w'\Lambda]) \xrightarrow{j} \mathbb{C}. \tag{13}$$

Let

$$V = i_{G,M}(\pi[w'_2\Lambda]) \text{ and } W = M(w'_2)(i_{G,M}(\pi[\Lambda])) \subset V.$$

Proposition 4. *If l_H is identically zero on $I(\pi, \Lambda)$ then*

$$\mathrm{Hom}_H(V/W, 1) \neq 0.$$

Proof. We assume l_H is identically zero on $I(\pi, \Lambda)$ and follow the diagram (13). The linear form j on $I(\pi, w'\Lambda)$ is H-invariant, since

$$\rho_P - 2\rho_{P_H} + w'\Lambda = (k-1, k-3, \ldots, 1-k, 1-k, \ldots, k-3, k-1)$$

lies in $(\mathfrak{a}_{M_H})^\perp = \{(x_1, \ldots, x_n, x_n, \ldots, x_1) \,|\, x_i \in \mathbb{R}\}$. We can argue as in ([JR92b] , Proposition 2) that j is not identically zero on $I(\pi, w'\Lambda)$. Since $R(w'_1, w'_2\Lambda)$ is an isomorphism, j induces a non-zero H-invariant form $l = j \circ R(w'_1, w'_2\Lambda)$ on $I(\pi, w'_2\Lambda)$. By our assumption it vanishes on W. Therefore, there exists an H-invariant linear form l on V/W. $\square$

The non-vanishing of l_H will therefore follow from the following.

Proposition 5.

$$\mathrm{Hom}_H(V/W, 1) = 0.$$

The proof of this proposition will occupy the rest of this section. The representation

$$\nu^{1/2}\sigma \times \nu^{-1/2}\sigma$$

has a unique irreducible quotient ([MW89], §I.11) which we denote by $L(\sigma)$. The representation $L(\sigma)$ is also the unique irreducible subrepresentation of $\nu^{-1/2}\sigma \times \nu^{1/2}\sigma$. More generally, if $\delta_1, \dots, \delta_t$ are square integrable representations and $e_1 \geq \cdots \geq e_t$, then the representation

$$\nu^{e_1}\delta_1 \times \cdots \times \nu^{e_t}\delta_t \tag{14}$$

has a unique irreducible quotient, which appears in the decomposition series of (14) with multiplicity one, [BW00]. It follows from §3.2.4 that σ can be expressed in the form (14) with exponents of absolute value strictly less then 1/2. Therefore the representation $\nu^{1/2}\sigma \times \nu^{-1/2}\sigma$ can also be written in the form (14). Thus, $L(\sigma)$ occurs with multiplicity one in the decomposition series of $\nu^{1/2}\sigma \times \nu^{-1/2}\sigma$ and hence also in the decomposition series of $\nu^{-1/2}\sigma \times \nu^{1/2}\sigma$. Set $T(\sigma) = (\nu^{-1/2}\sigma \times \nu^{1/2}\sigma)/L(\sigma)$. It follows that

$$\mathrm{Hom}_{G_{2r}}(T(\sigma), \nu^{-1/2}\sigma \times \nu^{1/2}\sigma) = 0. \tag{15}$$

The representation $L(\sigma)$ is also the image of the intertwining operator

$$M(s_1) : (\nu^{1/2}\sigma \times \nu^{-1/2}\sigma) \to (\nu^{-1/2}\sigma \times \nu^{1/2}\sigma),$$

where $s_1 = (1, 2)$ is the permutation interchanging the two diagonal $r \times r$-blocks. For $i = 1, \dots, k-1$, let $Q_i = L_i V_i$ be the parabolic of G of type

$$(\overbrace{r, \dots, r}^{2i-1}, 2r, \overbrace{r, \dots, r}^{2(k-i)-1}) \tag{16}$$

and let

$$\rho_i = \sigma \otimes \cdots \otimes \sigma \otimes T(\sigma) \otimes \sigma \otimes \cdots \otimes \sigma.$$

It is a representation of L_i. For a subset S of $\{1, \dots, k-1\}$ denote

$$w_S = \prod_{j \in S} (2j,\, 2j+1). \tag{17}$$

If $i \in S$ we denote

$$\mu_{S,i} = (s_{2i} w_S \Lambda)_{Q_i} = (w_S \Lambda)_{Q_i}.$$

Lemma 2. *If*

$$\mathrm{Hom}_H(V/W, 1) \neq 0$$

then there exist $i \in S \subset [1, k-1]$ *such that*

$$\mathrm{Hom}_H(i_{G,L_i}(\rho_i[\mu_{S,i}]), 1) \neq 0.$$

Proof. By definition

$$V = \nu^{k-\frac{1}{2}}\sigma \times (\nu^{k-\frac{5}{2}}\sigma \times \nu^{k-\frac{3}{2}}\sigma) \times \cdots \times (\nu^{\frac{3}{2}-k}\sigma \times \nu^{\frac{5}{2}-k}\sigma) \times \nu^{\frac{1}{2}-k}\sigma.$$

We can also make the image W of the map

$$\nu^{k-\frac{1}{2}}\sigma \times (\nu^{k-\frac{3}{2}}\sigma \times \nu^{k-\frac{5}{2}}\sigma) \times \cdots \times (\nu^{\frac{5}{2}-k}\sigma \times \nu^{\frac{3}{2}-k}\sigma) \times \nu^{\frac{1}{2}-k}\sigma$$

$$\downarrow M(w_2')$$

$$\nu^{k-\frac{1}{2}}\sigma \times (\nu^{k-\frac{5}{2}}\sigma \times \nu^{k-\frac{3}{2}}\sigma) \times \cdots \times (\nu^{\frac{3}{2}-k}\sigma \times \nu^{\frac{5}{2}-k}\sigma) \times \nu^{\frac{1}{2}-k}\sigma$$

more explicit.

$$W = \nu^{k-\frac{1}{2}}\sigma \times \nu^{k-2}L(\sigma) \times \cdots \times \nu^{2-k}L(\sigma) \times \nu^{\frac{1}{2}-k}\sigma.$$

If $\mathrm{Hom}_H(V/W, 1) \neq 0$ then, there is a choice of $\sigma_i \in \{\nu^{k-2i}L(\sigma), \nu^{k-2i}T(\sigma)\}$ for all $i \in [1, k-1]$, such that the set

$$S = \{i | \sigma_i = \nu^{k-2i}T(\sigma)\}$$

is not empty and there exists a non-zero H-invariant form on

$$\nu^{k-\frac{1}{2}}\sigma \times \sigma_1 \times \cdots \times \sigma_{k-1} \times \nu^{\frac{1}{2}-k}\sigma. \tag{18}$$

Let $i_0 \in S$ and define

$$\tau_i = \begin{cases} \nu^{k-2i-\frac{1}{2}}\sigma \times \nu^{k-2i+\frac{1}{2}}\sigma & \sigma_i = \nu^{k-2i}T(\sigma),\ i \neq i_0 \\ \nu^{k-2i_0}T(\sigma) & i = i_0 \\ \nu^{k-2i+\frac{1}{2}}\sigma \times \nu^{k-2i-\frac{1}{2}}\sigma & \sigma_i = \nu^{k-2i}L(\sigma) \end{cases}.$$

There is then a surjective G-morphism from

$$\tau = \nu^{k-\frac{1}{2}}\sigma \times \tau_1 \times \cdots \times \tau_{k-1} \times \nu^{\frac{1}{2}-k}\sigma$$

to the representation (18). To conclude the lemma, it is left to observe that

$$i_{G,L_{i_0}}(\rho_{i_0}[\mu_{S,i_0}]) = \tau.$$

□

To complete the proof of the local non-archimedean problem, we need to show that

$$\mathrm{Hom}_H(i_{G,L_i}(\rho_i[\mu_{S,i}]), 1) = 0 \tag{19}$$

for all pairs (i, S) such that $i \in S \subset [1, k-1]$. In order to show (19) we apply the geometric lemma of Bernstein-Zelevinsky twice. The representation $i_{G,L_i}(\rho_i[\mu_{S,i}])$ induced from Q_i is the image of the representation $i_{G,M}(\pi[w_S\Lambda])$ induced from P, under the intertwining operator $R(s_{2i}, w_S\Lambda)$ associated to the reflection $s_{2i} = (2i, 2i+1)$. We will show that $i_{G,L_i}(\rho_i[\mu_{S,i}])$ has no symplectic period, by first showing that, the H-filtration on $i_{G,M}(\pi[w_S\Lambda])$ obtained from the geometric lemma, has a unique factor that has a symplectic period and then showing that in the H-filtration of $i_{G,L_i}(\rho_i[\mu_{S,i}])$ obtained from the geometric lemma, the associated factor does not have a symplectic period. Let S be any subset of $[1,k-1]$. We apply the geometric lemma of Bernstein-Zelevinsky to the induced representation space V_S of

$$i_{G,M}(\pi[w_S\Lambda]) = \nu^{k+\frac{1}{2}-w_S^{-1}(1)}\sigma \times \cdots \times \nu^{k+\frac{1}{2}-w_S^{-1}(2k)}\sigma$$

where w_S is the permutation of $[1,2k]$ defined in (17). There is then an H-filtration of V_S by subspaces V_η parameterized by double coset representatives of $P\backslash G/H$, and associated sub-quotients X_η as explained in §3.2.3.

Proposition 6. *For a corresponding triple*

$$\xi \leftrightarrow \tau \leftrightarrow \eta$$

as in (10), we have

$$\mathrm{Hom}_H(X_\eta, 1) = 0$$

unless ξ is the unique element of $\mathfrak{I}_M \cap W(H)$ so that the corresponding τ (which is then an involution on $[1,2k]$) is defined by:

$$\tau(w_S(2j-1)) = w_S(2j),\ j \in [1,k]. \tag{20}$$

Proof. From Theorem 6, we may assume in particular that σ has the form

$$\sigma = \langle\Delta_1\rangle^t \times \cdots \times \langle\Delta_s\rangle^t$$

where

$$\Delta_i = \nu^{\beta_i}[\nu^{\frac{1-t_i}{2}}, \rho_i, \nu^{\frac{t_i-1}{2}}\rho_i]$$

$\rho_i \in C^u$ and $|\beta_i| < \frac{1}{2}$, $i \in [1,s]$. Let $\deg \rho_i = p_i$, and set $n_i = p_i t_i$. Let L be the Levi subgroup of G_r of type $(n_1,\ldots,n_s)$. Let the corresponding triple $\xi \leftrightarrow \tau \leftrightarrow \eta$ not satisfy (20). We must show that $Hom_H(X_\eta,1) = 0$. By (12), this is the same as showing that

$$\mathrm{Hom}_{M'_\xi}((r_{M',M}(\pi))[\rho_{Q'} - 2\rho_\xi + w_S\Lambda], 1) = 0. \tag{21}$$

