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Unitary periods, Hermitian forms and points on flag varieties

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Abstract Let E be an imaginary quadratic extension of $\mathbb{Q}$ of class number one. We examine certain representation numbers associated to Hermitian forms over E, which involve counting integral points on flag varieties.

1 Introduction

The study of representation numbers of integral quadratic and Hermitian forms is a topic of classical interest. For example, an identity of Jacobi says that the number of ways to write a positive integer n as a sum of four integer squares is equal to $8\sum d$ where the sum is over all divisors of n which are not divisible by 4. This result and many related results on representation numbers of quaternary quadratic forms were reinterpreted by Elstrodt et al. [4] as results about representation numbers of binary Hermitian forms over an imaginary quadratic number field E with ring of integers $\mathcal{O}$. They then related these to weighted sums of point evaluations of Eisenstein

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series for the group $PSL_2(\mathcal{O})$ acting on hyperbolic 3-space. These weighted sums can be interpreted adèlically as period integrals of the Eisenstein series over unitary groups.

More recently, formulas for unitary periods of Eisenstein series for the group $GL_n(E)$ have been obtained by Lapid and Rogawski [15] (for $n = 3$) and [16] (for general n). As in the work of [4], these formulas equate the period integral with a finite sum of Euler products. We remark however that the local terms in [4] are local densities that they compute explicitly at all places. Formulas for the same local densities were obtained by Hironaka [8]. Hironaka generalized the computation of local densities in a series of papers [7,9,10] and finally obtained a general formula for local densities of Hermitian forms in [11]. Though explicit, the formula is rather complicated. In [6], Hironaka introduced spherical functions on the space of Hermitian matrices associated to a quadratic extension of p-adic fields. She obtained a formula relating the spherical functions to local densities [6, Sect.2, Theorem]. Although the formula indicates a strong relation between spherical functions and local densities, it is not yet clear in general how explicit formulas for the latter can provide explicit formulas for the former. The local data that appears in the formula of [15,16] for the unitary period of an Eisenstein series is in terms of Hironaka's spherical functions, explicit formulas for which are available in [10, Theorem 1] for the case of an unramifed quadratic extension. For the case of a ramified extension, explicit formulas are only available if $n = 2$. Thus, in contrast to [4], the local terms in the results of [15,16] for $n > 2$ are explicit only outside a finite set of primes.

The purpose of the current work is to give an arithmetic application of the formula for the unitary period. For simplicity, we restrict our attention to an imaginary quadratic field E of class number one (see Remark 5). We express the unitary period of an Eisenstein series induced from a standard parabolic subgroup P of $G = GL_n$ as a Dirichlet series whose coefficients are certain representation numbers related to counting points on the (partial) flag variety $P\backslash G$. We then apply the formula obtained in [16] for the unitary period, to express the Dirichlet series as a finite sum of certain Euler products. Special cases reduce to more familiar representation numbers. For example, generalizing the setting of [4], consider the Eisenstein series E_P associated to the parabolic P of type $(n-1, 1)$ of G. Let $\mathcal{O}^n_{prim}$ be the set of column vectors $v = {}^t(v_1, \ldots, v_n) \in \mathcal{O}^n$ such that the ideal generated by the v_i's is $\mathcal{O}$. Let $g \in GL_n(\mathbb{C})$ be such that the associated positive definite Hermitian form

$$Q : v \mapsto {}^t\bar{v}g^t\bar{g}v$$

is integral. The Eisenstein series $E_P(g; \cdot)$ can be expressed as a Dirichlet series whose m-th coefficient is

$$\#\{v \in \mathcal{O}^n_{prim} : Q(v) = m\},$$

the number of ways to represent m by the Hermitian form Q with primitive integral vectors.

For a second example, let P be the parabolic of type $(1, n-2, 1)$. Then $E_P(g; \cdot)$ is a Dirichlet series in two complex variables whose (m_1, m_2) coefficient is

$$\begin{aligned}\#\{v \in \mathcal{O}^n_{prim},\ w \in |\det g|^2\, g^{-1}({}^t\bar{g})^{-1}\mathcal{O}^n_{prim} :\\ Q(v) = m_1,\ Q(w) = |\det g|^2\, m_2,\ Q(v, w) = 0\}\end{aligned} \tag{1.1}$$

where $Q(v, w) = {}^t\bar{v}g^t\bar{g}w$. In particular, if $g = e$ is the identity matrix, then this is the number of ways to represent the diagonal matrix $\mathrm{diag}(m_1, m_2)$ by Q with a $2 \times n$ integral matrix with primitive rows.

There exists a very general theory of representation numbers of one form by another, developed by Siegel for quadratic forms [19–21] and extended to Hermitian forms by Braun [3]. For more information, see the recent survey of Schulze-Pillot [22]. The representation numbers that arise from our formulas for parabolics other than those described in the above two examples, however, are not of the form considered by Siegel and Braun. For an example, take $n \geq 4$ with $P = B$ the standard Borel subgroup and U its unipotent radical. We have the "Plücker embedding"

$$U(\mathcal{O})\backslash SL_n(\mathcal{O}) \hookrightarrow \prod_{i=1}^{n-1} \mathcal{O}^{\binom{n}{i}} \tag{1.2}$$

$$h \mapsto (v_1(h), \ldots, v_{n-1}(h)) \tag{1.3}$$

where $v_i(h) \in \mathcal{O}^{\binom{n}{i}}$ is the vector of $i \times i$ minors in the bottom i rows of h. Let $\mathcal{I} \subset \prod_{i=1}^{n-1} \mathcal{O}^{\binom{n}{i}}$ be the image of this embedding. We define

$$\begin{aligned}r_B(Q; k_1, \ldots, k_{n-1}) = \#\{(v_1, \ldots, v_{n-1}) \in \mathcal{I} :\\ Q_i(v_i) = k_{n-i},\ i = 1, \ldots, n-1\}\end{aligned}$$

where Q_i is the Hermitian form on $\mathbb{C}^{\binom{n}{i}}$ associated to $\wedge^i(g^t\bar{g})$. This representation number is a coefficient of the Dirichlet series representing the value at g of the Eisenstein series induced from the Borel. Computing a unitary period of this Eisenstein series amounts to computing the weighted sum

$$\sum_{Q'} \frac{1}{\epsilon(Q')} \sum_{k_1, \ldots, k_{n-1} \geq 1} \frac{r_B(Q'; k_1, \ldots, k_{n-1})}{k_1^{s_1} \ldots k_{n-1}^{s_{n-1}}} \tag{1.4}$$

where the sum is over classes in the genus class of Q and $\epsilon(Q')$ is the size of the group of integral isometries preserving Q'. Our main result implies, in particular, the following.

Theorem 1.1 *Let g and Q be as above. Let $x = g^t\bar{g}$ and assume that x is in the $G(\mathcal{O}_{v_0})$-orbit of the identity for v_0 the place of E dividing the discriminant Δ_E of E.*

Then we have the identity

$$\sum_{Q'} \frac{1}{\epsilon(Q')} \sum_{(k_1\cdots k_{n-1},\Delta_E)=1} \frac{r_B(Q';k_1,\ldots,k_{n-1})}{k_1^{\lambda_1-\lambda_2+1}\cdots k_{n-1}^{\lambda_{n-1}-\lambda_n+1}}$$
$$= w_E^{-1} \det x^{-(\lambda_1+\frac{n-1}{2})} \prod_{p\nmid\Delta_E} P_{m(x_p)}(\nu_0;\lambda) \left(\prod_{i<j} \frac{L_p(\eta^{i+j+1},\lambda_i-\lambda_j)}{L_p(\eta^{i+j},\lambda_i-\lambda_j+1)}\right)$$

of the absolutely convergent multiple Dirichlet series on the left and Euler product on the right whenever $\mathrm{Re}(\lambda_i-\lambda_{i+1}) >> 1$ *for all* $i=1,\ldots,n-1$. *Furthermore, both sides admit a meromorphic continuation to all* $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$. *Here,* w_E *is the number of units in* $\mathcal{O}$, η *is the quadratic Dirichlet character associated to* $E/\mathbb{Q}$ *and* $L_p(\eta^i,s)=(1-\eta^i(p)p^{-s})^{-1}$ *for* $p\nmid\Delta_E$ *is the local Euler factor of the* L*-function* $L(\eta^i,s)$. *The expression* $P_{m(x_p)}(\nu_0;\lambda)$ *is a Laurant polynomial in* $p^{\lambda_1},\ldots,p^{\lambda_n}$ *given explicitly in* (3.11).

