A remark on the fundamental lemma of Jacquet

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Received *****; accepted after revision +++++

Abstract

We conjecture a generalization of the fundamental lemma of Jacquet in the context of GL_n over a quadratic extension. We provide a heuristic argument for our expectation and prove our conjecture for GL_2.

Résumé

Une remarque sur le lemme fondamental de Jacquet. Nous conjecturons une généralisation du lemme fondamental de Jacquet dans le contexte de GL_n sur une extension quadratique. Nous fournissons un argument heuristique pour notre attente et prouve la conjecture pour GL_2.

In [Jac04] and [Jac05] Jacquet obtained remarkable local identities between orbital integrals for GL_n with respect to a quadratic extension. In this work, we conjecture that these identities hold in a slightly more general setting and prove our conjecture when $n = 2$. Our expectations for matching functions are based on a spherical Fourier transform for symmetric spaces. We hope that this will shed some light on the right framework for the fundamental lemma in the context of Jacquet's relative trace formula.

Let E/F be an unramified quadratic extension of non-archimedean local fields with odd residual characteristic. Denote by $x \mapsto \bar{x}$ the associated Galois action and let ψ denote a character of F with conductor the ring of integers $\mathcal{O}_F$. We denote by $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ the associated character of E. Let $\mathfrak{p}_F$ denote the prime ideal in $\mathcal{O}_F$, ϖ a uniformizer for F and $q = \#(\mathcal{O}_F/\mathfrak{p}_F)$. Denote by $|\cdot|$ the standard absolute value on F so that $|\varpi| = q^{-1}$. Let $G = GL_n(E)$, $G' = GL_n(F)$, U (resp. U') the subgroup of upper triangular unipotent matrices in G (resp. G') and K (resp. K') the standard maximal compact of G (resp. G'). To ψ_E we associate the character $\psi_U(u) = \psi_E(u_{1,2} + \cdots + u_{n-1,n})$ on U and similarly we let $\psi_{U'}$ be the character of U' associated with ψ. Let $S = \{g \in G : {}^t\bar{g} = g\}$ be the space of hermitian matrices in G and denote by $g \cdot s = gs{}^t\bar{g}$ the action of G on S. For functions $\Phi \in C_c^\infty(G')$ and $\Psi \in C_c^\infty(S)$ and for a diagonal matrix $a = \mathrm{diag}(a_1, \ldots, a_n)$ in G' we define the orbital integrals

$$\Omega[\Phi, \psi; a] = \int_{U'}\int_{U'} \Phi({}^t u_1 a u_2)\psi_{U'}(u_1 u_2)du_1\, du_2 \text{ and } \Omega[\Psi, \psi; a] = \int_{U} \Psi({}^t\bar{u} a u)\psi_U(u)du.$$

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Haar measures are normalized as in [Jac05]. Let $\omega = \omega_{E/F}$ be the quadratic character of $F^\times$ associated to E/F by class field theory. For $\delta \in \{0,1\}$ we define the transfer factor $\gamma_\delta(a) = \omega^{\delta+1}(a_1)\omega^{\delta+2}(a_2)\cdots\omega^{\delta+n}(a_n)$. We say that Φ and Ψ δ-match and write $\Phi \overset{\delta}{\leftrightarrow} \Psi$ if $\Omega[\Phi,\psi;a] = \gamma_\delta(a)\Omega[\Psi,\psi;a]$ for all a.