Denote $M' = M'_1 \times \cdots \times M'_{2k}$ where M'_l is a Levi of G_r of type $(m^l_1,\ldots,m^l_{q_l})$. From ([Zel80], §1.5) we get that

$$r_{M',M}(\pi) = r_{M'_1,G_r}(\sigma) \otimes \cdots \otimes r_{M'_{2k},G_r}(\sigma).$$

In order to prove (21) it is enough to prove that

$$\mathrm{Hom}_{M'_\xi}((Z_1 \otimes \cdots \otimes Z_{2k})[\rho_{Q'} - 2\rho_\xi + w_S\Lambda], 1) = 0$$

for every choice of composition factors Z_l in a given filtration for the representation $r_{M'_l,G_r}(\sigma)$ for all $l \in [1,2k]$. Note that

$$r_{M'_l,G_r}(\sigma) = r_{M'_l,G_r} \circ i_{G_r,L}(\langle\Delta_1\rangle^t \otimes \cdots \otimes \langle\Delta_s\rangle^t).$$

We apply Proposition 2, to obtain a filtration of $r_{M'_l,G_r}(\sigma)$. By Lemma 1, the irreducibility assumption we made in §3.2.3 holds. We therefore get for each $l \in [1,2k]$ a filtration of $r_{M'_l,G_r}(\sigma)$ indexed by matrices $B_l = (b^l_{i,j})$ of non-negative integers that satisfy

$$\sum_{i=1}^{s} b^l_{i,j} = m^l_j\,;\ \sum_{j=1}^{q_l} b^l_{i,j} = n_i.$$

For every such choice of matrices we must show that

$$\mathrm{Hom}_{M'_\xi}((F_{B_1}(\sigma)\otimes\cdots\otimes F_{B_{2k}}(\sigma))[\rho_{Q'} - 2\rho_\xi + w_S\Lambda], 1) = 0 \tag{22}$$

where

$$F_{B_l}(\sigma) = \pi_1(\sigma, B_l)\otimes\cdots\otimes\pi_q(\sigma, B_l)$$

is defined in §2. Using Lemma 1 we can write $F_{B_l}(\sigma)$ explicitly. Unless for all i, j, l the entry $b^l_{i,j}$ is divisible by p_i, we have

$$F_{B_1}(\sigma)\otimes\cdots\otimes F_{B_{2k}}(\sigma) = 0.$$

Therefore, we may assume $b^l_{i,j} = p_i k^l_{i,j}$. When this is the case

$$\pi^l_j = \pi_j(\sigma, B_l) = \langle\Delta^l_{1,j}\rangle^t\times\cdots\times\langle\Delta^l_{s,j}\rangle^t$$

where

$$\Delta^l_{i,j} = \nu^{\beta_i}[\nu^{\frac{1-t_i}{2}+k^l_{i,j+1}+\cdots+k^l_{i,q_l}}\rho_i, \nu^{\frac{1-t_i}{2}+k^l_{i,j}+\cdots+k^l_{i,q_l}-1}\rho_i].$$

Denote by ω^l_j the central character of π^l_j and let $|\omega^l_j| = \nu^{x^l_j}$. From the explicit description of the group M'_ξ in §3.1.3, if (22) does not hold, we must have in particular

$$x^l_j + (\rho_{Q'} - 2\rho_\xi + w_S\Lambda)^l_j = x^{l'}_{j'} + (\rho_{Q'} - 2\rho_\xi + w_S\Lambda)^{l'}_{j'} \tag{23}$$

whenever $\tau(j,l) = (j',l') \neq (j,l)$. From Proposition 1 we get that

$$(\rho_{Q'} - 2\rho_\xi + w_S\Lambda)^l_j = k + \frac{1}{2} - w_S^{-1}(l) + \alpha^l_j(\tau).$$

Denote $\tau(1,1) = (j,l)$. Since $(M',\tau)\in\Gamma_M$, it follows that $j = q_l$. If we further have $l = 1$ it also follows that $q_1 = 1$, so $\tau(1,1) = (1,1)$. When this is the case, $m^1_1 = r$ must be even and $\pi^1_1 = \sigma$. A non-zero element of (22), will then provide a non-zero symplectic period of σ. Since σ is irreducible and generic, this stands in contradiction with [HR90]. We therefore must have $\tau(1,1) = (q_l, l)$ for some $l > 1$. Since $w_S^{-1}(1) = 1$ we have $w_S^{-1}(l) \geq 2$. Note also that $\alpha^1_1(\tau) = -\frac{1}{2} = -\alpha^l_{q_l}(\tau)$. We apply the identity (23) to the pair $(1,1)$ and get

$$0 \geq 2 - w_S^{-1}(l) = x^1_1 - x^l_{q_l}. \tag{24}$$

The absolute value of the central character of π_j^l may be computed explicitly, we have

$$x_j^l = \frac{1}{m_j^l}\sum_{i=1}^s p_i k_{i,j}^l \left(\beta_i + \frac{k_{i,j}^l - t_i}{2} + k_{i,j+1}^l + \cdots + k_{i,q_l}^l\right).$$

Since $m_{q_l}^l = m_1^1$ we get,

$$x_1^1 - x_{q_l}^l =$$

$$\frac{1}{m_1^1}\sum_{i=1}^s p_i \left(k_{i,1}^1\left(\beta_i + \frac{k_{i,1}^1 - t_i}{2} + k_{i,2}^1 + \cdots + k_{i,q_1}^1\right) - k_{i,q_l}^l\left(\beta_i + \frac{k_{i,q_l}^l - t_i}{2}\right)\right) =$$

$$\frac{1}{m_1^1}\sum_{i=1}^s p_i \left(k_{i,1}^1\left(\beta_i + \frac{t_i - k_{i,1}^1}{2}\right) + k_{i,q_l}^l\left(\frac{t_i - k_{i,q_l}^l}{2} - \beta_i\right)\right).$$

If $q_1 > 1$ then, $m_1^1 < r$ and therefore also $q_l > 1$. It then follows that $t_i - k_{i,1}^1 \geq 1$ and $t_i - k_{i,q_l}^l \geq 1$ for all i and therefore each summand on the right hand side is positive. This stands in contradiction with the inequality (24). Thus, we must have $q_1 = q_l = 1$. In this case $t_i = k_{i,1}^1 = k_{i,q_l}^l$ and the right hand side vanishes. This implies that $l = w_S(2)$. We complete the proof of the proposition by induction on k. The case $k = 1$ is proved by the above argument. If the proposition is false for some $k > 1$ then for some τ different then (20) and some $B_1, \dots, B_{2k}$ we have that (22) is false. From the above argument, by dropping the 1-st and $w_S(2)$-th blocks of M, we may also obtain a counter example for $k - 1$. $\square$

From now on we fix i_0 and S such that

$$i_0 \in S \subset [1, k-1].$$

We will denote $Q = LV = Q_{i_0}$, $\rho = \rho_{i_0}$ and $\mu = \mu_{S,i_0}$. We wish to show that $Hom_H(i_{G,L}(\rho[\mu]), 1) = 0$. Let us denote by

$$\xi_S \leftrightarrow \tau_S \leftrightarrow \eta_S$$

the corresponding triple, where τ_S is the τ defined in (20). Note that $\xi_S \in \mathfrak{I}_M \cap W(H)$. With respect to the parabolic Q, ξ_S is no longer admissible but we still have $\xi_S \in {}_L W_L \cap W(H)$. We wish to apply the geometric lemma §3.2.3 to $\mathrm{res}_{|H} \circ i_{G,L}(\rho[\mu])$. We fix an ordering $\{\eta_1, \dots, \eta_m\}$ for the representatives for $P\backslash G/H$, so that there is a subset of indices $\{j_1, \dots, j_l\}$ such that $\{\eta_{j_1}, \dots, \eta_{j_l}\}$ is a set of representatives for $Q\backslash G/H$ and

$$\bigcup_{i=1}^{j} P\eta_i H\,;\ \bigcup_{a=1}^{b} Q\eta_{j_a} H$$

are open in G for all $j \in [1,m]$, $b \in [1,l]$. We can then write the interval [1,m] of integers as a union of consecutive intervals $J_1 = [1,i_1], J_2 = [i_1+1,i_2],\dots,J_l = [i_{l-1}+1,m]$ such that $j_a \in J_a$ for all a and

$$Q\eta_{j_a}H = \cup_{j\in J_a} P\eta_j H.$$

The geometric lemma then defines a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m$$

of $i_{G,M}(\pi[w_S\Lambda])$ which is a refinement of the filtration

$$0 = V_0 \subset V_{i_1} \subset \cdots \subset V_{i_l}.$$

Similarly we get a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_l$$

of $i_{G,L}(\rho[\mu])$. Let l_0 be such that $\eta_S = \eta_{j_{l_0}}$.

Lemma 3. *The intertwining operator $M(s_{2i_0})$ maps V_{i_a} onto U_a for all $a \in [1,l]$.*

Proof. It is easy to see that the intertwining operator $M(s_{2i_0})$ maps V_{i_a} into U_a. We know that it maps V onto U. Since s_{2i_0} lies in the Weyl group of Q, for each f, $f' \in V$ we have that if $f_{|Qx} = f'_{|Qx}$ then $M(s_{2i_0})f(x) = M(s_{2i_0})f'(x)$. So if $\varphi \in U_a$ and $f \in V$ maps to φ, then the function f' that agrees with f on $\overset{a}{\underset{b=1}{\cup}} Q\eta_{j_b}H$ and is zero elsewhere, is in V_{i_a} and also maps to φ. □

Lemma 4.

$$\mathrm{Hom}_H(U_{l_0}/U_{l_0-1},1) = 0.$$

Proof. We observe that

$$M = L \cap \xi_S w_0 L w_0 \xi_S^{-1}$$

and that

$$V' = L \cap \xi_S w_0 V w_0 \xi_S^{-1}$$

is the unipotent radical of the parabolic $P \cap L$ of L. From (12) we get that

$$\mathrm{Hom}_H(U_{l_0}/U_{l_0-1},1) = \mathrm{Hom}_{M_{\xi_S}}(r_{M,L}(\rho)[\rho_P - 2\rho_{\xi_S} + \mu],1). \tag{25}$$

We denote $P_{r,r} = M_{r,r}U_{r,r}$ the parabolic of G_{2r} of type (r,r). Note that

$$r_{M,L}(\rho) = \sigma \otimes \cdots \otimes \sigma \otimes r_{M_{r,r},G_{2r}}(T(\sigma)) \otimes \sigma \otimes \cdots \otimes \sigma.$$