Remark 1 This theorem is Corollary 3.1 applied to the minimal parabolic B of G.

Remark 2 As we assume class number one, there is a unique prime $l|\Delta_E$ and therefore v_0 is well defined.

Remark 3 In some cases of small rank, there is a unique class in the genus class of the Hermitian form associated with the identity matrix. For example, in [5] Feit classifies all unimodular lattices over $\mathbb{Z}[\omega]$ of rank at most 12, where ω is a cube root of -1. Over $\mathbb{Z}[i]$ similar results were obtained by Iyanaga [12]. Schiemann has computed [18] more extensive tables of class numbers of positive definite unimodular Hermitian forms over the ring of integers of more general imaginary quadratic fields. These are available at the web page http://www.math.uni-sb.de/ag/schulze/Hermitian-lattices. We make use of the results of Feit and Iyanaga in Sect. 4, where we give examples of the representation numbers of a single Hermitian form in some special cases.

Remark 4 The expression $P_{m(x_p)}(\nu_0;\lambda)$ equals one whenever x_p is in the K_p-orbit of the identity, where $K_p=GL_n(\mathbb{Z}_p)\times GL_n(\mathbb{Z}_p)$ if p is split and $K_p=GL_n(\mathcal{O}_v)$ if p is inert and v is the place of E above p. Consequently, the product over primes appearing in the theorem is, up to finitely many local factors, a quotient of products of Dirichlet L-functions.

We fix here some notation regarding L-functions. First, $\zeta_E(s)$ is the Dedekind zeta function of E and $\zeta=\zeta_{\mathbb{Q}}$. We let $(\zeta_E)_{-1}=\mathrm{Res}_{s=1}(\zeta_E(s))$. For a Dirichlet character χ we let $L(\chi,s)=\prod_p L_p(\chi,s)$ be the (finite part of) the Dirichlet L-function. If $L(s)$ is either a Dirichlet L-function or a Dedekind zeta function we denote by $L^*(s)$ the completed L-function (including the archimedean factors) and by $L^{(D)}(s)$ the partial L-function away from primes dividing the integer D.

2 An anisotropic unitary period as a finite sum over a genus class

For a number field F, we denote by $\mathbb{A}_F$ the ring of adèles of F and by $\mathbb{A}_{F,f}$ its subring of finite adèles. We also let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. For an algebraic set Y defined over F and a place v of F we let $Y_v = Y(F_v)$ and $Y_{\mathbb{A}_F} = Y(\mathbb{A}_F)$.

Let E be an imaginary quadratic extension of $\mathbb{Q}$ of discriminant Δ_E. Throughout this work we assume that E has class number one. Denote by $\mathcal{O} = \mathcal{O}_E$ the ring of integers of E and let $w_E = \#\mathcal{O}^\times$. For any place p of $\mathbb{Q}$ let $E_p = E \otimes_F F_p$. Thus $E_\infty = \mathbb{C}$, $E_p = F_p \oplus F_p$ if p is split in E and $E_p/\mathbb{Q}_p$ is a quadratic extension if p is inert in E. Let G be the group GL_n regarded as an algebraic group defined over E. It will also be convenient to denote $G_\infty = GL_n(\mathbb{C})$, $G_p = GL_n(\mathbb{Q}_p) \times GL_n(\mathbb{Q}_p)$ for a split prime p and $G_p = GL_n(E_p)$ for an inert prime p. Let K be the standard maximal compact subgroup of $G_{\mathbb{A}_E}$, i.e.

$$K = U(n) \prod_{v<\infty} GL_n(\mathcal{O}_v)$$

where $U(n) = K_\infty$ is the unitary group in $GL_n(\mathbb{C})$ and the product is over all places of E. It will also be convenient to write $K = \prod_p K_p$ where for finite p we have $K_p = GL_n(\mathbb{Z}_p) \times GL_n(\mathbb{Z}_p)$ if p is split and $K_p = GL_n(\mathcal{O}_v)$ if p is inert and v is the place of E above p. For an object Y which is the restricted product $Y = \prod_p Y_p$ over all places of $\mathbb{Q}$, we let $Y_f = \prod_{p<\infty} Y_p$. Let

$$X = \{g \in G : {}^t\bar{g} = g\}$$

be the algebraic set defined over $\mathbb{Q}$ of Hermitian matrices in G. There is an action of G on X given by $g \cdot x = gx{}^t\bar{g}$. For $x \in X$ we let

$$H^x = \{g \in G : g \cdot x = x\}$$

be the unitary group associated with x. For $x \in X_{\mathbb{Q}}$ we define the class of x to be

$$[x] = GL_n(\mathcal{O}) \cdot x$$

and write $x \sim y$ if $y \in [x]$. Also define the genus class of x to be

$$[[x]] = X_{\mathbb{Q}} \cap (G_\infty K_f) \cdot x$$

and let $[[x]]/\sim$ be the set of classes in the genus class of x. Let X_∞^+ be the set of positive definite Hermitian matrices in X_∞. It is well known that if $x \in X_{\mathbb{Q}}$ is such that $x_\infty \in X_\infty^+$ then $[[x]]/\sim$ is a finite set. Let $x \in X_{\mathbb{Q}}$ be positive definite at infinity, and let $\theta \in G_\infty$ be such that

$$\theta \cdot e = x. \tag{2.1}$$

We let

$$\mathcal{E}(x) = \{g \in G_{\mathcal{O}} : g \cdot x = x\} \quad \text{and} \quad \epsilon(x) = \#\mathcal{E}(x).$$

Recall that since E is of class number one we have $G_{\mathbb{A}_E} = G_E G_\infty K_f$. It follows that the imbedding of G_∞ in $G_{\mathbb{A}_E}$ defines a bijection

$$G_E \backslash G_{\mathbb{A}_E}/K \simeq G_{\mathcal{O}} \backslash G_\infty / K_\infty = GL_n(\mathcal{O}) \backslash GL_n(\mathbb{C})/U(n).$$

The symmetric space $GL_n(\mathbb{C})/U(n)$ is identified with X_∞^+ via $g \mapsto g \cdot e$. Thus a function ϕ on $G_E \backslash G_{\mathbb{A}_E}/K$ can be regarded as a function ϕ^+ on $G_{\mathcal{O}} \backslash X_\infty^+$ by setting $\phi^+(g \cdot e) = \phi(g)$, $g \in G_\infty$. For the case of positive definite quadratic forms the analog of the following lemma follows from [2]. For the convenience of the reader we repeat the proof here.

Lemma 2.1 *Let ϕ be a function on $G_E \backslash G_{\mathbb{A}_E}/K$ then for all $x \in X_{\mathbb{Q}}$ such that $x_\infty \in X_\infty^+$ we have*

$$\int\limits_{H_{\mathbb{Q}}^x \backslash H_{\mathbb{A}}^x} \phi(h\theta)dh = \mathrm{vol}((H_{\mathbb{A}_f}^x \cap K_f)H_\infty^x) \sum_{[y] \in [[x]]/\sim} \epsilon(y)^{-1} \phi^+(y).$$

Proof First we define a map

$$\mathfrak{i} : H_{\mathbb{Q}}^x \backslash H_{\mathbb{A}}^x / (H_{\mathbb{A}_f}^x \cap K_f) H_\infty^x \to [[x]]/\sim$$

as follows. For any $h \in H_{\mathbb{A}}^x$ we write $h = N^{-1}M$ with $N \in G_{\mathbb{Q}}$ and $M \in G_\infty K_f$. We set

$$\mathfrak{i}(h) = [N \cdot x].$$

We check that the map is well defined. If $h = N'^{-1}M'$ is a second such decomposition then $N'N^{-1} \in G_E \cap G_\infty K_f \subset G_{\mathcal{O}}$ and therefore $[N \cdot x] = [N' \cdot x]$. Note also that if $\gamma \in H_{\mathbb{Q}}^x$ and $k \in (H_{\mathbb{A}_f}^x \cap K_f)H_\infty^x$ then $\gamma hk = (N\gamma^{-1})^{-1}(Mk)$ with $N\gamma^{-1} \in G_E$ and $Mk \in G_\infty K_f$. Since $\gamma^{-1} \cdot x = x$ we see that indeed $\mathfrak{i}$ is a well defined map on the double coset space. Let $y \in [[x]]$ and let $M \in G_\infty K_f$ be such that $y = M \cdot x$. By the local to global principle for Hermitian forms there exists $N \in G_E$ such that $y = N \cdot x$. Now let $h = N^{-1}M \in H_{\mathbb{A}}^x$ then clearly $\mathfrak{i}(h) = [y]$. This proves surjectivity. If h_1, $h_2 \in H_{\mathbb{A}}^x$ with respective decompositions $h_i = N_i^{-1}M_i$ are such that $[N_1 \cdot x] = [N_2 \cdot x]$ then there exists $\gamma \in G_{\mathcal{O}}$ such that $N_1 \cdot x = (\gamma N_2) \cdot x$. Note also that $M_i \cdot x = N_i \cdot x$ and therefore we get that $N_1^{-1}\gamma N_2 \in H_{\mathbb{Q}}^x$, that $M_2^{-1}\gamma^{-1}M_1 \in (H_{\mathbb{A}_f}^x \cap K_f)H_\infty^x$ and that

$$h_1 = (N_1^{-1}\gamma N_2)h_2(M_2^{-1}\gamma^{-1}M_1).$$

This proves injectivity of $\mathfrak{i}$. Note that $h \mapsto \phi(h\theta)$ is a function on the double coset space $H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}/(H^x_{\mathbb{A}_f} \cap K_f)H^x_\infty$ and therefore that

