# A METAPLECTIC CASSELMAN-SHALIKA FORMULA FOR $GL_r$

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Abstract. We provide formulas for various bases of spherical Whittaker functions on the $n$-fold metaplectic cover of $GL_r$ over a $p$-adic field and show that there is a basis of symmetric functions in the complex parameter. In addition we relate a specific spherical Whittaker function to the $p$-power part of the Weyl group multiple Dirichlet series for the root system of type $A_{r-1}$ constructed from $n$th order Gauss sums. We also show that the zonal spherical functions can be computed explicitly in terms of Hall-Littlewood polynomials as in Macdonald's formula for $GL_r$.

## Contents



## 1. Introduction

In this work we give formulas for the spherical Whittaker functions on the $n$-fold metaplectic cover of $GL_r$ over a $p$-adic field. Our formulas generalize the formula of Shintani and Casselman-Shalika which deals with the nonmetaplectic case ($n = 1$). Further we give an explicit relationship between $p$-adic metaplectic Whittaker functions and the local parts of Weyl group multiple Dirichlet series associated to root systems of type $A_{r-1}$ constructed by Chinta and Gunnells [13]. Weyl group multiple Dirchlet series, first introduced in [4],

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are Dirichlet series in several complex variables whose coefficients are built out of Gauss sums and the $n$th order power residue symbol.

Recently, Brubaker, Bump and Friedberg [6, 7] have given an alternative construction of Weyl group multiple Dirichlet series of type $A_{r-1}$, thereby confirming a conjecture made in [5]. Moreover, in [6], they have shown that their series coincide with the global Whittaker function of an Eisenstein series on the $n$-fold cover of $GL_r$ over a number field containing the $2n$-th roots of unity. The constructions of [6] and [7] involve a combinatorial description in terms of crystal graphs, while the construction of [13], generalizing an approach introduced in [12], involves averaging with respect to a certain Weyl group action. For this reason, the two approaches have not yet been shown to produce the same multiple Dirichlet series except in certain very special cases, see e.g. [11]. The results of the present work combined with the work of P. McNamara [23] lead to the resolution of this problem, as we now describe.

Using methods completely different from ours, McNamara also gives a formula for the $p$-adic metaplectic Whittaker functions on the $n$-fold cover of $GL_r$ (in fact of $SL_r$ but the two are comparable). As mentioned above, we use our formula to show that (a suitably chosen) spherical Whittaker function coincides with the $p$-part of the Weyl group multiple Dirichlet series constructed in Chinta-Gunnells [13]. This is the content of Theorem 4 in Section 9. On the other hand, McNamara's formula for the spherical Whittaker function shows that it coincides with the local part of the series constructed by Brubaker-Bump-Friedberg in [6]. Consequently, the approaches of Chinta-Gunnells and Brubaker-Bump-Friedberg do in fact produce the same series.

We now sketch the methods used in the proof of our generalization of the formula of Shintani and Casselman-Shalika. Let $n$ be a positive integer and let $F$ be a non-archimedean local field that contains a primitive $n$th root of unity. For every $c \in \mathbb{Z}/n\mathbb{Z}$ Kazhdan and Patterson associated in [17] the $c$-twisted $n$-fold metaplectic cover $\widetilde{GL}_r(F)^{(c)}$ of $GL_r(F)$. It is a central extension of $GL_r(F)$ by the group $\mu_n(F)$ of $n$th roots of unity in $F$. It is not, in general, the group of $F$ points of an algebraic group defined over $F$ but it is an $\ell$-group in the sense of [3]. We assume throughout the paper that $n$ is relatively prime to the residual characteristic of $F$. In a global setting over a number field this assumption is satisfied at almost all places.

Zonal spherical functions for a $p$-adic reductive group were computed explicitly by Macdonald [20]. In [9], Casselman reproved Macdonald's formulas using the theory of unramified principal series representations. This point of view was taken further by Casselman and Shalika in [10] where they explicitly compute the spherical Whittaker functions for a $p$-adic reductive group, generalizing Shintani's formula for $GL_r$ [30]. The method of Casselman and Shalika has since been applied in many cases to compute spherical functions on $p$-adic symmetric spaces or more generally on spherical varieties (e.g. [15], [25], [28]). See also [8], which extends the method to the metaplectic double cover of $Sp_{2r}(F)$.