Recall that $w_S(2i_0) = 2i_0+1$ and that by definition of τ_S

$$\tau_S(2i_0) = w_S(2(i_0+1))\,;\ \tau_S(2i_0+1) = w_S(2i_0-1).$$

We focus our attention on four blocks of M. Those associated with the indices

$$w_S(2i_0-1),\ 2i_0,\ 2i_0+1,\ w_S(2(i_0+1)).$$

Those blocks in the given order, form a copy of $(G_r)^4$ in M and its intersection with M_{ξ_S} is of matrices of the form

$$\mathrm{diag}(\tilde{b}, a, b, \tilde{a})$$

with $a, b \in G_r$. The entries of $\rho_P - 2\rho_{\xi_S} + \mu$ in those four blocks are

$$\left(k-2i_0+1, k-2i_0-\frac{1}{2}, k-2i_0+\frac{1}{2}, k-2i_0-1\right).$$

If (25) is not zero then we get that

$$\mathrm{Hom}_{M_{r,r}}(r_{M_{r,r},G_{2r}}(T(\sigma))[-\frac{1}{2},\frac{1}{2}] \otimes (\nu^1\tilde{\sigma} \otimes \nu^{-1}\tilde{\sigma}), 1) \neq 0$$

which is the same as saying that

$$\mathrm{Hom}_{M_{r,r}}(r_{M_{r,r},G_{2r}}(T(\sigma))[-\frac{1}{2},\frac{1}{2}], \nu^{-1}\sigma \otimes \nu^1\sigma) \neq 0.$$

Lifting to the group G_{2r} we get:

$$\mathrm{Hom}_{G_{2r}}(T(\sigma), \nu^{-\frac{1}{2}}\sigma \times \nu^{\frac{1}{2}}\sigma) \neq 0.$$

This stands in contradiction with (15). □

We can now complete the proof of (19). Assume that $A \neq 0$ is a symplectic period on $i_{G,L}(\rho[\mu])$. Let a be the smallest index such that A does not vanish on U_a. By Lemma 3, there is a symplectic period on V that vanishes on $V_{i_{a-1}}$ but does not vanish on V_{i_a}. This means that there exist t such that $i_{a-1} < t \leq i_a$ and a symplectic period on V_t/V_{t-1}. From Proposition 6 it follows that $t \in J_{l_0}$ which implies that $a = l_0$. This stands in contradiction with Lemma 4. □

4. The local problem (archimedean)

In this section the field F is either $\mathbb{R}$ or $\mathbb{C}$. Our goal is to show that l_H does not vanish on $I(\pi, \Lambda)$. The linear functional l_H is a decomposition of an intertwining operator $M(w')$ and a linear functional j on $I(\pi, w'\Lambda)$. We know that j does not vanish on this fully induced space, but the difficulty is to show that its restriction to the image of $M(w')$ does not vanish. For this purpose we construct a specific element of $I(\pi, w'\Lambda)$ that lies in the image of $M(w')$ and on which j is non-zero. The theory of minimal K-types due to Vogan (e.g. [Vog85]) is used to construct a specific element and to show that it indeed lies in the image of $M(w')$. The Cartan-Helgason theorem, which we state next, will show the non-vanishing of l_H on our specific construction. In our construction, we treat the real case and the complex case separately.

Finally, we remark that it will be interesting to compute the value of the functional j on the minimal K-type. It can be useful for estimates of the size of the symplectic period. This is not pursued in this work and we hope to come back to this issue in the future.

4.1. **The Cartan-Helgason Theorem.** In this section we recall a result usually referred to as the Cartan-Helgason Theorem [Hel70]. We apply the result as it appears in [Wal92], in two cases relevant to the real and the complex local non-vanishing problem. We start by describing the two settings we consider. The notation we use in this sub-section will be different then the rest of this work.

- **First setting ($F=\mathbb{R}$):**
  - $G'^{\mathbb{C}} = SO(2n,\mathbb{C}) = \{g \in SL(2n,\mathbb{C}) | {}^tgg = 1\}$,
  - $G' = SO^*(2n) = \{g \in SO(2n,\mathbb{C}) | {}^t\bar{g}\epsilon g = \epsilon\}$,
  - $U = SO(2n) = \{g \in SL(2n,\mathbb{R}) | {}^tgg = 1\}$,
  - $T = \{\mathrm{diag}(g_1,\dots,g_n) | g_i \in SO(2)\}$,
  - $H = Sp(2n,\mathbb{R}) = \{g \in SL(2n,\mathbb{R}) | {}^tg\epsilon g = \epsilon\}$,
- **Second setting ($F=\mathbb{C}$):**
  - $G'^{\mathbb{C}} = GL(2n,\mathbb{C})$,
  - $G' = U^*(2n) = \{g \in GL(2n,\mathbb{C}) | g\epsilon\bar{g}^{-1} = \epsilon\}$,
  - $U = U(2n) = \{g \in GL(2n,\mathbb{C}) | {}^t\bar{g}g = 1\}$,
  - $T = \{\mathrm{diag}(a_1,\dots,a_{2n}) | a_i \in \mathbb{C},\ |a_i| = 1\}$,
  - $H = Sp(2n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) | {}^tg\epsilon g = \epsilon\}$

We are interested in the (compact) symmetric space U/K where U is the maximal compact of $G = GL_{2n}(F)$ and $K = H \cap U$ is the maximal compact of H. Explicitly, $U = O(2n)$ if $F = \mathbb{R}$ and $U = U(2n)$ if $F = \mathbb{C}$ while K is $U(n)$ or $Sp(2n)$ compact, respectively. To study K-spherical representations of U we pass to the (non-compact) dual symmetric space G'/K. The role of G' is auxiliary and is only used in the proof of the Cartan-Helgason Theorem. The rest of the section applies to both settings. The groups U and G' are both imbedded into their universal complexification $G'^{\mathbb{C}}$. Let $\mathfrak{g}'$, $\mathfrak{u}$ and $\mathfrak{k}$ denote the Lie algebras of G', U and K, respectively and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}'$ with respect to the Killing form. Thus $\mathfrak{g}' = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition while $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Let (τ, V) be an irreducible representation of U equipped with a U-invariant scalar product $\langle\, ,\, \rangle$. The representation (τ, V) extends to a holomorphic representation of $G'^{\mathbb{C}}$. Its restriction to G' is an irreducible finite dimensional representation. The group T is a maximal torus of U. A character μ of T is called a weight of τ if there is a vector $v \neq 0$ in V such that $\tau(t)v = \mu(t)v$, $t \in T$. We then call v a weight vector for μ. Let $v_0 \in V$ be a weight vector with weight μ. Assume that μ has highest norm amongst weights of τ and that $\langle v_0, v_0 \rangle = 1$. There is then a minimal parabolic subgroup P_0 of G' with Langlands decomposition $P_0 = {}^0MAN$ so that μ is a dominant weight with respect to P_0. Here A is a vector group, ${}^0M$ is the centralizer of A in K and N is the unipotent radical of P_0. The weight vector v_0 is fixed by N. As in p. 537 of [Hel84] we have that

$$\langle \tau(g)u, v \rangle = \langle u, \bar{\tau}(\theta(g^{-1}))v \rangle,\ g \in G,\ u,\ v \in V,$$

where θ is the Cartan involution on G. It follows that for $\bar{n} \in \overline{N}$, the unipotent subgroup opposite to N, we have $\langle \tau(\bar{n})^{-1} v_0, v_0 \rangle = \langle v_0, v_0 \rangle$. Denote by v_K the vector

$$v_K = \int_K \tau(k) v_0 dk.$$

Lemma 5. *Assume μ is trivial on $T \cap K$, then*

$$\langle v_K, v_0 \rangle \neq 0.$$

Proof. In the proof of Theorem 4.1 in [Hel84], it is shown that in the situation of the lemma, ${}^0M$ acts trivially on v_0. The inner product $\langle v_K, v_0 \rangle$ can therefore be computed exactly as in the proof of Lemma 10.A.1.4 of [Wal92]. With the notation of [Wal92] we obtain

$$\langle v_K, v_0 \rangle = \int_{\bar{N}} a(\bar{n})^{\mu + 2\rho} d\bar{n},$$

where $\bar{n} = n(\bar{n})a(\bar{n})k(\bar{n})$ is the Iwasawa decomposition. The latter expression is strictly positive. The same computation is also carried out in the proof of Theorem 4.1 in [Hel84]. □

4.2. **The local problem (real).** Here $G = GL_{2n}(\mathbb{R})$, $K = O(2n) = \{g \in G | {}^t g g = 1\}$ and $K_0 = SO(2n)$ is the connected component of K. Also, $H = Sp(2n, \mathbb{R}) = \{g \in G | {}^t g \epsilon g = \epsilon\}$.

4.2.1. *Representations of K.* We summarize some basic facts that we will need about the representations of K_0 and of K. They can be found for example in §IV of [KV95]. For $\theta \in \mathbb{R}$ let

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then, $SO(2) = \{r(\theta) | \theta \in \mathbb{R}\}$ and the irreducible representations of $SO(2)$ are the characters

$$\chi_n(r(\theta)) = e^{in\theta},\ n \in \mathbb{Z}.$$

The irreducible representations ψ_m of $O(2)$ are parameterized by the set

$$\mathbb{Z}_+ \cup \{0^\pm\}.$$

If $m \geq 1$ then ψ_m is the two dimensional representation

$$\psi_m = \mathrm{Ind}_{SO(2)}^{O(2)}(\chi_m),$$

as a representation of $SO(2)$ it decomposes as

$$\chi_m \oplus \chi_{-m}.$$

As for the other two representations, ψ_{0^+} is the trivial representation and ψ_{0^-} is the one dimensional representation given by det. We define subgroups $T_0 \subset T \subset T_1$ of K that play a role in the classification of irreducible representations of K_0 and of K. Let

$$T_0 \simeq (SO(2))^n\,;\, T \simeq (SO(2))^{n-1} \times O(2)\,;\, T_1 \simeq (O(2))^n$$

be the subgroups of K, imbedded in diagonal (2×2)-blocks. The group T_0 is sometimes called a "small" Cartan subgroup and the group T a "large" Cartan subgroup of K. The choice of T amounts to a choice of positive roots to the root system associated with the complexified Lie algebras $(\mathfrak{k}, \mathfrak{t})$ of (K_0, T_0). If $\mathfrak{n}$ is the sum of all root spaces in $\mathfrak{k}$ associated with positive roots then T is the normalizer in K of $\mathfrak{t} \oplus \mathfrak{n}$. The irreducible representations $\chi_{\mathbf{m}}$ of T_0 are parameterized by $\mathbf{m} \in \mathbb{Z}^n$ and we refer to them as weights. For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ the associated weight is

$$\chi_{\mathbf{m}}(\mathrm{diag}(r(\theta_1), \dots, r(\theta_n))) = \prod_{i=1}^n \chi_{m_i}(r(\theta_i)).$$