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}} \phi(h\theta)dh = \mathrm{vol}((H^x_{\mathbb{A}_f} \cap K_f)H^x_\infty) \sum_t \frac{1}{\#(t^{-1}H^x_{\mathbb{Q}}t \cap (K_f H^x_\infty))} \phi(t\theta)$$

where the sum is over a set of representatives t for the double coset space $H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}/(H^x_{\mathbb{A}_f} \cap K_f)H^x_\infty$. Let $t = N^{-1}M$ be a decomposition as above, so that $\mathfrak{i}(t) = [N \cdot x]$. Then $\phi(t\theta) = \phi(M\theta) = \phi^+(M \cdot x) = \phi^+(N \cdot x) = \phi^+(\mathfrak{i}(t))$. Note also that

$$t^{-1}H^x_{\mathbb{Q}}t \cap (K_f H^x_\infty) = M^{-1}NH^x_{\mathbb{Q}}N^{-1}M \cap (K_f H^x_\infty)$$

is conjugate to

$$NH^x_{\mathbb{Q}}N^{-1} \cap M(K_f H^x_\infty)M^{-1} = H^{N\cdot x}_{\mathbb{Q}} \cap (K_f H^{N\cdot x}_\infty).$$

The latter equality is since $M_f \in K_f$ and $M_\infty \cdot x = N \cdot x$. But

$$H^{N\cdot x}_{\mathbb{Q}} \cap (K_f H^{N\cdot x}_\infty) = \mathcal{E}(N \cdot x)$$

and therefore

$$\#(t^{-1}H^x_{\mathbb{Q}}t \cap (K_f H^x_\infty)) = \epsilon(N \cdot x) = \epsilon(\mathfrak{i}(t)).$$

The lemma now follows. □

3 Periods of Eisenstein series and representation numbers

3.1 Eisenstein series, classical and adelic

Here we set up some notation and define the Eisenstein series that we consider. We will only consider Eisenstein series induced from characters on standard parabolic subgroups. Let $B = TU$ be the standard Borel subgroup of G with its standard Levi decomposition and let $P = MV$ be a parabolic of type $(n_1, \dots, n_t)$ containing B with its standard Levi decomposition. For integers $a \le b$ let $[a, b] = \{a, a+1, \dots, b\}$. Let

$$I_i = [n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_i], \quad i = 1, \dots, t$$

be the segments determined by P and let

$$N_i = n_{i+1} + \cdots + n_t, \quad i = 1, \dots, t-1.$$

We will view $\mathbb{C}^t$ as a subspace of $\mathbb{C}^n$ as follows. For $\mu = (\mu_1, \dots, \mu_t) \in \mathbb{C}^t$, when convenient, we will also denote by μ the n-tuple $(\mu_1^{(n_1)}, \dots, \mu_t^{(n_t)})$, where $a^{(m)}$ is the m-tuple $(a, \dots, a)$. From now on, we will always consider t-tuples μ so that $n_1\mu_1 + \cdots + n_t\mu_t = 0$. Denote by $I_P^G(\mu) = \mathrm{Ind}_{P_{\mathbb{A}_E}}^{G_{\mathbb{A}_E}}(\mu)$ the representation of $G_{\mathbb{A}_E}$ parabolically induced from the character

$$\mathrm{diag}(m_1, \dots, m_t) \mapsto \prod_{i=1}^t |\det m_i|_{\mathbb{A}_E}^{\mu_i}$$

on $M_{\mathbb{A}_E}$. For $\varphi \in I_P^G(\mu)$ we consider the Eisenstein series $E_P(\varphi, \mu)$ defined as the meromorphic continuation to all $\mu \in \mathbb{C}^t$ of the series

$$E_P(g, \varphi, \mu) = \sum_{\gamma \in P_E \backslash G_E} \varphi(\gamma g)$$

that converges if $\mathrm{Re}(\mu_i - \mu_{i+1})$ is large enough for all $i = 1, \dots, t-1$. Let

$$\varphi_\mu(mvk) = \prod_{i=1}^t |\det m_i|^{\mu_i + \frac{1}{2}(n_{i+1} + \cdots + n_t - (n_1 + \cdots + n_{i-1}))}$$

where $m = \mathrm{diag}(m_1, \dots, m_t) \in M_{\mathbb{A}_E}$, $v \in V_{\mathbb{A}_E}$ and $k \in K$, be the K-invariant element of $I_P^G(\mu)$ normalized so that $\varphi_\mu(e) = 1$. Let

$$E_P(g; \mu) = E_P(g, \varphi_\mu, \mu).$$

Since the field E has class number one, the embedding of $G_{\mathcal{O}}$ in G_E defines a bijection $P_{\mathcal{O}} \backslash G_{\mathcal{O}} \simeq P_E \backslash G_E$. As a function on $G_{\mathcal{O}} \backslash X_\infty^+$, i.e. with $E^+(g \cdot e; \mu) = E_P(g; \mu)$, it can therefore be expressed by

$$E_P^+(x; \mu) = \det x^{\mu_1 + \frac{n_2 + \cdots + n_t}{2}} \sum_{\delta \in P_{\mathcal{O}} \backslash G_{\mathcal{O}}} \prod_{i=1}^{t-1} d_{N_i}(\delta \cdot x)^{-(\mu_i - \mu_{i+1} + \frac{n_i + n_{i+1}}{2})} \tag{3.1}$$

where $d_i(x)$ is the determinant of the lower right $i \times i$ block of x. In particular, we have

$$E_B^+(x; \lambda) = \det x^{\lambda_1 + \frac{n-1}{2}} \sum_{\delta \in B_{\mathcal{O}} \backslash G_{\mathcal{O}}} \prod_{i=1}^{n-1} d_i(\delta \cdot x)^{-(\lambda_{n-i} - \lambda_{n-i+1} + 1)}. \tag{3.2}$$

Again (3.1) and (3.2) are only valid for $\mathrm{Re}(\mu_i - \mu_{i+1})$ and $\mathrm{Re}(\lambda_{n-i} - \lambda_{n-i+1})$ sufficiently large. The Eisenstein series $E_P^+(x; \mu)$ can be expressed as a residue of the Eisenstein series $E_B^+(x; \lambda)$. Let f be a function on $\mathbb{C}^n$ and let $\mu \in \mathbb{C}^t$. Whenever well-defined, we define the residue operator $\mathrm{Res}_P(\mu)$ by

$$\mathrm{Res}_P(\mu)f = \lim_{\lambda\to\mu+\Lambda(P)} f(\lambda) \prod_{\substack{j\in[1,n-1]\\ j\notin\{n-N_i:i=1,\dots,t-1\}}} (\lambda_j - \lambda_{j+1} - 1)$$

where

$$\Lambda(P) = (\Lambda_{n_1},\dots,\Lambda_{n_t}) \quad \text{and} \quad \Lambda_n = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}) \in \mathbb{C}^n.$$

We shall also set $\mathrm{Res}_G = \mathrm{Res}_G(0)$. It is well known that

$$\mathrm{Res}_G\, E_B^+(x;\cdot) \equiv c_n$$

is a constant and by computations of Langlands in [13] we have

$$c_n = \frac{((\zeta_E^*)_{-1})^{n-1}}{\zeta_E^*(2)\zeta_E^*(3)\cdots\zeta_E^*(n)}.$$

We also set

$$c(P) = \prod_{i=1}^{t} c_{n_i}.$$

Using Langlands computation it can then be shown that

$$\mathrm{Res}_P(\mu)E_B^+(x;\cdot) = c(P)E_P^+(x;\mu). \tag{3.3}$$

3.2 Eisenstein series and representation numbers

For $x \in X_{\mathbb{Q}}$ we let Q_x denote the Hermitian form associated with the matrix x, i.e.