Let $\mathcal{H}_G$ (resp. $\mathcal{H}_{G'}$) denote the convolution Hecke algebra of bi-K (resp. K')-invariant, compactly supported functions on G (resp. G') and denote by $\mathcal{H}_S$ the space of K-invariant, compactly supported functions on S. It is an $\mathcal{H}_G$-module under the convolution $f * \Psi(s) = \int_G f(g)\Psi(g^{-1}\cdot s)ds$. The standard theory of zonal spherical functions and the spherical Fourier transform identifies $\mathcal{H}_{G'}$ with the ring of symmetric Laurant polynomials in n variables. If ω_x denotes the spherical function with complex parameter $x \in (\mathbb{C}^\times)^n$ then the map

$$\Phi \mapsto \hat{\Phi}(x) = \int\limits_{G'} \Phi(g)\omega_x(g^{-1})dg$$

is an isomorphism from $\mathcal{H}_{G'}$ to the algebra $\mathbb{S}'_n$ of symmetric elements in $\mathbb{C}[x_1,\dots,x_n,(x_1\cdots x_n)^{-1}]$. Similarly, under the spherical Fourier transform $f \mapsto \hat{f}$ the Hecke algebra $\mathcal{H}_G$ is identifies with the algebra $\mathbb{S}_n$ of symmetric elements in $\mathbb{C}[x_1^2,\dots,x_n^2,(x_1^2\cdots x_n^2)^{-1}]$. The embedding $\mathbb{S}_n \hookrightarrow \mathbb{S}'_n$ defines a monomorphism of $\mathcal{H}_G$ into $\mathcal{H}_{G'}$ (usually referred to as quadratic base-change) and we may view $\mathcal{H}_{G'}$ as an $\mathcal{H}_G$-module of rank 2^n. We denote by $f * \Phi$ the associated action of $\mathcal{H}_G$ on $\mathcal{H}_{G'}$. In [Hir99], Y. Hironaka accomplished her project of a theory of spherical functions and a spherical Fourier transform for S. Denote by Ω_x Hironaka's spherical function with parameter x on S and let

$$\Psi \mapsto \hat{\Psi}(x) = \int\limits_{S} \Psi(s)\Omega_x(s^{-1})ds$$

be the spherical Fourier transform on $\mathcal{H}_S$. It is an isomorphism of $\mathcal{H}_G$-modules from $\mathcal{H}_S$ to $\mathbb{S}'_n$ (Theorem 2, [Hir99]). We therefore obtain an isomorphism of $\mathcal{H}_G$-modules between $\mathcal{H}_S$ and $\mathcal{H}_{G'}$. The function Ψ corresponds to Φ whenever $\hat{\Psi} = \hat{\Phi}$.

Conjecture 1 There exist a $\delta = \delta(n) \in \{0,1\}$ depending only on n, such that $\Phi \overset{\delta}{\leftrightarrow} \Psi$ whenever $\Phi \in \mathcal{H}_{G'}$ and $\Psi \in \mathcal{H}_S$ are such that $\hat{\Phi} = \hat{\Psi}$.

Let $\Phi_0 \in \mathcal{H}_{G'}$ be the characteristic function of K' and let $\Psi_0 \in \mathcal{H}_S$ be the characteristic function of $S \cap K$. When $\Phi \in \mathcal{H}_G * \Phi_0$ and $\Psi \in \mathcal{H}_G * \Psi_0$ the matching was already conjectured in [JY90] and proved by Jacquet in [Jac05]. Note that $\Phi \overset{\delta}{\leftrightarrow} \Psi$ if and only if $\Phi \overset{1-\delta}{\leftrightarrow} \Psi(\omega \circ \det)$ and that for $\Psi \in \mathcal{H}_G * \Psi_0$ the support of Ψ is contained in the kernel of $\omega \circ \det$. For this reason Jacquet's matching works for either δ in $\{0,1\}$. The more general matching we suggest here will determine the transfer factor γ_δ uniquely.