The norm of a weight is given by $|\chi_{\mathbf{m}}|^2 = |\mathbf{m}|^2 = m_1^2 + \cdots + m_n^2$. A weight is called dominant if $m_1 \geq \cdots \geq m_{n-1} \geq |m_n|$. The irreducible representations of K_0 are parameterized by dominant weights. For a dominant weight $\mathbf{m}$ denote by $\sigma_{\mathbf{m}}$ the associated irreducible representation of K_0. It is characterized by the fact that it contains with multiplicity one a vector with weight $\mathbf{m}$ and all other dominant weights that appear have strictly lesser norm. We let ρ_K be half the sum of positive roots for $(\mathfrak{k}, \mathfrak{t})$. We have,

$$2\rho_K = (2n-2, \dots, 2, 0).$$

We define a norm on the irreducible representations of K_0 by

$$\|\sigma_{\mathbf{m}}\| = |\mathbf{m} + 2\rho_K|.$$

The irreducible representations of T are parameterized by n-tuples $(m_1,\dots,m_n)$ so that $m_1,\dots,m_{n-1}\in\mathbb{Z}$ and $m_n\in\mathbb{Z}_+\cup\{0^\pm\}$. The representation associated to such an n-tuple $\mathbf{m}$ is

$$\psi_{\mathbf{m}}=\chi_{m_1}\otimes\cdots\otimes\chi_{m_{n-1}}\otimes\psi_{m_n}.$$

We call such an n-tuple $\mathbf{m}$, and the representation $\psi_{\mathbf{m}}$ of T associated to it, dominant if it satisfies $m_1\ge\cdots\ge m_{n-1}\succeq m_n$, where by $m_{n-1}\succeq m_n$ we mean $m_{n-1}\ge m_n$ if m_n is a positive integer and $m_{n-1}\ge 0$ otherwise. Let

$$\hat{T}_{\mathrm{dom}}=\{(m_1,\dots,m_n)\in\mathbb{Z}^{n-1}\times(\mathbb{Z}_+\cup\{0^\pm\})|m_1\ge\cdots\ge m_{n-1}\succeq m_n\}.$$

The irreducible representations $\tau_{\mathbf{m}}$ of K are parameterized by $\mathbf{m}\in\hat{T}_{\mathrm{dom}}$. The correspondence is as follows. If V is the space of $\tau_{\mathbf{m}}$ and $V^{\mathfrak{n}}$ is the subspace killed by $\mathfrak{n}$ then $\psi_{\mathbf{m}}$ is isomorphic to the representation $(\tau_{\mathbf{m}}|_T,V^{\mathfrak{n}})$. For $\mathbf{m}\in\hat{T}_{\mathrm{dom}}$ we associate the n-tuple

$$\mathbf{m}'=(m_1,\dots,m_{n-1},m_n')\in\mathbb{Z}^n \tag{26}$$

with $m_n'=m_n$ if $m_n\ge 1$ and $m_n=0$ otherwise. Then, the weights of $\tau_{\mathbf{m}}$ with maximal norm are the orbit of $\mathbf{m}'$ under the signed permutation group in n variables and they all appear in $\tau_{\mathbf{m}}$ with multiplicity one. For $\mathbf{m}\in\hat{T}_{\mathrm{dom}}$ and $\mathbf{m}'\in\mathbb{Z}^n$ associated to it as in (26) we define the norm of $\tau_{\mathbf{m}}$ to be

$$\|\tau_{\mathbf{m}}\|=|\mathbf{m}'+2\rho_K|.$$

The irreducible representations $\varphi_{\mathbf{m}}$ of T_1 are parameterized by $(\mathbb{Z}_+\cup\{0^\pm\})^n$. For $\mathbf{m}=(m_1,\dots,m_n)\in(\mathbb{Z}_+\cup\{0^\pm\})^n$ the associated representation is

$$\varphi_{\mathbf{m}}=\psi_{m_1}\otimes\cdots\otimes\psi_{m_n}.$$

It will be important to us to interpret representations as induced from characters when possible. For $\mathbf{m}\in(\mathbb{Z}_+\cup\{0^\pm\})^n$ let

$$T_{\mathbf{m}}=\{\mathrm{diag}(k_1,\dots,k_n)|k_i\in SO(2)\text{ if }m_i\ge 1,\ k_i\in O(2)\text{ otherwise}\}$$

we then have

$$\varphi_{\mathbf{m}}=\mathrm{Ind}_{T_{\mathbf{m}}}^{T_1}(\chi_{\mathbf{m}}) \tag{27}$$

where $\chi_{\mathbf{m}}=x_1\otimes\cdots\otimes x_n$ and $x_i=\chi_{m_i}$ if $m_i\ge 1$ and $x_i=\psi_{m_i}$ otherwise. It is a representation of dimension 2^s where s is the number of m_i's which are positive integers. We describe these representations more explicitly for the sake of the lemma that follows. To simplify notation assume that $\mathbf{m}=(m_1,\dots,m_n)$ is such that $m_1,\dots,m_s$ are positive integers and $m_{s+1},\dots,m_n\in\{0^\pm\}$. As a representation of T_0, the space V of $\varphi_{\mathbf{m}}$ decomposes into weight spaces as follows.

$$V=\oplus\,\mathbb{C}v_S$$

where the sum is over all subsets $S \subset \{1, \dots, s\}$, v_S is a weight vector of weight $\chi_{\mathbf{m}(\mathbf{S})}$ and $\mathbf{m}(\mathbf{S}) = (m'_1, \dots, m'_s, 0, \dots, 0)$ with

$$m'_i = \begin{cases} -m_i & i \in S \\ m_i & i \notin S \end{cases}.$$

To characterize the representation it is now enough to say where an element of the form $\alpha = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ sends v_S, where $\alpha_i \in \{1_2, \mathrm{diag}(1, -1)\}$. Note that $\varphi_{\mathbf{m}} = \varphi_{(m_1,\dots,m_s)} \otimes \varphi_{(m_{s+1},\dots,m_n)}$ and that $\varphi_{(m_{s+1},\dots,m_n)}$ is a character. It is therefore enough to describe the action of elements α as above with $\alpha_{s+1} = \dots = \alpha_n = 1_2$. For $S \subset \{1, \dots, s\}$ let $\alpha_S = \mathrm{diag}(\alpha_1, \dots, \alpha_s, 1_2, \dots, 1_2)$ with

$$\alpha_i = \begin{cases} \mathrm{diag}(1, -1) & i \in S \\ 1_2 & i \notin S \end{cases}.$$

The action is given by

$$\varphi_{\mathbf{m}}(\alpha_{S'}) v_S = v_{S''}$$

where $S'' = (S \cup S') \backslash (S \cap S')$.

Lemma 6. *Let $\mathbf{m} = (m_1, \dots, m_n) \in \hat{T}_{dom}$, and let s be the number of i's such that $m_i \geq 1$. Let $\mathbf{m}''$ be the weight*

$$\mathbf{m}'' = (m_1, \dots, m_s, \overbrace{0, \dots, 0}^{(n-s)-times}),$$

let w be a signed permutation in n variables and set $w\mathbf{m}'' = (m''_1, \dots, m''_n)$. Denote further

$$w\mathbf{m}' = (m'_1, \dots, m'_n) \in (\mathbb{Z}_+ \cup \{0^\pm\})^n$$

where $m'_i = m''_i$ if $m''_i \geq 1$ and $m'_i = m_n$ if $m''_i = 0$. Let v'_0 be a weight vector of weight $w\mathbf{m}''$ for the representation $\tau_{\mathbf{m}}$ of K. Then the space $\tau_{\mathbf{m}}(T_1)v'_0$ realizes the representation $\varphi_{w\mathbf{m}'}$ in $\tau_{\mathbf{m}}$.

Proof. Let $v_0 = \tau_{\mathbf{m}}(w^{-1})v'_0$ and let

$$\mathbf{m}' = (m_1, \dots, m_s, \overbrace{m_n, \dots, m_n}^{(n-s)-times}) \in (\mathbb{Z}_+ \cup \{0^\pm\})^n.$$

It is not hard to see that if $\tau_{\mathbf{m}}(T_1)v_0$ realizes the representation $\varphi_{\mathbf{m}'}$ then $\tau_{\mathbf{m}}(w)\tau_{\mathbf{m}}(T_1)v_0 = \tau_{\mathbf{m}}(T_1)v'_0$ realizes the representation $\varphi_{w\mathbf{m}'}$. It is therefore enough to assume that $w = 1$. Let U be the space of $\tau_{\mathbf{m}}$ then, $U^{\mathfrak{n}} = \mathbb{C}\, v_0$ is the space of the representation $\psi_{\mathbf{m}}$ of T. Let $V = \tau_{\mathbf{m}}(T_1)U^{\mathfrak{n}} = \tau_{\mathbf{m}}(T_1)v_0$. We need to show that V is a realization of the representation $\varphi_{\mathbf{m}'}$. We will use the notation introduced in the discussion before the lemma. Set $v_\phi = v_0$ and let $v_S = \tau_{\mathbf{m}}(\alpha_S)v_\phi$, $S \subset \{1,\dots,s\}$. Note that for all $t \in T_0$ we have

$$\tau_{\mathbf{m}}(t)v_S = \tau_{\mathbf{m}}(\alpha_S)\tau_{\mathbf{m}}(\alpha_S^{-1}t\alpha_S)v_\phi =$$

$$\chi_{\mathbf{m}}(\alpha_S^{-1}t\alpha_S)\tau_{\mathbf{m}}(\alpha_S)v_\phi = \chi_{\mathbf{m(S)}}(t)v_S.$$

Thus v_S spans the one dimensional subspace of U of weight $\mathbf{m}(\mathbf{S})$. If

$$\alpha = \mathrm{diag}(1_2,\dots,1_2,\alpha_{s+1},\dots,\alpha_n)$$

with $\alpha_i \in \{1_2, \mathrm{diag}(1,-1)\}$, the same yoga shows that $\tau_{\mathbf{m}}(\alpha)v_S$ has weight $\mathbf{m}(\mathbf{S})$ and hence lies in $\mathbb{C}v_S$. It follows that

$$V = \oplus\, \mathbb{C}\, v_S$$

is the weight space decomposition of V. Since $\alpha_{S'}\alpha_S = \alpha_{S''}$ for any two subsets S' and S of $\{1,\dots,s\}$ (here as before $S'' = (S \cup S')\backslash(S \cap S')$), we get that $\tau_{\mathbf{m}}(\alpha_{S'})v_S = v_{S''}$. If $s = n$ the lemma is now complete. If $s < n$, it is left to show that elements of the form $\alpha = \mathrm{diag}(1_2,\dots,1_2,\alpha_{s+1},\dots,\alpha_n)$ as above act on V according to the character

$$\alpha \mapsto \prod_{i=s+1}^{n} \psi_{m_n}(\alpha_i).$$

Note that in this case $U^{\mathfrak{n}} = \mathbb{C}v_\phi$ and that T acts on v_ϕ by the character $\psi_{\mathbf{m}}$ and hence on v_S be the character $\psi_{\mathbf{m(S)}}$. Let i be such that $s+1 \le i \le n-1$, let $\alpha = \mathrm{diag}(1_2,\dots,1_2,\alpha_{s+1},\dots,\alpha_n)$ as above with $\alpha_j = 1_2$ for all $j \ne i$ and let $w \in K$ be the permutation that interchanges the i-th and n-th 2×2 diagonal blocks. Then, $\tau_{\mathbf{m}}(w)v_\phi = ev_\phi$ with $e \in \{\pm 1\}$, since $\tau_{\mathbf{m}}(w)v_\phi$ is again of weight $\mathbf{m}$. Since w commutes with all α_S it follows that $\tau_{\mathbf{m}}(w)v_S = ev_S$ for all $S \subset \{1,\dots,s\}$. Therefore,