The main result of this work, specialized to the case $n = 1$, recovers the Shintani, Casselman-Shalika formula for the spherical Whittaker functions of $GL_r(F)$ in terms of the symmetric Schur polynomials. For general $n$, a central difficulty is the failure of uniqueness of Whittaker functionals (and therefore of spherical Whittaker functions of a fixed Hecke eigenvalue). In [14], Y. Hironaka computed explicitly the spherical functions on the space of non-singular Hermitian matrices with respect to an unramified quadratic extension of $p$-adic fields. This is a case where multiplicity one fails. Hironaka's approach to the Casselman-Shalika method in case of multiplicities (see §1 of [ibid.]) is our guideline for this work. (See [26] for an earlier treatment of multiplicities by different methods.)

Roughly speaking, a spherical function can be expressed as the value of a certain linear form applied to translates of the unramified vector in an unramified principal series representation. The idea behind the Casselman-Shalika method is to reduce the computation for the value of the linear form on a translate of the element invariant under the maximal compact subgroup to the computation of simpler expressions for elements invariant under a smaller open compact — the Iwahori subgroup. There are three main steps in carrying out the method.

The first has to do solely with the group and not with the particular linear form we consider. It is an expansion of the unramified element of a principal series representation in terms of a "well-chosen" basis of the Iwahori invariant subspace — the Casselman-Shalika basis. In [27], Sakellaridis provides a formula for spherical functions in the general setting of spherical varieties for a split reductive group (this, however, does not contain our case as long as $n > 1$). His characterization of the Casselman-Shalika basis simplifies the computation. In Subsection 3.3 below we take this approach and construct the Casselman-Shalika basis for the unramified principal series of $\widetilde{G}L_r^{(c)}(F)$.

The second step is to obtain Weyl group functional equations between the spherical functions. The unramified principal series representations are parameterized by a variable, say $s$, in some complex variety on which a related Weyl group acts. Crucial to the computation of the spherical functions is to relate explicitly between the spherical functions associated to $s$ and those associated to $ws$ for any Weyl element $w$. When $n = 1$, the space of Whittaker functionals of an unramified principal series representation is one dimensional. There is then a one dimensional space of spherical Whittaker functions for a given parameter $s$ (i.e. for a fixed Hecke eigenvalue) and the functional equations are therefore scalar valued. For general $n$ the space of Whittaker functionals for an unramified principal series representation of $\widetilde{G}L_r^{(c)}(F)$ is of dimension $\frac{n^r}{\gcd(n,2rc+r-1)}$. This complicates the computation of the functional equations. Once a basis of Whittaker functionals for any parameter $s$ has been fixed, there is a matrix associated to any Weyl element $w$ that expresses the basis for $ws$ in terms of the basis for $s$. Such a functional equation is provided in [17, Lemma 1.3.3]. We present it again in Section 5 below, correcting some minor errors already pointed out in [1]. It is further pointed out in [ibid.] that in order to justify the computation of Kazhdan-Patterson one has to choose the metaplectic group defined by a *block compatible* 2-cocycle of $GL_r(F)$ as constructed in [2].

The third step is the evaluation of the linear forms on translates of the Iwahori invariant functions in the Casselman-Shalika basis. This, in our case, is not much more complicated than in the $n = 1$ case and is the content of Lemma 7. This lemma leads in turn to Theorems 1 and 2 (at the ends of Sections 4 and 6, resp.) which give two different expressions for the spherical Whittaker functions as a sum over the Weyl group.