$$Q_x(\xi) = {}^t\bar{\xi}x\xi \text{ for } \xi \in \mathbb{C}^n.$$

We let $x \in X_{\mathbb{Q}}$ be such that $x_\infty \in X_\infty^+$ and Q_x is integral (i.e. $Q_x(\xi) \in \mathbb{Z}$ for all $\xi \in \mathcal{O}^n$). We will show that for such x, the Eisenstein series $E_P^+(x;\mu)$ is a Dirichlet series in the variables $(\mu_1-\mu_2,\dots,\mu_{t-1}-\mu_t)$. We interpret the coefficients in terms of a type of representation number, which counts certain points on the (partial) flag variety $P_E\backslash G_E$. To define the representation numbers we will use the Plücker coordinates of the flag variety. To any $g \in G_E$ we associate the vectors $v_1(g),\dots,v_{n-1}(g)$ where $v_i(g) \in E^{\binom{n}{i}}$ is the vector of all $i \times i$ minors in the bottom i rows of g. For a vector $v \in E^m$ we denote by $[v]$ its $E^\times$-orbit in the projective space $\mathbb{P}_E^{m-1}$. The map

$$P_E g \mapsto ([v_{N_1}(g)],\dots,[v_{N_{t-1}}(g)])$$

is an embedding

$$P_E\backslash G_E \hookrightarrow \prod_{i=1}^{t-1} \mathbb{P}_E^{\binom{n}{N_i}-1}.$$

It will be more convenient for us to use the identification $P_{\mathcal{O}}\backslash G_{\mathcal{O}} \simeq P_E\backslash G_E$ and work with integral coordinates. The map

$$g \mapsto (v_{N_1}(g), \dots, v_{N_{t-1}}(g))$$

also defines an embedding

$$P_{\mathcal{O}}\backslash G_{\mathcal{O}} \hookrightarrow \prod_{i=1}^{t-1} \left(\mathcal{O}^{\binom{n}{N_i}}/\mathcal{O}^\times\right).$$

We let

$$\mathcal{I}(P;\mathcal{O}) = \left\{(v_1, \dots, v_{t-1}) \in \prod_{i=1}^{t-1} \mathcal{O}^{\binom{n}{N_i}} : \exists g \in G_{\mathcal{O}}, \right.$$
$$\left. v_{N_i}(g) = v_i, \quad \forall i = 1, \dots, t-1 \right\}.$$

Thus, a $t-1$ tuple is in $\mathcal{I}(P;\mathcal{O})$ if it satisfies the relations imposed by $P_E\backslash G_E$.

To define the representation numbers we need some more notation. For any matrix $g \in M_{n\times k}(E)$ and integers $1 \le i \le n$, $1 \le j \le k$ we denote, as usual, the $(i, j)^{\text{th}}$ component of g by g_{ij}. We extend this notation as follows. Let

$$I_m(n) = \{(i_1, \dots, i_m) \in \mathbb{Z}^m : 1 \le i_1 < \dots < i_m \le n\}.$$

For $i = (i_1, \dots, i_r) \in I_r(n)$ and $j = (j_1, \dots, j_q) \in I_q(k)$ we denote by $g_{ij} \in M_{r\times q}(E)$ the matrix so that $(g_{ij})_{lm} = g_{i_l j_m}$ for $l = 1, \dots, r$ and $m = 1, \dots, q$. Later on it will also be convenient, when $q \le n$ to let $g^{(j)} = g_{[n+1-q,n],j}$.

Note that for $g \in G_E$ the linear operator $\wedge^k g : E^{\binom{n}{k}} \to E^{\binom{n}{k}}$ is represented by the matrix $(\det g_{ij})_{i,\, j\in I_k(n)}$ with respect to the basis $E_i = e_{i_1} \wedge \dots \wedge e_{i_k}$, $i = (i_1, \dots, i_k) \in I_k(n)$ of $E^{\binom{n}{k}}$, where e_i, $i = 1, \dots, n$ is the standard basis of E^n. From now on when we write $\wedge^k g$ we will mean the matrix $(\det g_{ij})_{i,\, j\in I_k(n)}$.

The representation numbers that we consider are defined for positive integers $k_1, \dots, k_{t-1}$ by

$$r_P(x; k_1, \dots, k_{t-1}) = \#\{(v_1, \dots, v_{n-1}) \in \mathcal{I}(P;\mathcal{O}) : Q_{\wedge^{N_i} x}(v_i) = k_i,$$
$$i = 1, \dots, t-1\}. \tag{3.4}$$

For every integer D define the Dirichlet series

$$Z_P^{(D)}(x; s_1, \dots, s_{t-1}) = w_E^{-(t-1)} \sum_{(k_1 k_2 \cdots k_{n-1}, D)=1} \frac{r_P(x; k_1, \dots, k_{t-1})}{k_1^{s_1} k_2^{s_2} \cdots k_{t-1}^{s_{t-1}}}.$$

We also define the genus representation numbers

$$r_P(\mathrm{gen}(x); k_1, \dots, k_{t-1}) = \sum_{y \in [[x]]/\sim} \epsilon^{-1}(y) r_P(y; k_1, \dots, k_{t-1}) \tag{3.5}$$

and the associated Dirichlet series

$$Z_P^{(D)}(\mathrm{gen}(x); s_1, \dots, s_{t-1}) = w_E^{-(t-1)} \sum_{(k_1 k_2 \cdots k_{t-1}, D)=1} \frac{r_P(\mathrm{gen}(x); k_1, \dots, k_{t-1})}{k_1^{s_1} k_2^{s_2} \cdots k_{t-1}^{s_{t-1}}}.$$

The series $Z_P^{(D)}(x; \cdot)$ and $Z_P^{(D)}(\mathrm{gen}(x); \cdot)$ converge absolutely for $\mathrm{Re}(s_1), \dots, \mathrm{Re}(s_{t-1})$ sufficiently large. If $D = 1$ we will sometimes omit the superscript.

We now express special values of the Eisenstein series (3.1) in terms of the Dirichlet series $Z_P(x; s_1, \dots, s_{t-1})$. We need the following two Lemmas. The first is an elementary exercise in computation of a determinant, which we leave to the reader.

Lemma 3.1 *Let A and B be $k \times n$ matrices with $k \le n$. Then*

$$\det(A^t B) = \sum_{j \in I_k(n)} \det(A^{(j)} B^{(j)}).$$

Lemma 3.2 *For $\delta \in G_{\mathcal{O}}$ we have*

$$d_i(\delta \cdot x) = Q_{\wedge^i x}(v_i(\delta)).$$

Proof We parameterize the coordinates of the vector $v_i(\delta)$ by $(v_j)_{j \in I_i(n)}$ where $v_j = \det(\delta^{(j)})$. Note that

$$d_i(\delta \cdot x) = d_i(\delta x^t \bar{\delta}) = \det((\delta x)_{[n+1-i,n],[1,n]}{}^t (\bar{\delta}_{[n+1-i,n],[1,n]})).$$

By Lemma 3.1 we get that

$$d_i(\delta \cdot x) = \sum_{j \in I_i(n)} \det((\delta x)^{(j)} (\bar{\delta})^{(j)}). \tag{3.6}$$

We apply Lemma 3.1 once more to obtain

$$\det((\delta x)^{(j)}) = \sum_{k \in I_i(n)} \det(\delta^{(j)} x_{kj}). \tag{3.7}$$

Plugging (3.7) into (3.6) we obtain that

$$d_i(\delta \cdot x) = \sum_{j,k \in I_i(n)} v_k \bar{v}_j x_{kj}.$$

□

Applying Lemma 3.2 we may now rewrite (3.1) as

$$\begin{aligned}
&\det x^{-(\mu_1+\frac{n_2+\cdots+n_t}{2})} E_P^+(x,\mu) \\
&= w_E^{-(t-1)} \sum_{(v_1,\ldots,v_{t-1})\in\mathcal{I}(P;\mathcal{O})} \prod_{i=1}^{t-1} Q_{\wedge^{N_i} x}(v_i)^{-(\mu_i-\mu_{i+1}+\frac{n_i+n_{i+1}}{2})} \\
&= w_E^{-(t-1)} \sum_{k_1,\ldots,k_{t-1}\geq 1} \frac{r_P(x;k_1,\ldots,k_{t-1})}{k_1^{-(\mu_1-\mu_2+\frac{n_1+n_2}{2})}\cdots k_{t-1}^{-(\mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2})}}
\end{aligned}$$

in the region of absolute convergence. We have proven

Proposition 3.1 *Let $x \in X_{\mathbb{Q}}$ be such that $x_\infty \in X_\infty^+$ and Q_x is integral. Then for $\mu \in \mathbb{C}^t$ such that* $\mathrm{Re}(\mu_i - \mu_{i+1}) >> 1,\ i = 1,\ldots,t-1$ *we have*

$$\begin{aligned}
&E_P^+(x;\mu) \\
&= \det x^{\mu_1+\frac{n_2+\cdots+n_t}{2}} Z_P\left(x; \mu_1-\mu_2+\frac{n_1+n_2}{2},\ldots,\mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2}\right).
\end{aligned}$$

In particular, the Dirichlet series $Z_P^{(D)}(x;s_1,\ldots,s_{t-1})$ converges in some positive cone and admits a meromorphic continuation to $(s_1,\ldots,s_{t-1}) \in \mathbb{C}^{t-1}$.