$$\tau_{\mathbf{m}}(\alpha)v_S = \tau_{\mathbf{m}}(w)\tau_{\mathbf{m}}(w\alpha w)\tau_{\mathbf{m}}(w)v_S =$$

$$e\tau_{\mathbf{m}}(w)\tau_{\mathbf{m}}(w\alpha w)v_S = e\psi_{\mathbf{m}}(w\alpha w)\tau_{\mathbf{m}}(w)v_S =$$

$$\psi_{\mathbf{m}}(w\alpha w)v_S = \psi_{m_n}(\alpha_i)v_S.$$

The lemma follows. $\square$

4.2.2. *Standard modules and minimal K-types.* Let $\mathbf{r} = (r_1,\dots,r_s)$ be a partition of $2n$ with all $r_i \in \{1,2\}$, and let $P_{\mathbf{r}} = M_{\mathbf{r}}A_{\mathbf{r}}N_{\mathbf{r}}$ be the standard parabolic of G of type $\mathbf{r}$ with its Langlands decomposition. Thus $A_{\mathbf{r}}$ is a vector group isomorphic to $\mathbb{R}^s$ and $M_{\mathbf{r}}$ consists of elements of the form $\mathrm{diag}(g_1,\dots,g_s)$, where $g_i \in SL_2^{\pm}(\mathbb{R})$ if $r_i = 2$ and $g_i \in \{\pm 1\}$ if $r_i = 1$. Here $SL_2^{\pm}(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) | \det g = \pm 1\}$. Thus $M_{\mathbf{r}} \simeq (SL_2^{\pm}(\mathbb{R}))^{s_1} \times \{\pm 1\}^{s_2}$, where s_1 and s_2 are the number of 2's and of 1's in the partition $\mathbf{r}$, respectively. The group $N_{\mathbf{r}}$ is the unipotent radical of $P_{\mathbf{r}}$. Let δ be an irreducible discrete series representation of M and let $\nu = (\nu_1, \dots, \nu_s) \in \mathbb{R}^s$ with $\nu_1 \ge \cdots \ge \nu_s$. We consider ν as a character of $A_{\mathbf{r}}$. Let $I(M_{\mathbf{r}}A_{\mathbf{r}}, \delta, \nu)$ be the representation of G obtained from $\delta \otimes \nu \otimes 1$ by parabolic induction from $P_{\mathbf{r}}$ to G. We refer to such a representation as a standard module. The K-finite subspace of $I(M_{\mathbf{r}}A_{\mathbf{r}}, \delta, \nu)$ may be identified independently of ν with $\mathrm{Ind}_{M_{\mathbf{r}} \cap K}^{K}(\delta|_{M_{\mathbf{r}} \cap K})$ by restricting functions from G to K. The study of the K-finite subspace of a standard module will suffice for our needs. Our next goal is to construct, for those δ relevant to us, a specific element $f^\delta \in \mathrm{Ind}_{M_{\mathbf{r}} \cap K}^{K}(\delta|_{M_{\mathbf{r}} \cap K})$ so that the span $R(K)f^\delta$ of its K translates, is a minimal K-type of $\mathrm{Ind}_{M_{\mathbf{r}} \cap K}^{K}(\delta|_{M_{\mathbf{r}} \cap K})$. We will start with the irreducible representations of $M_{\mathbf{r}} \cap K$.

The group $M_{\mathbf{r}} \cap K$ is a maximal compact subgroup of $M_{\mathbf{r}}$. It consists of elements of the form $\mathrm{diag}(g_1, \dots, g_s)$ with $g_i \in O(2)$ if $r_i = 2$ and $g_i \in \{\pm 1\}$ if $r_i = 1$. Thus $M_{\mathbf{r}} \cap K \simeq (O(2))^{s_1} \times \{\pm 1\}^{s_2}$. The $(M_{\mathbf{r}} \cap K)$-types are parameterized by s-tuples $\mathbf{d} = (d_1, \dots, d_s)$ with $d_i \in \mathbb{Z}_+ \cup \{0^\pm\}$ if $r_i = 2$ and $d_i \in \{e, \mathrm{sgn}\}$ if $r_i = 1$. Here e and sgn are the trivial and sign characters of $\{\pm 1\}$, respectively. The representation $\phi_{\mathbf{d}}$ associated with $\mathbf{d}$ is given by

$$\phi_{\mathbf{d}} = \phi_{d_1} \otimes \cdots \otimes \phi_{d_s}$$

where $\phi_{d_i} = \psi_{d_i}$ if $r_i = 2$ and $\phi_{d_i} = d_i$ if $r_i = 1$. We may also express $\phi_{\mathbf{d}}$ as a representation induced from a character. Let $M_{\mathbf{r},\mathbf{d}}$ be the subgroup of $M_{\mathbf{r}}$ of all elements of the form $\mathrm{diag}(g_1, \dots, g_s)$ where

$$g_i \in \begin{cases} SL_2(\mathbb{R}) & r_i = 2 \text{ and } d_i \ge 1 \\ SL_2^{\pm}(\mathbb{R}) & r_i = 2 \text{ and } d_i \in \{0^\pm\} \\ \{\pm 1\} & r_i = 1 \end{cases}$$

and let $\chi_{\mathbf{d}} = x_1 \otimes \cdots \otimes x_s$ be the character of $M_{\mathbf{r},\mathbf{d}}$ with

$$x_i = \begin{cases} \chi_{d_i} & r_i = 2 \text{ and } d_i \ge 1 \\ \psi_{d_i} & r_i = 2 \text{ and } d_i \in \{0^\pm\} \\ d_i & r_i = 1 \end{cases} .$$

Then,

$$\phi_{\mathbf{d}} = \mathrm{Ind}_{M_{\mathbf{r},\mathbf{d}} \cap K}^{M_{\mathbf{r}} \cap K}(\chi_{\mathbf{d}}). \tag{28}$$

For each $\mathbf{d}$ as above define its norm to be

$$\|\phi_{\mathbf{d}}\| = \sum_{\{i | r_i = 2,\, d_i \ge 1\}} d_i^2.$$

Definition 3. For an admissible representation π of $M_{\mathbf{r}}$, we say that ϕ is a minimal $(M_{\mathbf{r}} \cap K)$-type of π if $\|\phi\|$ is minimal amongst the $(M_{\mathbf{r}} \cap K)$-types of π.

We will need to analyze some relation between representations of $M_{\mathbf{r}}\cap K$ and representations of T_1. The relation will rely on the minimal $O(2)$-types of the representations I_1, I_2 and J_{e_2} defined as follows. Let

$$I_1 = \mathrm{Ind}_{\{\mathrm{diag}(\pm 1,\pm 1)\}}^{O(2)}(\mathrm{sgn}\otimes e),$$

$$I_2 = \mathrm{Ind}_{\{\mathrm{diag}(\pm 1,\pm 1)\}}^{O(2)}(e\otimes \mathrm{sgn})$$

and

$$J_{e_2} = \mathrm{Ind}_{\{\mathrm{diag}(\pm 1,\pm 1)\}}^{O(2)}(e_2\otimes e_2)$$

where $e_2 \in \{e, \mathrm{sgn}\}$. Define the functions c_j and d_j on $O(2)$ by,

$$c_j(\mathrm{diag}(\alpha_1,\alpha_2)r(\theta)) = \alpha_j\cos\theta\,;\; d_j(\mathrm{diag}(\alpha_1,\alpha_2)r(\theta)) = \alpha_j\sin\theta$$

for $\alpha_i \in \{\pm 1\}$, $\theta\in\mathbb{R}$, $j = 1, 2$. It is easy to see that c_j, $d_j \in I_j$ and $\mathbb{C}\, c_j \oplus \mathbb{C}\, d_j$ spans a realization of ψ_1 in I_j. We denote this imbedding by

$$\iota_j : \psi_1 \hookrightarrow I_j. \tag{29}$$

The function

$$k \mapsto \psi_{0^{e_2}}(k),\ k\in O(2)$$

spans a realization of the character $\psi_{0^{e_2}}$ in J_{e_2}, where 0^{e_2} denotes 0^+ (resp. 0^-) if $e_2 = e$ (resp. $e_2 = \mathrm{sgn}$). We denote this imbedding

$$\jmath_{e_2} : \psi_{0^{e_2}} \hookrightarrow J_{e_2}. \tag{30}$$

Let $\delta = \delta_1\otimes\cdots\otimes\delta_s$ be a discrete series representation of $M_{\mathbf{r}}$, i.e. δ_i is a discrete series representation of $SL_2^{\pm}(\mathbb{R})$ if $r_i = 2$ and $\delta_i \in \{e,\ \mathrm{sgn}\}$ if $r_i = 1$. The class of $\mathrm{Ind}_{M_{\mathbf{r}}\cap K}^{K}(\delta|_{M_{\mathbf{r}}\cap K})$ depends on $(M_{\mathbf{r}},\delta)$, only up to a permutation. Let $w\in\mathfrak{S}_s$ be a permutation in s variables. We also view w as a permutation matrix in G that permutes the blocks of $M_{\mathbf{r}}$. In particular $w \in K$. Thus, $wM_{\mathbf{r}}w^{-1} = M_{w\mathbf{r}}$ is the standard Levi subgroup of G associated with the partition $w\mathbf{r} = (r_{w^{-1}(1)},\ldots,r_{w^{-1}(s)})$. Denote by $V = V_1\otimes\cdots\otimes V_s$ the space of δ, where V_i is the space of δ_i. We also denote by w the isomorphism $w(v_1\otimes\cdots\otimes v_s) = v_{w^{-1}(1)}\otimes\cdots\otimes v_{w^{-1}(s)}$ from V to $w(V) = V_{w^{-1}(1)}\otimes\cdots\otimes V_{w^{-1}(s)}$ and let $w(\delta)$ be the representation of $M_{w\mathbf{r}}$ on $w(V)$ given by

$$w(\delta)(wmw^{-1})(w(v)) = w(\delta(m)v),\ m\in M_{\mathbf{r}},\ v\in V.$$

For an element $f \in \mathrm{Ind}_{M_{\mathbf{r}}\cap K}^{K}(\delta|_{M_{\mathbf{r}}\cap K})$ we set

$$w(f)(k) = w(f(w^{-1}k)).$$

The map $f\mapsto w(f)$ is an intertwining operator and provides an isomorphism

$$\mathrm{Ind}_{M_{\mathbf{r}}\cap K}^{K}(\delta|_{M_{\mathbf{r}}\cap K}) \xrightarrow{w} \mathrm{Ind}_{M_{w\mathbf{r}}\cap K}^{K}(w(\delta)|_{M_{w\mathbf{r}}\cap K}).$$