Once Theorem 2 has been established, it is a simple matter to relate the spherical Whittaker functions to the local parts of the type $A$ Weyl group multiple Dirichlet series constructed by Chinta and Gunnells. After a short preparation in Section 8 this is done in Section 9. Along the way, in Section 7 we show that a basis of spherical Whittaker functions can be chosen, so that their values at each point of $\widetilde{G}L_r^{(c)}$ are symmetric functions of the complex variable $s$. The Chinta-Gunnells local component is not symmetric. Nevertheless, the functional equation satisfied by Eisenstein series suggests that such a basis may play a role in the global theory. Finally in Section 10 we show that the zonal spherical functions can be computed explicitly in terms of Hall-Littlewood polynomials as in Macdonald's formula for $GL_r$. Though zonal spherical functions are not the main object of study in this paper, we include this short section as a further application of the utility of the Casselman-Shalika basis computed in Subsection 3.3.

We now provide a brief description of McNamara's work [23]. McNamara directly computes the spherical Whittaker function as an integral of the spherical vector $\phi_K$ in the principal series representation over the unipotent group $U$. He decomposes $U$ into cells on which $\phi_K$ is constant. Remarkably, he shows a bijection between this collection of cells and elements of a canonical crystal basis for the quantized universal enveloping algebra $U_q(\mathfrak{g})$, which then allows him to reproduce the Gelfand-Tsetlin description of Brubaker-Bump-Friedberg [6] for the $p$-part of a type $A$ Weyl group multiple Dirichlet series. Equating our formula for the spherical Whittaker function (given in Theorem 2) with his produces a purely combinatorial identity: a sum over a Weyl group equals a sum over a crystal basis. It is striking that to date the only means of proving this identity is via the theory of Whittaker functions on the metaplectic group $\widetilde{G}L_r^{(c)}(F)$. It would be desirable to have a more direct combinatorial proof of this identity. The insight gained by such a proof may allow us to generalize the crystal basis description to other root systems.

Finally we remark that in an upcoming work McNamara generalized our work to the setting of a metaplectic covering of any unramified reductive $p$-adic group [21]. The Kazhdan-Patterson theory of principal series representations and intertwining operators for the metaplectic cover of $GL_r$ plays a key role in our current work. This theory has recently been extended to metaplectic coverings of reductive groups of general type by Savin [29], Loke-Savin [19] and McNamara [22, 21].

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## 2. Notation and preliminaries

Let $F$ denote a non archimedean local field, $\mathcal{O}$ the ring of integers of $F$, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, $q$ the size of the residue field $\mathcal{O}/\mathfrak{p}$ of $F$ and $\varpi$ a uniformizer in $\mathfrak{p}$. Let $|\cdot|_F$ be the normalized absolute value on $F$ and let $v : F \to \mathbb{Z} \cup \{\infty\}$ be the valuation on $F$ such that $|x|_F = q^{-v(x)}$. Denote by $F^\times$ the multiplicative group of $F$ and for every integer $n$ let

$$F^{\times n} = \{x^n : x \in F^\times\}.$$

Fix a nontrivial character $\psi$ of $F$ and let $dx$ be the self-dual Haar measure on $F$ with respect to $\psi$. We shall assume throughout that $\psi$ has conductor $\mathcal{O}$. Thus $\mathrm{vol}(\mathcal{O}) = 1$. Fix a positive integer $r$ and let $G = GL_r(F)$. The Iwasawa decomposition gives $G = BK = AUK$ where $U$ is the group of upper triangular unipotent matrices in $G$, $A$ the group of diagonal matrices in $G$, $B = AU$ the standard Borel subgroup of upper triangular matrices in $G$ and $K = GL_r(\mathcal{O})$ the standard maximal compact subgroup of $G$. We denote by $\psi_U$ the character of $U$ defined by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}),\ u = (u_{i,j}) \in U.$$