The conjecture is motivated by certain spectral identities obtained in [Off]. Roughly speaking, the orbital integrals we consider here are the local components of global distributions that appear on the geometric expansion of the relevant relative trace formula identity. We therefore refer to the above δ-matching as a geometric matching. Similarly, a spectral matching will refer to an identity between the local components of the distributions on the spectral expansion of the trace identity. Using the spherical Fourier transform on S, it can be shown that Φ and Ψ as in Conjecture 1 match spectrally (by this we mean that Proposition 5 of [Off] holds for Φ and Ψ). The more general spectral identities obtained in [loc. cit.] make use of the geometric matching of Φ_0 and Ψ_0. It will be interesting to see if a geometric matching can be derived from a spectral one. This may require a local relative trace identity. In any case, our conjecture would refine the results of [Off]. More specifically, it would determine the ambiguity in defining the transfer factor, the δ in Theorem 2 and Theorem 3 of [loc. cit.] would become $\delta(n)$ of Conjecture 1. In particular $\delta(2) = 1$ is determined in what follows.

From now on set $n = 2$. In this case we perform direct computations. They require no new ideas and are done merely for the sake of verifying our conjecture in the simplest non-trivial case. We do not expect the straight forward approach to generalize and we will spear the reader from the computations. We only bring here the results. We do hope however, that the beautiful and sophisticated machinery developed by Jacquet will suffice to prove our conjecture for a general n.

We introduce different sets of basis to $\mathbb{S}_2'$. Denote by $\sigma_{(m,k)}(x,y)$ the monomial symmetric polynomial $x^m y^k + x^k y^m$ if $m > k$ and $x^k y^k$ if $m = k$. For a variable t let $P_{m,k}(x,y;t)$ be the Hall-Littlewood polynomial. We have the following relation

$$P_{(m,k)}(x,y;t) = \sigma_{(m,k)}(x,y) + (1-t) \sum_{i=1}^{\lfloor \frac{m-k-1}{2} \rfloor} \sigma_{m-i,k+i}(x,y).$$

Inverting this triangular relation we also find that

$$\sigma_{(m,k)}(x,y) = P_{(m,k)}(x,y;t) - (1-t) \sum_{i=1}^{\lfloor \frac{m-k-1}{2} \rfloor} t^{i-1} P_{(m-i,k+i)}(x,y;t).$$

Thus, for every $t \in \mathbb{C}$ the set $\{P_{(m,k)}(x,y;t) : m \geq k\}$ forms a basis of $\mathbb{S}_2'$. Let $c_{(m,k)} \in \mathcal{H}_{G'}$ be the characteristic function of $K' \operatorname{diag}(\varpi^m, \varpi^k) K'$ and let $C_{(m,k)} \in \mathcal{H}_S$ be the characteristic function of $K \cdot \operatorname{diag}(\varpi^m, \varpi^k)$. Formulas for the spherical Fourier transform on $\mathcal{H}_{G'}$ can be found in ([Mac95], p. 299) and on $\mathcal{H}_S$ in ([Hir99], p. 571) (in fact for $n = 2$ Hironaka already carries out this computation in [Hir89] but we apply here her formulas as normalized in [Hir99]). We have

$$\hat{c}_{(m,k)}(x,y) = q^{\frac{m-k}{2}} P_{(m,k)}(x,y;q^{-1}) \text{ and } \hat{C}_{(m,k)}(x,y) = (-1)^m q^{\frac{m-k}{2}} P_{(m,k)}(x,y;-q^{-1}).$$

We set

$$\Phi_{(m,k)} = c_{(m,k)} - (q-1) \sum_{i=1}^{\lfloor \frac{m-k-1}{2} \rfloor} c_{(m-i,k+i)} \, ; \, \Psi_{(m,k)} = (-1)^m \left[C_{(m,k)} + (q+1) \sum_{i=1}^{\lfloor \frac{m-k-1}{2} \rfloor} C_{(m-i,k+i)} \right].$$

Thus, $\hat{\Phi}_{(m,k)} = \hat{\Psi}_{(m,k)} = q^{\frac{m-n}{2}} \sigma_{(m,k)}$ and $\{\Phi_{(m,k)} : m \geq k\}$ (resp. $\{\Psi_{(m,k)} : m \geq k\}$) forms a basis of $\mathcal{H}_{G'}$ (resp. $\mathcal{H}_S$). Conjecture 1 for $n = 2$ follows from the following.