The situation we need to study is the following. Let $\mathbf{r_0} = (r_1, \dots, r_s)$ be a partition of r with $r_i \in \{1,2\}$ and let $\delta_0 = \delta_1 \otimes \cdots \otimes \delta_s$ be an irreducible, discrete series representation of $M_{\mathbf{r_0}}$. For the longest element w_s of $\mathfrak{S}_s$ we have $w_s(\delta_0) = \delta_s \otimes \cdots \otimes \delta_1$. Let

$$\mathbf{r} = (\overbrace{\mathbf{r_0}, \dots, \mathbf{r_0}}^{m-times}, \overbrace{w_s\mathbf{r_0}, \dots, w_s\mathbf{r_0}}^{m-times})$$

and let $\delta = \delta_0^{\otimes m} \otimes (w_s(\delta_0))^{\otimes m}$ be a discrete series representation of $M_{\mathbf{r}}$. Our specific construction will be for the representation $\mathrm{Ind}_{M_{\mathbf{r}} \cap K}^{K}(\delta|_{M_{\mathbf{r}} \cap K})$. The decomposition of a discrete series representation of $SL_2^{\pm}(\mathbb{R})$ into its $O(2)$-types is well known. In particular the decomposition is multiplicity free and if ψ_d is a minimal $O(2)$-type then $d \geq 2$. Let $\phi_{\mathbf{d_0}}$ be the minimal $(M_{\mathbf{r_0}} \cap O(r))$-type of δ_0 with $\mathbf{d_0} = (d_1, \dots, d_s)$. Thus, $d_i \geq 2$ whenever $r_i = 2$. Let $e_1 \neq e_2$ be the two different characters $\{e, \mathrm{sgn}\}$ of $\{\pm 1\}$. Let t_1 be the number of d_i's that equal e_1 and let $t_1 + t_2$ be the number of d_i's that equal e_2. With this notation we need not specify which of e and sgn occur more then the other in $\mathbf{d_0}$ (if any). Set

$$\mathbf{d} = (\overbrace{\mathbf{d_0}, \dots, \mathbf{d_0}}^{m-times}, \overbrace{w_s\mathbf{d_0}, \dots, w_s\mathbf{d_0}}^{m-times}).$$

Then, $\phi_{\mathbf{d}}$ is the minimal $(M_{\mathbf{r}} \cap K)$-type of δ. We construct from $\mathbf{d}$ a parameter $\mathbf{m}(\mathbf{d}) \in \hat{T}_{\mathrm{dom}}$, a parameter $\mathbf{m}'(\mathbf{d})$ for a representation of T_1 and a weight $\mathbf{m}''(\mathbf{d})$ as follows. Let $s_1 = s - 2t_1 - t_2$ be the number of 2's in the partition $\mathbf{r_0}$ and let $d'_1 \geq \cdots \geq d'_{s_1}$ be a rearrangement of the d_i's with $r_i = 2$ in decreasing order. For a tuple with consecutive repetitions we will use the notation

$$(a_1^{(k_1)}, \dots, a_p^{(k_p)}) = (\overbrace{a_1, \dots, a_1}^{k_1-times}, \dots, \overbrace{a_p, \dots, a_p}^{k_p-times}).$$

If $t_2 = 0$ we set

$$\mathbf{m}(\mathbf{d}) = ({d'_1}^{(2m)}, \dots, {d'_{s_1}}^{(2m)}, 1^{(2mt_1)}) \tag{31}$$

and if $t_2 > 0$ we set

$$\mathbf{m}(\mathbf{d}) = ({d'_1}^{(2m)}, \dots, {d'_{s_1}}^{(2m)}, 1^{(2mt_1)}, 0^{(t_2m-1)}, 0^{e_2}). \tag{32}$$

We remark here that the definition of $\mathbf{m}(\mathbf{d})$ in (31) and (32) remains unchanged when replacing $(M_{\mathbf{r}}, \delta)$ with $(M_{w\mathbf{r}}, w(\delta))$ for a permutation $w \in \mathfrak{S}_{2ms}$. The n-tuple $\mathbf{m}'(\mathbf{d})$ is defined by

$$\mathbf{m}'(\mathbf{d}) = ({d'_1}^{(m)}, \dots, {d'_{s_1}}^{(m)}, 1^{(mt_1)}, (0^{e_2})^{(t_2m)}, 1^{(mt_1)}, {d'_{s_1}}^{(m)}, \dots, {d'_1}^{(m)}).$$

The weight $\mathbf{m}''(\mathbf{d})$ is given by

$$\mathbf{m}''(\mathbf{d}) = ({d'_1}^{(m)}, \dots, {d'_{s_1}}^{(m)}, 1^{(mt_1)}, 0^{(t_2m)}, 1^{(mt_1)}, {d'_{s_1}}^{(m)}, \dots, {d'_1}^{(m)}).$$

Let $w_0 \in \mathfrak{S}_{ms}$ be a permutation so that

$$w_0(\overbrace{\mathbf{d_0}, \dots, \mathbf{d_0}}^{m-times}) = ({d'_1}^{(m)}, \dots, {d'_{s_1}}^{(m)}, (e, \mathrm{sgn})^{mt_1}, e_2^{(mt_2)}).$$

Let $w_1 = \mathrm{diag}(w_0, \tilde{w}_0) \in \mathfrak{S}_{2ms}$. Note that as an element of K we have $w_1 \in K_H$. We denote

$$\mathbf{r}' = w_1 \mathbf{r} = (2^{(ms_1)}, 1^{(2ms_2)}, 2^{(ms_1)})$$

where $s_2 = 2t_1 + t_2$ and

$$\mathbf{d}' = w_1 \mathbf{d} = ({d'_1}^{(m)}, \dots, {d'_{s_1}}^{(m)}, (e, \mathrm{sgn})^{mt_1}, {e_2}^{(2t_2m)}, (\mathrm{sgn}, e)^{mt_1}, {d'_{s_1}}^{(m)}, \dots, {d'_1}^{(m)}).$$

The representation $\phi_{\mathbf{d}'}$ is the minimal $(M_{\mathbf{r}'} \cap K)$-type of $\delta' = w_1(\delta)$. We have a commutative diagram of K-maps.

$$\begin{array}{ccc} \mathrm{Ind}_{M_{\mathbf{r}'} \cap K}^K(\phi_{\mathbf{d}'}) & \underset{\simeq}{\xrightarrow{w_1^{-1}}} & \mathrm{Ind}_{M \cap K}^K(\phi_{\mathbf{d}}) \\ \cap & & \cap \\ \mathrm{Ind}_{M_{\mathbf{r}'} \cap K}^K(\delta'|_{M_{\mathbf{r}'} \cap K}) & \underset{\simeq}{\xrightarrow{w_1^{-1}}} & \mathrm{Ind}_{M_{\mathbf{r}} \cap K}^K(\delta|_{M_{\mathbf{r}} \cap K}) \end{array}.$$

The special element $f^\delta \in \mathrm{Ind}_{M_{\mathbf{r}} \cap K}^K(\delta|_{M_{\mathbf{r}} \cap K})$ that we construct will lie in $\mathrm{Ind}_{M_{\mathbf{r}} \cap K}^K(\phi_{\mathbf{d}})$. Note that

$$\begin{aligned} \mathrm{Ind}_{M_{\mathbf{r}'} \cap K}^{T_1}(\phi_{\mathbf{d}'}) = (\psi_{d'_1})^{\otimes m} \otimes \cdots \otimes (\psi_{d'_{s_1}})^{\otimes m} \otimes (I_1)^{\otimes t_1 m} \\ \otimes (J_{e_2})^{\otimes t_2 m} \otimes (I_2)^{\otimes t_1 m} \otimes (\psi_{d'_{s_1}})^{\otimes m} \otimes \cdots \otimes (\psi_{d'_1})^{\otimes m}. \end{aligned}$$

Using the imbedding (29) and the imbedding (30) we define the imbedding

$$\iota : \varphi_{\mathbf{m}'(\mathbf{d})} \hookrightarrow \mathrm{Ind}_{M_{\mathbf{r}'} \cap K}^{T_1}(\phi_{\mathbf{d}'}) \tag{33}$$

of T_1 representations, to be

$$\begin{aligned} (\mathrm{Id}_{\psi_{d'_1}})^{\otimes m} \otimes \cdots \otimes (\mathrm{Id}_{\psi_{d'_{s_1}}})^{\otimes m} \otimes (\iota_1)^{\otimes t_1 m} \otimes (\jmath_{e_2})^{\otimes t_2 m} \\ \otimes (\iota_2)^{\otimes t_1 m} \otimes (\mathrm{Id}_{\psi_{d'_{s_1}}})^{\otimes m} \otimes \cdots \otimes (\mathrm{Id}_{\psi_{d'_1}})^{\otimes m}. \end{aligned}$$

As in (28) we have

$$\mathrm{Ind}_{M_{\mathbf{r}'} \cap K}^{T_1}(\phi_{\mathbf{d}'}) = \mathrm{Ind}_{M_{\mathbf{r}',\mathbf{d}'} \cap K}^{T_1}(\chi_{\mathbf{d}'}).$$

Let $u_0 \in \mathrm{Ind}_{M_{\mathbf{r}',\mathbf{d}'} \cap K}^{T_1}(\chi_{\mathbf{d}'})$ be the scalar valued function on T_1 given by

$$\begin{aligned} u_0 = (u_1)^{\otimes m} \otimes \cdots \otimes (u_{s_1})^{\otimes m} \otimes (c_1 + id_1)^{\otimes mt_1} \otimes \\ (\psi_{0^{e_2}})^{\otimes mt_2} \otimes (c_2 + id_2)^{\otimes mt_1} \otimes (u_{s_1})^{\otimes m} \otimes \cdots \otimes (u_1)^{\otimes m} \end{aligned} \tag{34}$$

where

$$u_i(k) = \chi_{d'_i}(k \ \mathrm{diag}(1, \det k)),\ k \in O(2).$$

It is easy to check that u_0 is a weight vector for the weight $\mathbf{m}''(\mathbf{d})$ and that it lies in the image of ι. Thus, there exist a unique v_0 in the space of $\varphi_{\mathbf{m}'(\mathbf{d})}$ which is a weight vector of weight $\mathbf{m}''(\mathbf{d})$ such that $\iota(v_0) = u_0$. It will be important to us that

$$u_0(1) = 1 \neq 0. \tag{35}$$

The map ι provides an imbedding $\tilde{\iota}$

$$\mathrm{Ind}_{T_1}^K(\varphi_{\mathbf{m}'(\mathbf{d})}) \overset{\tilde{\iota}}{\hookrightarrow} \mathrm{Ind}_{T_1}^K(\mathrm{Ind}_{M_{\mathbf{r}'}\cap K}^{T_1}(\phi_{\mathbf{d}'})) = \mathrm{Ind}_{M_{\mathbf{r}'}\cap K}^K(\phi_{\mathbf{d}'}).$$