Denote by $\mathfrak{W}$ the group of permutation matrices in $G$ and identify it with the Weyl group of $G$. We will also identify $\mathfrak{W}$ with the group of permutations of $\{1, \ldots, r\}$ via

$$w = (\delta_{i,w(j)}).$$

Let

$$\Phi = \{(i, j) : 1 \le i,\ j \le r,\ \ i \ne j\}$$

be the root system associated with $G$. It is a root system of type $A_{r-1}$ with Weyl group $\mathfrak{W}$. The action of $\mathfrak{W}$ on $\Phi$ is given by $w(i, j) = (w(i), w(j))$. Let

$$\Delta = \{(i, i + 1) : i = 1, \ldots, r - 1\}$$

be the set of simple roots with respect to $B$. We will sometimes find it convenient to think of the roots as embedded in an $r$-dimensional lattice $\Lambda$ which we identify with $\mathbb{Z}^r$. Under this embedding the simple root $(i, j)$ corresponds to the vector $e_i - e_j \in \Lambda$, where the $e_i$ are the standard basis vectors in $\Lambda$. The action of $\mathfrak{W}$ on $\Lambda$ via permutation matrices is compatible with this embedding and the action of $\mathfrak{W}$ on $\Phi$.

If $i < j$ the root $\alpha = (i, j)$ is positive and we write $\alpha > 0$. Otherwise we write $\alpha < 0$. Let

$$\Phi^-(w) = \{\alpha > 0 : w^{-1}\alpha < 0\}$$

and let $\ell(w) = |\Phi^-(w)|$ be the length of the permutation $w$. If $w_\alpha$ denotes the simple reflection associated to $\alpha \in \Delta$ then

$$\ell(w_\alpha w) = \begin{cases} \ell(w) + 1 & \alpha \notin \Phi^-(w) \\ \ell(w) - 1 & \alpha \in \Phi^-(w). \end{cases}$$

Denote by $w_0 \in \mathfrak{W}$ the longest Weyl element, i.e. the unique $w \in \mathfrak{W}$ that takes all positive roots to negative roots. For every root $\alpha \in \Phi$ let $u_\alpha : F \to G$ be the associated one parameter subgroup and let $U_\alpha$ be its image. If $\alpha > 0$ then $U_\alpha$ is a subgroup of $U$. Otherwise $U_\alpha$ is a subgroup of the group $\overline{U}$ of lower triangular unipotent matrices. For every $w \in \mathfrak{W}$ we denote by $U_w$ the group generated by $U_\alpha$, $\alpha \in \Phi^-(w)$. Then, the imbedding of $U_w$ in $U$ defines a bijection

$$U_w \simeq (U \cap wUw^{-1})\backslash U.$$

The Haar measure on $U_\alpha$ will be taken according to the isomorphism with $F$. On $U_w$ (and in particular on $U = U_{w_0}$) we will use accordingly the product Haar measure. It is normalized by the requirement that

$$\mathrm{vol}(U_w \cap K) = 1.$$

2.1. **The metaplectic $n$-fold covers of $GL_r$.** Fix a positive integer $n$ and let

$$\mu_n(k) = \{x \in k : x^n = 1\}$$

be the group of $n$th roots of unity in a field $k$. Assume from now on that $F$ is such that $|\mu_n(F)| = n$ and let $( \, , \, ) = ( \, , \, )_{F,n} : F^\times \times F^\times \to \mu_n(F)$ be the $n$th order Hilbert symbol. It is a bilinear form on $F^\times$ that defines a nondegenerate bilinear form on $F^\times / F^{\times n}$ and satisfies

$$(x, -x) = (x, y)(y, x) = 1, \; x, \; y \in F^\times.$$

In particular $(x, -1) = (x, x) \in \{\pm 1\}$ is a sign which is also an $n$th root of unity (and therefore always equals 1 if $n$ is odd). We denote by $\varrho = \varrho_{n,F}$ the sign determined by

$$\varrho = (\varpi, \varpi) = (\varpi, -1).$$

Associated to every 2-cocycle $\sigma : G \times G \to \mu_n(F)$ there is a central extension $\tilde{G}$ of $G$ by $\mu_n(F)$ satisfying an exact sequence

$$1 \longrightarrow \mu_n(F) \xrightarrow{\iota} \tilde{G} \xrightarrow{\mathbf{p}} G \longrightarrow 1.$$

We call $\tilde{G}$ a metaplectic $n$-fold cover of $G$. As a set, we can realize $\tilde{G}$ as

$$\tilde{G} = G \times \mu_n(F) = \{\langle g, \zeta \rangle : g \in G, \zeta \in \mu_n(F)\}.$$