From now on we shall use the notation $Z_P^{(D)}(x;s_1,\ldots,s_{t-1})$ and $Z_P^{(D)}(\mathrm{gen}(x);s_1,\ldots,s_{t-1})$ to denote the meromorphic continuation of the corresponding Dirichlet series.

Remark 5 If we remove our assumption on the class number of E, an analog of Lemma 2.1 still holds, i.e. the anisotropic unitary period of an Eisenstein series can be expressed as a finite weighted sum of point evaluations. The sum is over the genus class of the unitary group H^x. The genus class of an algebraic group H defined over F is the pointed set $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}/((H_{\mathbb{A}_f} \cap K_f)H_\infty)$. More generally, if S is a finite set of places of $\mathbb{Q}$ containing ∞ and if superscript S denotes a product over places not in S and a subscript of places in S, then $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}/((H_{\mathbb{A}^S} \cap K^S)H_S)$ is the S-genus of H. Since there exists S such that the S-genus of G has a single element, every coset in the flag variety $P_E\backslash G_E$ has an $\mathcal{O}_S$-integral representative. At an integral point x, we may therefore express the Eisenstein series as a Dirichlet series. The coefficients relatively prime to S, can then be interpreted as representation numbers counting points in the flag variety. Our formula for the H^x-period of an Eisenstein series can then be applied—as we proceed to do in the class number one case in what follows—to provide information on the weighted sum of these representation numbers over the genus of H^x.

3.3 The unitary period of an Eisenstein series

In [16], we obtained the following formula for the unitary period of an Eisenstein series:

$$\int_{H^x_{\mathbb{Q}}\backslash H^x_{\mathbb{A}}} E_B(h,\varphi,\lambda)\,dh = 2^{-n}\,\mathrm{vol}((E^1_{\mathbb{Q}}\backslash E^1_{\mathbb{A}})^n)\sum_{\nu} J^{st,x}(\nu,\varphi,\lambda). \tag{3.8}$$

Here E^1 is the algebraic group defined over $\mathbb{Q}$ as the kernel of the norm map $N = N_{E/\mathbb{Q}} : E^\times \to \mathbb{Q}^\times$. We view $(E^1)^n$ as a subgroup of T and denote by T' the subgroup of G' defined over $\mathbb{Q}$ of diagonal matrices in G'. We shall denote by N also the extension of the norm map from $\mathbb{A}_E^\times$ to $\mathbb{A}^\times$ (with kernel $E^1_{\mathbb{A}}$) and by abuse of notation also the norm map $N : T_{\mathbb{A}_E} \to T'_{\mathbb{A}}$. The term $J^{st,x}(\nu,\varphi,\lambda)$ is a factorizable linear functional on $I_B^G(\lambda)$ parameterized by the group of Hecke characters ν on $T'_{\mathbb{A}}/N(T_{\mathbb{A}_E})$, i.e. characters ν of $T'_{\mathbb{Q}}\backslash T'_{\mathbb{A}}$ such that $\nu\circ N = \mathbf{1}_T$ is the trivial character on $T_{\mathbb{A}_E}$. Thus the sum on the right hand side of (3.8) is over the 2^n characters $\nu = (\nu_1,\dots,\nu_n)$ where $\nu_i \in \{\mathbf{1}_{T'},\eta\}$ and η is the quadratic Hecke character associated to $E/\mathbb{Q}$ by class field theory. Let θ be as in (2.1). Applying the linear functional to the right shift $R(\theta)\varphi_\lambda$ of φ_λ by θ we have

$$J^{st,x}(\nu,R(\theta)\varphi_\lambda,\lambda) = J^{st,x_\infty}(\nu_\infty,R(\theta)\varphi_\infty,\lambda)\prod_{p<\infty} J^{st,x_p}(\nu_p,\varphi_p,\lambda).$$

For a precise definition of $J^{st,x}(\nu,\varphi,\lambda)$ and its local factors, we refer to [16]. Recall that all unitary groups H^x are inner forms. We fix once and for all a Haar measure on $H^e_{\mathbb{A}}$ and choose compatible measures on the other unitary groups. The volume element appears in the formula for the period because the $J^{st,x}$ functionals on the right hand side are proportional to the volume of $H^x_{\mathbb{A}}$ and inverse proportional to the volume on $(E^1_{\mathbb{A}})^n$. Globally, the functionals also satisfy

$$J^{st,x}(\eta\nu,\varphi,\lambda) = \eta(\det x)J^{st,x}(\nu,\varphi,\lambda) \tag{3.9}$$

where $\eta\nu = (\eta\nu_1,\dots,\eta\nu_n)$. We remark that up to a finite product of local terms, the right hand side of (3.8) is expressed explicitly as the meromorphic continuation of a Dirichlet series in the variables $(\lambda_1-\lambda_2,\dots,\lambda_{n-1}-\lambda_n)$ that converges on some positive cone. We recall here the explicit formulas that we know for the local terms. Let

$$J^{x_p}(\nu_p;\lambda) = \frac{\mathrm{vol}((E^1_p)^n\cap K_p)}{\mathrm{vol}(H^e_p\cap K_p)} J^{st,x_p}(\nu_p,\varphi_p,\lambda)$$

where φ_p is the K_p-invariant section in $\mathrm{Ind}_{B_p}^{G_p}(\lambda)$ normalized so that $\varphi_p(e) = 1$. If $p < \infty$ is either split or such that $E_p/\mathbb{Q}_p$ is an unramified quadratic extension then

$$J^{x_p}(\nu_p; \lambda) = P_{m(x_p)}(\nu_p; \lambda) \prod_{1 \le i < j \le n} \frac{L_p(\nu_i \nu_j \eta, \lambda_i - \lambda_j)}{L_p(\nu_i \nu_j, \lambda_i - \lambda_j + 1)} \tag{3.10}$$

where $P_m(\nu; \lambda)$ is, up to a scalar, the mth Hall-Littlewood polynomial in the variables $\nu_1(p)p^{\lambda_1}, \dots, \nu_n(p)p^{\lambda_n}$ with parameter $\eta(p)p^{-1}$. Explicitly, if $x_p \in K_p \cdot e$ then $m(x_p) = (0, \dots, 0) \in \mathbb{Z}^n$ and $P_{m(x_p)}(\nu; \lambda) = 1$. More generally, for any $x_p \in X_p$ there exists a unique $m = m(x) = (m_1, \dots, m_n) \in \mathbb{Z}^n$ with $m_1 \ge \dots \ge m_n$ such that $x_p \in K_p \cdot p^m$ where $p^m = \mathrm{diag}(p^{m_1}, \dots, p^{m_n})$. We then have

$$\begin{aligned} P_m(\nu; \lambda) = \nu_0(p^m) &\frac{\prod_{i=1}^n L_p(\eta^i, i)}{L_p(\eta, 1)^n} \\ &\times \sum_{\sigma \in S_n} \sigma \left(p^{\langle \lambda - \Lambda_n, m \rangle} \prod_{i<j} \frac{L_p(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j)}{L_p(\nu_i \nu_j^{-1} \eta, \lambda_i - \lambda_j + 1)} \right) \end{aligned} \tag{3.11}$$

where $\nu_0 = (\eta, \eta^2, \dots, \eta^n)$ and the permutation σ acts on an expression in (ν, λ) by permuting the indices of both ν and λ. In the case where $E_p/\mathbb{Q}_p$ is ramified there are no explicit formulas available for $J^{x_p}(\nu_p; \lambda)$, but if $x_p \in K_p \cdot e$ then we have an asymptotic formula

$$\lim_{\lambda \mapsto \infty} J^{x_p}(\nu_p; \lambda) = 2^{n-1} \mathrm{ch}_{\{\nu_0, \eta\nu_0\}}(\nu) \tag{3.12}$$

where for any set A we denote by ch_A the characteristic function of A. In any case $J^{x_p}(\nu_p, \varphi_p, \lambda)$ is a rational function in $p^{\lambda_1}, \dots, p^{\lambda_n}$. The formulas (3.9)–(3.12) can be found in [16]. In [14] we also observed that

$$J^{st, x_\infty}(\nu, R(\theta)\varphi_{\lambda,\infty}, \lambda) = \frac{\mathrm{vol}(H_\infty^e \cap K_\infty)}{\mathrm{vol}((E_\infty^1)^n \cap K_\infty)}.$$