Theorem 0.1 $\quad \Phi_{(m,k)} \overset{1}{\leftrightarrow} \Psi_{(m,k)}$.

As already mentioned, this matching is proved by straight forward computation. We set

$$\omega_{(m,k)}(a,b) = \Omega[c_{(m,k)}, \psi; \operatorname{diag}(a,b)] \text{ and } \Omega_{(m,k)}(a,b) = \Omega[C_{(m,k)}, \psi; \operatorname{diag}(a,b)].$$

Crucial to the computation of $\omega_{(m,k)}(a,b)$ is the integral $\xi(i,v)$ and to the computation of $\Omega_{(m,k)}(a,b)$ the integral $\eta(i,v)$ for $i > 0$ and $v \in F$ where

$$\xi(i,v) = \int_{x \in F, \, |x| = q^i} \psi(x - \frac{v}{x}) \, dx \text{ and } \eta(i,v) = \int_{x \in E, \, x\bar{x} \in v(1+\mathfrak{p}_F^i)} \psi_E(x) \, dx.$$

Let $|v| = q^m$. We use the fact that if $m > i$ and $m \neq 2i$ then $\xi(i,v) = 0$, that can be proved by the methods of ([Jac87], p. 144-146), and the fact that if $m > 2i$ then $\eta(i,v) = 0$ that can be proved by similar methods. Denote by $\phi^{(m,k)}$ the characteristic function of the set $\{(a,b) \in F^2 : |a| = q^{-m}, \, |b| = q^{-k}\}$. We obtain the following formulas (here $m - k \geq 3$):

$$\omega_{(k,k)}(a,b) = \phi^{(k,k)}(a,b) + \sum_{i=1}^{\infty} \xi(i, -\frac{b}{a}) \phi^{(k+i,n-i)}$$

$$\omega_{(k+1,k)}(a,b) = \phi^{(k,k+1)}(a,b) - \phi^{(k+1,k)}(a,b)$$

$$\omega_{(k+2,k)}(a,b) = \phi^{(k,k+2)}(a,b) - 2\phi^{(k+1,k+1)}(a,b) - \sum_{i=0}^{\infty} \xi(i+1,-\frac{b}{a})\phi^{(k+2+i,k-i)}$$

$$\omega_{(m,k)}(a,b) = \phi^{(k,m)}(a,b) - 2\Phi^{(k+1,m-1)}(a,b) + \phi^{(k+2,m-2)}(a,b).$$

$$\Omega_{(k,k)}(a,b) = \phi^{(k,k)}(a,b) + \sum_{i=1}^{\infty} \eta(i,-\frac{b}{a})\phi^{(k+i,k-i)}(a,b)$$

$$\Omega_{(k+1,k)}(a,b) = \phi^{(k+1,n)}(a,b) + \phi^{(k,k+1)}(a,b)$$

$$\Omega_{(k+2,k)}(a,b) = \phi^{(k,k+2)}(a,b) - \sum_{i=0}^{\infty} \left[\eta(i+1,-\frac{b}{a})\right] \Phi^{(k+2+i,k-i)}(a,b)$$

$$\Omega_{(m,k)}(a,b) = \phi^{(k,m)}(a,b) - \phi^{(k+2,m-2)}(a,b).$$

To obtain Theorem 0.1 we note that Jacquet's matching $\Phi_0 \overset{1}{\leftrightarrow} \Psi_0$ implies that $\xi(i,v) = (-1)^i \eta(i,v)$ whenever $|v| = q^{2i} > 1$. The rest is elementary and requires carefully plugging in our formulas for all $m \geq k$ to verify that $\Omega[\Phi_{(m,k)}, \psi; \mathrm{diag}(a,b)] = \omega(b)\Omega[\Psi_{(m,k)}, \psi; \mathrm{diag}(a,b)]$.

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