We may view both ι and $\tilde{\iota}$ as intertwining operators between spaces of scalar valued functions (see (27) and (28)), namely

$$\iota : \mathrm{Ind}_{T_{\mathbf{m}'(\mathbf{d})}}^{T_1}(\chi_{\mathbf{m}'(\mathbf{d})}) \to \mathrm{Ind}_{M_{\mathbf{r}',\mathbf{d}'}\cap K}^{T_1}(\chi_{\mathbf{d}'}) \tag{36}$$

and

$$\tilde{\iota} : \mathrm{Ind}_{T_{\mathbf{m}'(\mathbf{d})}}^{K}(\chi_{\mathbf{m}'(\mathbf{d})}) \to \mathrm{Ind}_{M_{\mathbf{r}',\mathbf{d}'}\cap K}^{K}(\chi_{\mathbf{d}'}).$$

Note that for $f \in \mathrm{Ind}_{T_{\mathbf{m}'(\mathbf{d})}}^{K}(\chi_{\mathbf{m}'(\mathbf{d})})$ and $k \in K$ the map $t \mapsto f(tk)$, $t \in T_1$ lies in $\mathrm{Ind}_{T_{\mathbf{m}'(\mathbf{d})}}^{T_1}(\chi_{\mathbf{m}'(\mathbf{d})})$. The map $\tilde{\iota}$ is then given by

$$(\tilde{\iota}(f))(k) = \iota(t \mapsto f(tk))(1). \tag{37}$$

Let $V_{\mathbf{m}(\mathbf{d})}$ be the space of the representation $\tau_{\mathbf{m}(\mathbf{d})}$, equipped with a K-invariant hermitian inner product $\langle\, ,\rangle$. By Lemma 6 there is an imbedding of $\varphi_{\mathbf{m}'(\mathbf{d})}$ in $\tau_{\mathbf{m}(\mathbf{d})}|_{T_1}$. Let $U_{\mathbf{m}'(\mathbf{d})} \subset V_{\mathbf{m}(\mathbf{d})}$ be the subspace in which $\varphi_{\mathbf{m}'(\mathbf{d})}$ is realized. Using (33) there is an imbedding which we also denote by $\iota : U_{\mathbf{m}'(\mathbf{d})} \to \mathrm{Ind}_{M_{\mathbf{r}'}\cap K}^{T_1}(\phi_{\mathbf{d}'})$. Let $v_0 \in U_{\mathbf{m}'(\mathbf{d})}$ be the (unique) weight vector of $\tau_{\mathbf{m}(\mathbf{d})}$ of weight $\mathbf{m}''(\mathbf{d})$ so that $\langle v_0, v_0\rangle = 1$. Thus $\iota(v_0)$ is a non-zero multiple of u_0. Define

$$f_0(k) = \langle \tau_{\mathbf{m}(\mathbf{d})}(k)v_0, v_0\rangle.$$

It follows (see (27)) that

$$f_0 \in \mathrm{Ind}_{T_{\mathbf{m}'(\mathbf{d})}}^{K}(\chi_{\mathbf{m}'(\mathbf{d})}) = \mathrm{Ind}_{T_1}^K(\varphi_{\mathbf{m}'(\mathbf{d})})$$

and therefore $\tilde{\iota}(f_0) \in \mathrm{Ind}_{M_{\mathbf{r}'}\cap K}^K(\phi_{\mathbf{d}'})$. Finally, we define

$$f^\delta = w_1^{-1} \circ \tilde{\iota}(f_0) \in \mathrm{Ind}_{M_{\mathbf{r}}\cap K}^K(\phi_{\mathbf{d}}).$$

It is clear that $R(K)f_0$ realizes $\tau_{\mathbf{m}(\mathbf{d})}$ in $\mathrm{Ind}_{T_{\mathbf{m}'(\mathbf{d})}}^{K}(\chi_{\mathbf{m}'(\mathbf{d})})$. Since $w_1^{-1} \circ \tilde{\iota}$ is an intertwining operator it follows that the space $R(K)f^\delta$ realizes $\tau_{\mathbf{m}(\mathbf{d})}$ in $\mathrm{Ind}_{M_{\mathbf{r}}\cap K}^K(\phi_{\mathbf{d}})$.

Definition 4. Let π be an admissible representation of G. We say that $\tau \in \hat{K}$ is a K-type of π if τ occurs in $\pi|_K$. A minimal K-type of π is one for which $\|\tau\|$ is minimal amongst all K-types of π.

Lemma 7. *Let δ be a discrete series representation of M and let $\phi_{\mathbf{d}}$ be its minimal $(M \cap K)$-type. Let $\mathbf{m}(\mathbf{d})$ be defined as in (31) and (32). Then, $\tau_{m(\mathbf{d})}$ is a minimal K-type of $\mathrm{Ind}_{M\cap K}^{K}(\delta|_{M\cap K})$.*

Proof. We have constructed above an element f^{δ} that shows that $\tau_{\mathbf{m}(\mathbf{d})}$ indeed occurs in $\mathrm{Ind}_{M\cap K}^{K}(\delta|_{M\cap K})$. It is not difficult to show directly that it is a minimal K-type. Alternatively, it follows from the analogue statement for K_0, which is stated in [Vog85]. □

4.2.3. *The non-vanishing.* We are now back to our setting, where $P = MU$ is the parabolic of G of type $(r, \dots, r)$ and $2n = 2mr$. We consider the induced representation $I(\pi, \Lambda) = I_P^G(\pi[\Lambda])$ where $\pi = \sigma^{\otimes 2m}$ and σ is an irreducible, unitary, generic representation of $GL_r(\mathbb{R})$. Such a representation σ can be expressed as a standard module of the following form. There is a partition $\mathbf{r_0} = (r_1, \dots, r_s)$ of r with $r_i \in \{1, 2\}$, an irreducible, discrete series representation $\delta_0 = \delta_1 \otimes \cdots \otimes \delta_s$ of $M_{\mathbf{r_0}}$ and exponents $\beta_0 = (\beta_1, \dots, \beta_s) \in \mathbb{R}^s$ so that $\beta_1 \geq \cdots \geq \beta_s$ and $|\beta_i| < 1/2$ for all i such that

$$\sigma = I(M_{\mathbf{r_0}} A_{\mathbf{r_0}}, \delta_0, \beta_0).$$

Let

$$\mathbf{r}' = (\overbrace{\mathbf{r_0}, \dots, \mathbf{r_0}}^{2m-times})$$

and let $\beta = (\beta_0, \dots, \beta_0) \in \mathbb{R}^{2ms}$. The exponent

$$\nu = \beta + \Lambda \in \mathbb{R}^{2ms}$$

lies in the positive Weyl chamber with respect to $P_{\mathbf{r}'}$. Thus, $\delta' = \delta_0^{\otimes 2m}$ is a discrete series representation of $M_{\mathbf{r}'}$ and

$$I(\pi, \Lambda) = I(M_{\mathbf{r}'} A_{\mathbf{r}'}, \delta', \nu)$$

is a standard module. Our goal is to show that the linear form l_H does not vanish on this standard module. The theory of minimal K-types will serve us for this purpose. But before we explain how, let us write j more explicitly. For $f \in I(\pi, w'\Lambda)$ we have

$$jf = \int_{K_H} l_{M_H}(f(k))dk$$

where l_{M_H} is the (unique up to a scalar) M_H-invariant form on π. Here we view f as a function on G taking values in the space of π. With our new description of σ we have

$$I(\pi, w'\Lambda) = I(M_{\mathbf{r}'} A_{\mathbf{r}'}, \delta', w'\nu)$$

and we may view f as a function on G with values in the space of δ'. Doing so, we may write the pairing l_{M_H} and hence j more explicitly. Let

$$w = \mathrm{diag}(\overbrace{1_r, \dots, 1_r}^{m-times}, \overbrace{w_s, \dots, w_s}^{m-times}),$$

denote $\mathbf{r} = w\mathbf{r}'$ and $\delta = w(\delta')$ then,

$$jf = \int_{K_H} l_{M_{\mathbf{r}} \cap H}((M(w)f)(k))dk$$

where $l_{M_{\mathbf{r}} \cap H}$ is the (unique up to a scalar) $(M_{\mathbf{r}} \cap H)$-invariant form on $\delta = \delta_0^{\otimes m} \otimes (w_s(\delta_0))^{\otimes m}$. We may therefore describe l_H as follows

$$I(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', \nu) \overset{M(w')}{\rightarrow} I(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', w'\nu) \overset{M(w)}{\rightarrow} I(M_{\mathbf{r}}A_{\mathbf{r}}, \delta, ww'\nu) \overset{j_0}{\rightarrow} \mathbb{C}$$

where

$$j_0 f = \int_{K_H} l_{M_{\mathbf{r}} \cap H}(f(k))dk. \tag{38}$$

We have constructed in the previous section an element f^δ in $\mathrm{Ind}_{M_{\mathbf{r}} \cap K}^K(\delta|_{M_{\mathbf{r}} \cap K})$. We will show that this element lies in the image of the intertwining operator $M(w) \circ M(w')$ and that $j_0 f^\delta \neq 0$. This will prove the non-vanishing of l_H. We recall that $f^\delta \in \mathrm{Ind}_{M_{\mathbf{r}} \cap K}^K(\phi_{\mathbf{d}})$ where $\phi_{\mathbf{d}}$ is the minimal $(M \cap K)$-type of δ. By Lemma 7, the space $R(K)f^\delta$ realizes the minimal K-type $\tau_{\mathbf{m}(\mathbf{d})}$ of $\mathrm{Ind}_{M_{\mathbf{r}} \cap K}^K(\delta|_{M_{\mathbf{r}} \cap K})$. The standard module $I(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', \nu)$ has a unique irreducible quotient, the Langlands quotient $J(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', \nu)$. Theorem 11.253 of [KV95] implies that all minimal K-types of $I(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', \nu)$ survive the projection to $J(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', \nu)$. In fact, the theorem states that for connected groups all minimal K-types of a standard module survive the Langlands quotient. We will briefly explain how it can be deduced for G from its validity to $SL_{2n}(\mathbb{R})$. The discrete series representation $\delta'|_{M_{\mathbf{r}'} \cap SL_{2n}(\mathbb{R})}$ is either irreducible or decomposes as a sum of two irreducible discrete series representations. It follows that the standard module $I(M_{\mathbf{r}'}A_{\mathbf{r}'}, \delta', \nu)|_{SL_{2n}(\mathbb{R})}$ is either a standard module or a sum of two standard modules. Intertwining operators for G and for $SL_{2n}(\mathbb{R})$ are defined by the same formula and therefore the fact that all minimal K_0-types of a standard module for $SL_{2n}(\mathbb{R})$ survive the Langlands quotient imply that so do all minimal K-types of a standard module for G. It now follows that f^δ lies in the image of the intertwining operator $M(w) \circ M(w')$, since the projection to the Langlands quotient factors through $M(w) \circ M(w')$. It is left to show the non-vanishing of $j_0 f^\delta$. Since $\phi_{\mathbf{d}}$ occurs in δ with multiplicity one, the restriction of $l_{M_{\mathbf{r}} \cap H}$ to $\phi_{\mathbf{d}}$ is the (unique up to a scalar) non-zero $(M_{\mathbf{r}} \cap H)$-invariant form on $\phi_{\mathbf{d}}$. Realizing $\phi_{\mathbf{d}}$ as in (28) as a representation induced from a character, we have that

$$l_{M_{\mathbf{r}}\cap H}(v) = \int_{M_{\mathbf{r}}\cap H\cap K} v(k)dk,\ v \in \operatorname{Ind}_{M_{\mathbf{r},\mathbf{d}}\cap K}^{M_{\mathbf{r}}\cap K}(\chi_{\mathbf{d}}) = \phi_{\mathbf{d}}.$$