We obtain that

$$\begin{aligned} \int_{H_{\mathbb{Q}}^x \backslash H_{\mathbb{A}}^x} E_B(h\theta; \lambda)\, dh = 2^{-n} &\frac{\mathrm{vol}((E_{\mathbb{Q}}^1 \backslash E_{\mathbb{A}}^1)^n)}{\mathrm{vol}((E_{\mathbb{A}}^1)^n \cap K)} \mathrm{vol}((H_{\mathbb{A}_f}^x \cap K_f) H_\infty^e) \\ &\times \sum_\nu \left[\left(\prod_{p \nmid \Delta_E} P_{m(x_p)}(\nu; \lambda) \right) \prod_{i<j} \frac{L^{S_E}(\nu_i \nu_j \eta, \lambda_i - \lambda_j)}{L^{S_E}(\nu_i \nu_j, \lambda_i - \lambda_j + 1)} \right. \\ &\left. \times \prod_{p \mid \Delta_E} J^{x_p}(\nu_p; \lambda) \right] \end{aligned} \tag{3.13}$$

where Δ_E is the discriminant of E and S_E is the set of all prime numbers that divide Δ_E.

Lemma 3.3

$$\frac{\mathrm{vol}((E^1_{\mathbb{Q}}\backslash E^1_{\mathbb{A}})^n)}{\mathrm{vol}((E^1_{\mathbb{A}})^n \cap K)} = w_E^{-n}.$$

Proof The quotient of volumes is of course independent of a choice of measure on $E^1_{\mathbb{A}}$. We fix the decomposable Haar measure on $E^1_{\mathbb{A}}$ as chosen in [14] with respect to an additive character $\psi = \psi_0 \circ \mathrm{Trace}_{E/\mathbb{Q}}$ where ψ_0 is an additive character on $\mathbb{Q}\backslash\mathbb{A}$. The local measure on E^1_p is determined by the exact sequence $1 \to E^1_p \to E^\times_p \to \mathbb{Q}^\times_p$ and the Haar measure $d_{E_p^\times} x = L(1, \mathbf{1}_{E_p^\times})\frac{d^{\psi_p} x}{|x|_{E_p}}$ (resp. $d_{\mathbb{Q}_p^\times} x = L(1, \mathbf{1}_{\mathbb{Q}_p^\times})\frac{d^{(\psi_0)_p} x}{|x|_{\mathbb{Q}_p}}$) on $E^\times_p$ (resp. $\mathbb{Q}^\times_p$), where $d^{\psi_p} x$ (resp. $d^{(\psi_0)_p} x$) is the self dual Haar measure on E_p (resp. $\mathbb{Q}_p$) with respect to ψ_p (resp. $(\psi_0)_p$). As explained in [14], if we set

$$\mathfrak{d}_{E_p} = \mathfrak{d}^{\psi}_{E_p} = \begin{cases} \mathrm{vol}(\mathcal{O}_{E_p}) & E_p \text{ non-archimedean,} \\ \frac{1}{2}\,\mathrm{vol}(\{x + iy : 0 \le x, y \le 1\}) & E_p \text{ complex} \end{cases}$$

where the volume is taken with respect to d^{ψ_p} then $\prod_p \mathfrak{d}_{E_p} = |\Delta_E|^{-\frac{1}{2}}$ is independent of ψ. By Ono's formula for the Tamagawa number of a torus [17] we have $\mathrm{vol}(E^1_{\mathbb{Q}}\backslash E^1_{\mathbb{A}}) = 2L^*(1, \eta)$. By Dirichlet's class number formula

$$L^*(1, \eta) = \frac{2h_E}{w_E\,|\Delta_E|^{\frac{1}{2}}}$$

where h_E is the class number of E. Since we assume class number one, we see that $L^*(1, \eta) = 2w_E^{-1}\,|\Delta_E|^{-\frac{1}{2}}$ and therefore that

$$\mathrm{vol}((E^1_{\mathbb{Q}}\backslash E^1_{\mathbb{A}})^n) = (4w_E^{-1}\,|\Delta_E|^{-\frac{1}{2}})^n.$$

The volume on the denominator can be computed as the product over all primes of its local counterparts. We leave it to the reader to verify that

$$\mathrm{vol}((E^1_p)^n \cap K_p) = \begin{cases} \mathfrak{d}^n_{E_p} & p \text{ is either split or unramified} \\ (2\mathfrak{d}_{E_p})^n & p = \infty \text{ or } p \text{ is a ramified prime.} \end{cases}$$

In all nine cases of imaginary quadratic fields of class number one we have $\sum_{p|\Delta_E} 1 = 1$. We therefore have

$$\mathrm{vol}((E_1)^n_{\mathbb{A}} \cap K) = (4\,|\Delta_E|^{-\frac{1}{2}})^n.$$

∎

Applying Lemmas 2.1 and 3.3 to (3.13) we get that

$$\sum_{y\in[[x]]/\sim} \epsilon(y)^{-1} E_B^+(y;\lambda) = (2w_E)^{-n} \times \sum_\nu \Bigg[\prod_{p\nmid\Delta_E} P_{m(x_p)}(\nu;\lambda) \times \left(\prod_{i<j} \frac{L_p(\nu_i\nu_j\eta, \lambda_i-\lambda_j)}{L_p(\nu_i\nu_j, \lambda_i-\lambda_j+1)} \right) \prod_{p\mid\Delta_E} J^{x_p}(\nu_p;\lambda) \Bigg]. \tag{3.14}$$

Combined with Proposition 3.1, (3.14) gives for $x \in X_{\mathbb{Q}}$ such that $x_\infty \in X_\infty^+$ and Q_x is integral the identity

$$Z_B(\mathrm{gen}(x); \lambda_1-\lambda_2+1, \ldots, \lambda_{n-1}-\lambda_n+1) = (2w_E)^{-n} \det x^{-(\lambda_1+\frac{n-1}{2})} \times \sum_\nu \Bigg[\prod_{p\nmid\Delta_E} P_{m(x_p)}(\nu;\lambda) \left(\prod_{i<j} \frac{L_p(\nu_i\nu_j\eta, \lambda_i-\lambda_j)}{L_p(\nu_i\nu_j, \lambda_i-\lambda_j+1)} \right) \prod_{p\mid\Delta_E} J^{x_p}(\nu_p;\lambda) \Bigg]. \tag{3.15}$$

Similarly, applying Res_P to (3.14) and taking (3.3) into consideration we have proven

Theorem 3.4 *Let $x \in X_{\mathbb{Q}}$ be such that $x_\infty \in X_\infty^+$ and Q_x is integral. Then for any parabolic subgroup P of G containing B and for $\mu \in \mathbb{C}^t$ we have*

$$\begin{aligned} &Z_P\left(\mathrm{gen}(x); \mu_1-\mu_2+\frac{n_1+n_2}{2}, \ldots, \mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2}\right) \\ &= (2w_E)^{-n} c(P)^{-1} \det x^{-(\mu_1+\frac{n_2+\cdots+n_t}{2})} \times \mathrm{Res}_P(\mu) \\ &\quad\times \sum_\nu \Bigg[\prod_{p\nmid\Delta_E} P_{m(x_p)}(\nu;\lambda) \left(\prod_{i<j} \frac{L_p(\nu_i\nu_j\eta, \lambda_i-\lambda_j)}{L_p(\nu_i\nu_j, \lambda_i-\lambda_j+1)} \right) \prod_{p\mid\Delta_E} J^{x_p}(\nu_p;\lambda) \Bigg] \end{aligned}$$

where the Euler product in each summand of the right hand side, is only convergent for μ in some positive cone, but the identity holds in the sense of analytic continuation.

If x is such that x_l is in the K_l-orbit of the identity for the prime l dividing Δ_E then we can obtain more explicit formulas for the representation numbers $r_P(x; k_1, \ldots, k_t)$, for integers k_i not divisible by l, by using the asymptotic formula (3.12). In view of (3.9) we have

Corollary 3.1 *If in addition to the assumptions in Theorem 3.4 we have $x_l \in K_l \cdot e$ where l is the unique prime dividing Δ_E then in the sense of analytic continuation we* *have the identity*

$$
\begin{aligned}
&Z_P^{(\Delta_E)}\left(\mathrm{gen}(x);\, \mu_1-\mu_2+\frac{n_1+n_2}{2},\ldots,\mu_{t-1}-\mu_t+\frac{n_{t-1}+n_t}{2}\right)\\
&\quad = w_E^{-n}c(P)^{-1}\det x^{-(\mu_1+\frac{n_2+\cdots+n_t}{2})}\prod_{p\nmid\Delta_E}P_{m(x_p)}(\nu_0;\mu+\Lambda(P))\\
&\qquad \times \mathrm{Res}_P(\mu)\prod_{p\nmid\Delta_E}\left(\prod_{i<j}\frac{L_p(\eta^{i+j+1},\lambda_i-\lambda_j)}{L_p(\eta^{i+j},\lambda_i-\lambda_j+1)}\right).
\end{aligned}
$$

4 Explicit examples

4.1 The mirabolic parabolic

Assume here that P is the parabolic subgroup of G of type $(n-1, 1)$. As explained in Sect. 1, the representation number $r_P(x; k)$ is then the number of ways to represent k by the Hermitian form Q_x with primitive vectors. We also define

$$r(x;k) = \#\{v \in \mathcal{O}^n : Q_x(v) = k\}$$

and

$$r(\mathrm{gen}(x);k) = \sum_{y\in[[x]]/\sim} \epsilon(y)^{-1} r(y;k).$$

Let

$$\hat{Z}^{(D)}(x;s) = w_E^{-1}\sum_{(k,D)=1}\frac{r(x;k)}{k^s}$$

and

$$\hat{Z}^{(D)}(\mathrm{gen}(x);s) = w_E^{-1}\sum_{(k,D)=1}\frac{r(\mathrm{gen}(x);k)}{k^s}.$$

Then it is easy to see that

$$\hat{Z}^{(D)}(x;s) = \zeta_E^{(D)}(s) Z_P^{(D)}(x;s)$$

and

$$\hat{Z}^{(D)}(\mathrm{gen}(x);s) = \zeta_E^{(D)}(s) Z_P^{(D)}(\mathrm{gen}(x);s).$$

Applying Corollary 3.1 and setting $\mu(s) = (\frac{s}{n} - \frac{1}{2}, (1-n)(\frac{s}{n} - \frac{1}{2}))$ we get that whenever $x_l \in K_l \cdot e$ for the prime l dividing Δ_E we have

$$\hat{Z}^{(\Delta_E)}(\mathrm{gen}(x); s) = w_E^{-n} \frac{\zeta_E^*(2)\zeta_E^*(3)\cdots\zeta_E^*(n-1)}{(\zeta_E^*)_{-1}^{n-2}} \det x^{-\frac{s}{n}} \left(\frac{\zeta_{-1}^{(\Delta_E)}}{L^{(\Delta_E)}(\eta, 2)}\right)^{n-2}$$
$$\times \prod_{k=2}^{n-2} \left(\frac{L^{(\Delta_E)}(\eta^{k+1}, k)}{L^{(\Delta_E)}(\eta^k, k+1)}\right)^{n-(k+1)} \prod_{p \nmid \Delta_E} P_{m(x_p)}(\nu_0; \mu(s) + \Lambda(P))$$
$$\times \zeta_E^{(\Delta_E)}(s) \prod_{i=1}^{n-1} \frac{L^{(\Delta_E)}(\eta^{i+n+1}, s-i)}{L^{(\Delta_E)}(\eta^{i+n}, s+1-i)}. \tag{4.1}$$

4.2 The parabolic $(1, n-2, 1)$

Here we assume that $n \geq 3$ and that P is the standard parabolic subgroup of G of type $(1, n-2, 1)$. The Plücker coordinates of a matrix g are given by

$$v = v_1(g) = (v_1, v_2, \ldots, v_n)$$
$$w = v_{n-1}(g) = (w_1, w_2, \ldots, w_n)$$

where $v_i = g^{(i)}$ and $w_i = g^{([1,n]-\{i\})}$. We leave it to the reader to verify that

$$\mathcal{I}(P; \mathcal{O}) = \left\{ v, w \in \mathcal{O}_{prim}^n : \sum_{i=1}^{n} (-1)^i v_i w_i = 0 \right\}.$$

In order to interpret $r_P(x; m_1, m_2)$ as more familiar representation numbers we will use the change of variables $(v, w) \mapsto (v, w')$ where $w' = (w_1', \ldots, w_n')$ with $w_i' = (-1)^i \bar{w}_i$. Note then that

$$Q_{\wedge^{n-1}x}(w) = Q_{(\det x)x^{-1}}(w').$$

Therefore, the representation number $r_P(x; m_1, m_2)$ is the size of the set

$$\{v, w \in \mathcal{O}_{prim}^n : {}^t\bar{v}w = 0,\ Q_{(\det x)x^{-1}}(w) = m_1,\ Q_x(v) = m_2\}.$$

Note further, that the map $(v, w) \mapsto (v, (\det x)x^{-1}w)$ is a bijection from this set to the set in (1.1). We also define

$$r(x; m_1, m_2) = \{v, w \in \mathcal{O}^n : {}^t\bar{v}w = 0,\ Q_{(\det x)x^{-1}}(w) = m_1,\ Q_x(v) = m_2\}$$

and $r(\text{gen}(x); m_1, m_2) = \sum_{y\in[[x]]/\sim} \epsilon(y)^{-1} r(y; m_1, m_2)$. Let

$$\hat{Z}^{(D)}(x; s_1, s_2) = w_E^{-2} \sum_{(m_1 m_2, D)=1} \frac{r(x; m_1, m_2)}{m_1^{s_1} m_2^{s_2}}$$

and

$$\hat{Z}^{(D)}(\text{gen}(x); s_1, s_2) = w_E^{-2} \sum_{(m_1 m_2, D)=1} \frac{r(\text{gen}(x); m_1, m_2)}{m_1^{s_1} m_2^{s_2}}$$

then it is easy to see that

$$\hat{Z}^{(D)}(x; s_1, s_2) = \zeta_E^{(D)}(s_1)\zeta_E^{(D)}(s_2) Z_P^{(D)}(x; s_1, s_2)$$

and

$$\hat{Z}^{(D)}(\text{gen}(x); s_1, s_2) = \zeta_E^{(D)}(s_1)\zeta_E^{(D)}(s_2) Z_P^{(D)}(\text{gen}(x); s_1, s_2).$$

Applying Corollary 3.1 and setting

$$\mu(s_1, s_2) = \left(\frac{(n-1)s_1 + s_2}{n} - \frac{n-1}{2}, \frac{s_2 - s_1}{n}, \frac{n-1}{2} - \frac{s_1 + (n-1)s_2}{n} \right)$$

we get that whenever $x_l \in K_l \cdot e$ for the prime $l | \Delta_E$ we have

$$\begin{aligned}
\hat{Z}^{(\Delta_E)}(\text{gen}(x); s_1, s_2) = {} & w_E^{-n} \frac{\zeta_E^*(2)\zeta_E^*(3)\cdots\zeta_E^*(n-2)}{(\zeta_E^*)_{-1}^{n-3}} \det x^{-\frac{(n-1)s_1+s_2}{n}} \\
& \times \left(\frac{\zeta_{-1}^{(\Delta_E)}}{L^{(\Delta_E)}(\eta, 2)} \right)^{n-3} \prod_{k=2}^{n-3} \left(\frac{L^{(\Delta_E)}(\eta^{k+1}, k)}{L^{(\Delta_E)}(\eta^k, k+1)} \right)^{n-(k+2)} \\
& \times \zeta_E^{(\Delta_E)}(s_1)\zeta_E^{(\Delta_E)}(s_2) \\
& \times \prod_{p \nmid \Delta_E} P_{m(x_p)}(\nu_0; \mu(s_1, s_2) + \Lambda(P)) \\
& \times \frac{L^{(\Delta_E)}(\eta^n, s_1 + s_2 + 1 - n)}{L^{(\Delta_E)}(\eta^{n+1}, s_1 + s_2 + 2 - n)} \\
& \times \prod_{i=2}^{n-1} \frac{L^{(\Delta_E)}(\eta^{i+n+1}, s_2 + 1 - i)}{L^{(\Delta_E)}(\eta^{i+n}, s_2 + 2 - i)} \\
& \times \frac{L^{(\Delta_E)}(\eta^i, s_1 + i - n)}{L^{(\Delta_E)}(\eta^{i+1}, s_1 + i + 1 - n)}.
\end{aligned} \tag{4.2}$$