It follows that for $f \in \operatorname{Ind}_{M_{\mathbf{r},\mathbf{d}}\cap K}^{K}(\chi_{\mathbf{d}}) = \operatorname{Ind}_{M_{\mathbf{r}}\cap K}^{K}(\phi_{\mathbf{d}})$ we have

$$j_0 f = \int_{K_H} f(k)dk.$$

Thus $j_0 = E \circ P^{K_H}$ where P^{K_H} is the projection to the K_H-invariant space, and E is the evaluation at the identity linear form. We now recall that $f^\delta = w_1^{-1} \circ \tilde{\iota}(f_0)$. Realizing the representations involved as spaces induced from characters, ι is the imbedding (36) and $\tilde{\iota}$ is as in (37). Since $w_1^{-1} \circ \tilde{\iota}$ is an intertwining operator we get that

$$j_0 f^\delta = (w_1^{-1} \circ \tilde{\iota}(f_{K_H}))(1) = \tilde{\iota}(f_{K_H}))(w_1) \tag{39}$$

where

$$f_{K_H}(k_1) = \int_{K_H} f_0(k_1 k)dk = \langle \tau_{\mathbf{m}(\mathbf{d})}(k_1)v_{K_H}, v_0\rangle$$

and

$$v_{K_H} = \int_{K_H} \tau_{\mathbf{m}(\mathbf{d})}(k)v_0 dk.$$

From the definition of $\tilde{\iota}$ in (37) we see that the right hand side of (39) equals $\iota(a_{K_H})(1)$ where $a_{K_H}(t) = f_{K_H}(tw_1)$, $t \in T_1$. Since v_0 is an eigenfunction of weight $\mathbf{m}''(\mathbf{d})$ for $\tau_{\mathbf{m}(\mathbf{d})}$ we see that

$$a_{K_H}(t) = \langle \tau_{\mathbf{m}(\mathbf{d})}(tw_1)v_{K_H}, v_0\rangle = \langle \tau_{\mathbf{m}(\mathbf{d})}(t)v_{K_H}, v_0\rangle$$

also transforms as the character $\chi_{\mathbf{m}''(\mathbf{d})}$ under T_0. Since the weight $\mathbf{m}''(\mathbf{d})$ appears with multiplicity one in $\varphi_{\mathbf{m}'(\mathbf{d})}$ and since the function a_0 on T_1 defined by $a_0(t) = f_0(t)$ is a non-zero weight function of weight $\mathbf{m}''(\mathbf{d})$ in $\varphi_{\mathbf{m}'(\mathbf{d})}$ we have $a_{K_H} = \langle v_{K_H}, v_0\rangle a_0$. From the definition of ι it follows that $\iota(a_0)$ is a non-zero multiple of u_0. Plugging all this into (39) we see that $j_0 f^\delta$ is a non-zero multiple of

$$\langle v_{K_H}, v_0\rangle u_0(1).$$

Since $K_H \subset K_0$ there is an irreducible representation of K_0 contained in $\tau_{\mathbf{m}(\mathbf{d})}$ that contains both v_0 and v_{K_H}. The non-vanishing now follows from (35) and the Cartan-Helgason Theorem (Lemma 5).

4.3. **The local problem (complex).** In this section $F = \mathbb{C}$. The group G is the complex Lie group $GL_{2n}(\mathbb{C})$ and its maximal compact K is the unitary group

$$U(2n) = \{g \in G | {}^t\bar{g}g = 1\}.$$

The symplectic group H is the complex Lie group

$$Sp(2n, \mathbb{C}) = \{g \in G | {}^t g\epsilon g = 1\}.$$

The non-vanishing of l_H is proved in the complex case, in a similar way to the real case, but things are much simpler here. Denote by G_r the group $GL_r(\mathbb{C})$ and by B_r its subgroup of upper triangular matrices. We will further put $B = B_{2n}$ and let T be the subgroup of diagonal matrices and U_0 be the subgroup of unipotent matrices in B. We now set up the background we need concerning K-types. Let $T_K = T \cap K$ be the maximal torus of K. Note that $T_K = \{\mathrm{diag}(z_1, \ldots, z_{2n}) || z_i| = 1\}$. For $\mathbf{m} \in \mathbb{Z}^{2n}$ let $\chi_{\mathbf{m}}$ be the character of T_K defined by

$$\chi_{\mathbf{m}}(\mathrm{diag}(z_1, \ldots, z_{2n})) = \prod_{i=1}^{2n} z_i^{m_i}.$$

The T_K-types are precisely all the characters $\chi_{\mathbf{m}}$ parameterized by $\mathbf{m} \in \mathbb{Z}^{2n}$. We call them weights. We call a weight $\chi_{\mathbf{m}}$ or $\mathbf{m} = (m_1, \ldots, m_{2n})$ dominant if $m_1 \geq \cdots \geq m_{2n}$. The K-types are parameterized by dominant weights. For a dominant weight $\mathbf{m}$ we denote by $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ the irreducible representation of K of highest weight $\mathbf{m}$. Let ρ_K denote half the sum of positive roots, thus

$$2\rho_K = (2n-1, 2n-3, \ldots, 1-2n).$$

We define the norm of $\tau_{\mathbf{m}}$ to be

$$\|\tau_{\mathbf{m}}\| = |\mathbf{m} + 2\rho_K|$$

where $|\cdot|$ stands for the standard Euclidean norm on $\mathbb{R}^{2n}$. It is easy to see that

$$\|\tau_{\mathbf{m}}\| = \max_{\mathbf{m}'} |\mathbf{m}' + 2\rho_K| \tag{40}$$

where the maximum is taken over all weights $\mathbf{m}'$ of $\tau_{\mathbf{m}}$.

Definition 5. For an admissible representation π of G we say that $\tau_{\mathbf{m}}$ is a minimal K-type of π, if $\|\tau_{\mathbf{m}}\|$ is minimal amongst all K-types of π.

Let $\mathbf{m} \in \mathbb{Z}^{2n}$ and denote $I(\chi_{\mathbf{m}}) = \mathrm{Ind}_{T_K}^K(\chi_{\mathbf{m}})$. We define an element $f^{\mathbf{m}} \in I(\chi_{\mathbf{m}})$ as follows. Let w_1 be the permutation so that $w_1\mathbf{m}$ is a dominant weight, and let v_0 be the weight vector of weight $\mathbf{m}$ in $\tau_{w_1\mathbf{m}}$. We define

$$f^{\mathbf{m}}(k) = \langle \tau_{w_1\mathbf{m}}(k)v_0, v_0\rangle.$$

The space $R(K)f^{\mathbf{m}}$ is a realization of $\tau_{w_1(\mathbf{m})}$ in $I(\chi_{\mathbf{m}})$.

Lemma 8. *The representation $\tau_{w_1\mathbf{m}}$ is a minimal K-type of $I(\chi_{\mathbf{m}})$.*

Proof. The above construction of $f^{\mathbf{m}}$ shows that $\tau_{w_1\mathbf{m}}$ indeed occurs in $I(\chi_{\mathbf{m}})$. If $\mathbf{m}'$ is a dominant weight so that $\tau_{\mathbf{m}'}$ occurs in $I(\chi_{\mathbf{m}})$ then Frobenius reciprocity shows that

$$\mathrm{Hom}_{T_K}(\tau_{\mathbf{m}'}|_{T_K}, \chi_{\mathbf{m}}) \neq 0,$$

i.e. that $\mathbf{m}$ occurs in $\tau_{\mathbf{m}'}$. It follows that the Weyl orbit of $\mathbf{m}$ all occur and therefore the dominant weight $w_1\mathbf{m}$ is a weight of $\tau_{\mathbf{m}'}$. It then follows from (40) that $\|\tau_{\mathbf{m}'}\| \geq \|\tau_{w_1\mathbf{m}}\|$. □

For a character χ of B_r we denote $I(\chi) = \mathrm{Ind}_{B_r}^{G_r}(\chi)$. An irreducible, unitary generic representation σ of G_r is always of the form $\sigma = I(\chi_0)$ for some character $\chi_0 = (\chi_1, \dots, \chi_r)$ of B_r, such that $\chi_i = \nu^{\beta_i}\delta_i$, β_i are real numbers of absolute value less then $1/2$ and δ_i are unitary characters. We may and do assume further that the β_i's are in decreasing order. For $\pi = \sigma^{\otimes 2m}$ we therefore have $I(\pi, \Lambda) = I(\chi)$ where χ is the character of B given by

$$\chi = \bigotimes_{i=1}^{2m} (\nu^{m-i+1/2}\chi_0).$$

Note that the character χ of T, lies in the positive Weyl chamber with respect to B. It follows that $I(\chi)$ has a unique irreducible quotient which we denote by $J(\chi)$. We observe that after realizing $I(\pi, \Lambda)$ as the space $I(\chi)$ of scalar valued functions

$$l_H f = \int_{K_H} \left(M(w)M(w')f\right)(k)dk,\ f \in I(\chi)$$

where

$$w = \mathrm{diag}(\overbrace{1_r, \dots, 1_r}^{m\text{-times}}, \overbrace{w_r, \dots, w_r}^{m\text{-times}}).$$

Thus l_H is the decomposition

$$I(\chi) \overset{M(w)\circ M(w')}{\rightarrow} I(ww'(\chi)) \overset{j_0}{\rightarrow} \mathbb{C}$$

where

$$j_0 f = \int_{K_H} f(k)dk,\ f \in I(ww'(\chi)).$$

Let $\mathbf{m} \in \mathbb{Z}^{2n}$ be the weight so that $ww'(\chi)|_{T_K} = \chi_{\mathbf{m}}$. The K-finite space of $I(w(\chi))$ is identified with $I(\chi_{\mathbf{m}})$. It follows from Theorem 11.253 of

[KV95] that $f^{\mathbf{m}}$ lies in the image of $M(w) \circ M(w')$ and therefore to prove the non-vanishing of l_H it is enough to show that $j_0 f^{\mathbf{m}} \neq 0$. But $j_0 f^{\mathbf{m}} = \langle v_{K_H}, v_0 \rangle$ where

$$v_{K_H} = \int_{K_H} \tau_{w_1 \mathbf{m}}(k) v_0 dk.$$

The non-vanishing is now a consequence of the Cartan-Helgason Theorem (Lemma 5).

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