Assume now that $n=3$. We apply this formula to obtain an explicit expression for $r(e;m_1,m_2)$. We have

$$\sum_{(m_1m_2,\Delta_E)=1}\frac{r(\mathrm{gen}(e);m_1,m_2)}{m_1^{s_1}m_2^{s_2}}=w_E^{-1}\zeta^{(\Delta_E)}(s_1-1)\zeta^{(\Delta_E)}(s_1)\zeta^{(\Delta_E)}(s_2-1)\zeta^{(\Delta_E)}(s_2)$$
$$\times\frac{L^{(\Delta_E)}(\eta,s_1+s_2-2)}{\zeta^{(\Delta_E)}(s_1s_1+s_2-1)}.$$

We expand the right hand side as a Dirichlet series and equate coefficients with the Dirichlet series on the left hand side. Doing this, we find that whenever $\gcd(m_1m_2,\Delta_E)=1$,

$$r(\mathrm{gen}(e);m_1,m_2)=w_E^{-1}\sum_{d|gcd(m_1,m_2)}d\sigma_1\left(\frac{m_1}{d}\right)\sigma_1\left(\frac{m_2}{d}\right)\phi_\eta(d)$$

where

$$\phi_\eta(d)=\sum_{d_0|d}\mu(d/d_0)\eta(d_0)d_0=d\prod_{p|d}\left(1-\frac{\eta(p)}{p}\right)$$

is a twisted Euler function and $\sigma_1(d)=\sum_{d_0|d}d_0$.

If the field E is such that $[[e]]=[e]$ (as is the case for example if $E=\mathbb{Q}(\sqrt{-1})$ or $E=\mathbb{Q}(\sqrt{-3})$) then we obtain explicitly the representation number $r(e;m_1,m_2)$. It is easy to see that $\mathcal{E}(e)$ consists of scaled permutation matrices with unit scales and therefore that $\epsilon(e)=6w_E^3$. It follows that if E is a field of class number one for which the genus class of the identity consists of a unique class, then whenever m_1 and m_2 are relatively prime to the discriminant of E, the number $r(e;m_1,m_2)$ of pairs of orthogonal, $\mathcal{O}$-integral vectors lying on the complex 3-dimensional spheres of radius $\sqrt{m_1}$ and $\sqrt{m_2}$, respectively, is

$$6w_E^2\sum_{d|gcd(m_1,m_2)}d\sigma_1\left(\frac{m_1}{d}\right)\sigma_1\left(\frac{m_2}{d}\right)\phi_\eta(d).$$

For $E=\mathbb{Q}(\sqrt{-1})$ and m_1m_2 odd, the number

$$96\sum_{d|gcd(m_1,m_2)}d\sigma_1\frac{m_1}{d}\sigma_1\frac{m_2}{d}\phi_\eta(d)$$

counts the pairs of 6-tuples $(a_1, a_2, \dots, a_6), (b_1, b_2, \dots, b_6) \in \mathbb{Z}^6$ satisfying the equations

$$\begin{aligned} a_1^2 + a_2^2 + \cdots + a_6^2 &= m_1 \\ b_1^2 + b_2^2 + \cdots + b_6^2 &= m_2 \\ a_1 b_1 + a_2 b_2 + \cdots + a_6 b_6 &= 0 \\ a_1 b_2 - b_1 a_2 + a_3 b_4 - a_4 b_3 + a_5 b_6 - a_6 b_5 &= 0. \end{aligned}$$

4.3 The case of GL_4 and the Borel

Assume here that $n = 4$. In this section we give an explicit description of the incidence relations and representation numbers arising from the minimal parabolic Eisenstein series. Our description of the incidence relations is taken from [1].

Given a 4×4 matrix g and a subset S of $\{1, 2, 3, 4\}$ with r elements, we let $A_S(g) = \det g^{(S)}$ be the minor of the matrix obtained by taking the bottom r rows of g and the columns indexed by the elements of S. Then the Plücker coordinates $v_i(g)$ are given by

$$\begin{aligned} v_1 &= {}^t(A_1, A_2, A_3, A_4) \\ v_2 &= {}^t(A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}) \\ v_3 &= {}^t(A_{123}, A_{124}, A_{134}, A_{234}). \end{aligned}$$

These coordinates satisfy the following incidence relations:

$$\begin{pmatrix} 0 & -A_{34} & A_{24} & -A_{23} \\ A_{34} & 0 & -A_{14} & A_{13} \\ -A_{24} & A_{14} & 0 & -A_{12} \\ A_{23} & -A_{13} & A_{12} & 0 \end{pmatrix} v_1 = 0 \tag{4.3}$$

$$\begin{pmatrix} 0 & -A_{12} & A_{13} & -A_{14} \\ A_{12} & 0 & -A_{23} & A_{24} \\ -A_{13} & A_{23} & 0 & -A_{34} \\ A_{14} & -A_{24} & A_{34} & 0 \end{pmatrix} v_3 = 0 \tag{4.4}$$

$$A_1 A_{234} - A_2 A_{134} + A_3 A_{124} - A_4 A_{123} = 0 \tag{4.5}$$

$$A_{12} A_{34} - A_{13} A_{24} + A_{14} A_{23} = 0 \tag{4.6}$$

Furthermore, for $g \in G_{\mathcal{O}}$ the vectors v_i are obviously *primitive:*

$$\begin{aligned} gcd(A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}) &= gcd(A_1, A_2, A_3, A_4) \\ &= gcd(A_{123}, A_{124}, A_{134}, A_{234}) = 1. \end{aligned} \tag{4.7}$$

Conversely, we have the following result.

Theorem 4.1 [1] If $(v_1, v_2, v_3) \in \mathcal{O}^4 \times \mathcal{O}^6 \times \mathcal{O}^4$ satisfies (4.3), (4.4) and (4.7), then $(v_1, v_2, v_3) \in \mathcal{I}(B, \mathcal{O})$. In particular, (4.5) and (4.6) are automatically satisfied.

This allows us to be explicit about the representation numbers arising from the $GL_4(\mathcal{O})$ minimal parabolic Eisenstein series. For $x \in X_{\mathbb{Q}}$ such that Q_x is integral, we have

$$\begin{aligned} r_B(x; j, k, l) = \#\{(v_1, v_2, v_3) \in \mathcal{O}^4 \times \mathcal{O}^6 \times \mathcal{O}^4 : (4.3), (4.4), (4.7) \\ \text{are satisfied and } Q_x(v_1) = l,\, Q_{\wedge^2 x}(v_2) = k,\, Q_{\wedge^3 x}(v_3) = j\}. \end{aligned} \tag{4.8}$$

When E is equal to the field of discriminant -4 or -3, the 4×4 identity matrix e is the only class in its genus [5,12]. Therefore in these cases we have

$$r_B(e; j, k, l) = 24 w_E^4 r_B(\mathrm{gen}(e); j, k, l)$$

and using Corollary 3.1 we get

$$\begin{aligned} &Z_B^{(\Delta_E)}(e; s_1, s_2, s_3) \\ &= 24 \left[\frac{\zeta^{(\Delta_E)}(s_1 - 2)}{L^{(\Delta_E)}(\eta, s_1 - 1)} \frac{L^{(\Delta_E)}(\eta, s_1 + s_2 - 1)}{\zeta^{(\Delta_E)}(s_1 + s_2)} \frac{\zeta^{(\Delta_E)}(s_1 + s_2 + s_3)}{L^{(\Delta_E)}(\eta, s_1 + s_2 + s_3 + 1)} \right. \\ &\quad \left. \times \frac{\zeta^{(\Delta_E)}(s_2 + 1)}{L^{(\Delta_E)}(\eta, s_2 + 2)} \frac{L^{(\Delta_E)}(\eta, s_2 + s_3 + 2)}{\zeta^{(\Delta_E)}(s_2 + s_3 + 3)} \frac{\zeta^{(\Delta_E)}(s_3 + 1)}{L^{(\Delta_E)}(\eta, s_3 + 2)} \right]. \end{aligned} \tag{4.9}$$

Expanding out the Dirichlet series on the right hand side will give an expression for $r_B(e; j, k, l)$ when $\gcd(jkl, \Delta_E) = 1$ in terms of divisor sums involving the Möbius function and the character η.

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