The embedding $\iota$ and the projection $\mathbf{p}$ are given by

$$\iota(\zeta) = \langle e, \zeta \rangle \text{ and } \mathbf{p}(\langle g, \zeta \rangle) = g$$

where $e$ denotes the identity element of $G$. The multiplication is defined in terms of $\sigma$ as follows,

$$\langle g_1, \zeta_1 \rangle \langle g_2, \zeta_2 \rangle = \langle g_1 g_2, \zeta_1 \zeta_2 \, \sigma(g_1, g_2) \rangle .$$

For any subset $X \subseteq G$ let

$$\tilde{X} = \mathbf{p}^{-1}(X) \subseteq \tilde{G}.$$

We also fix the section $\mathbf{s} : G \to \tilde{G}$ of $\mathbf{p}$ given by $\mathbf{s}(g) = \langle g, 1 \rangle$. Then for $g_1, \; g_2 \in G$ we have

$$\mathbf{s}(g_1)\mathbf{s}(g_2) = \langle g_1 g_2, \sigma(g_1, g_2) \rangle .$$

For 2-cocycles in the same cohomology class the associated metaplectic coverings are isomorphic. Kazhdan and Patterson provided in [17] 2-cocycles $\sigma^{(c)}$ parameterized by $c \in \mathbb{Z}/n\mathbb{Z}$ that exhaust all cohomology classes (but do not necessarily all lie in different cohomology classes). They are related by

$$\sigma^{(c)}(g_1, g_2) = \sigma^{(0)}(g_1, g_2)(\det g_1, \det g_2)^c, \; g_1, \; g_2 \in G. \tag{2.1}$$

We take $\sigma^{(0)} = \sigma_r^{(0)}$ to be the block compatible 2-cocycle on $G$ constructed in [2] and let $\sigma^{(c)} = \sigma_r^{(c)}$ be related to $\sigma_r^{(0)}$ by (2.1). It is the unique family of 2-cocycles that satisfies the three properties (2.2), (2.3) and (2.4) below.

If $r = r_1 + \cdots + r_k$ and $g_i,\, g_i' \in GL_{r_i}(F)$ for $i = 1, \dots, k$ then

$$\sigma_r^{(c)}(\mathrm{diag}(g_1, \dots, g_k), \mathrm{diag}(g_1', \dots, g_k')) = \left[\prod_{i=1}^{k} \sigma_{r_i}^{(c)}(g_i, g_i')\right] \cdot \left[\prod_{i<j} (\det g_i, \det g_j')^{c+1} (\det g_j, \det g_i')^c\right]. \tag{2.2}$$

The 2-cocycle $\sigma_1^{(0)}$ is the trivial one, i.e.

$$\sigma_1^{(c)}(x, y) = (x, y)^c,\ x,\ y \in F^\times. \tag{2.3}$$

The 2-cocycle $\sigma_2^{(0)}$ is the one explicitly described by Kubota. That is,

$$\sigma_2^{(c)}(g_1, g_2) = \left(\frac{\chi(g_1 g_2)}{\chi(g_1)}, \frac{\chi(g_1 g_2)}{\chi(g_2) \det g_1}\right) (\det g_1, \det g_2)^c \tag{2.4}$$

where

$$\chi(g) = \begin{cases} c & c \neq 0 \\ d & c = 0 \end{cases} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Throughout the paper we fix the positive integers $r$ and $n$ and the modulus class $c \in \mathbb{Z}/n\mathbb{Z}$ and let $\sigma = \sigma_r^{(c)}$. Note that the restriction of $\sigma$ to $A$ is given by

$$\sigma(a, a') = \left[\prod_{i<j} (a_i, a_j')\right] \cdot \prod_{i,j} (a_i, a_j')^c \tag{2.5}$$

for $a = \mathrm{diag}(a_1, \dots, a_r)$ and $a' = \mathrm{diag}(a_1', \dots, a_r')$.

The group $U$ splits in $\tilde{G}$. In fact $\mathbf{s}_{|U}$ is an imbedding of $U$ in $\tilde{G}$. Furthermore, we have

$$\sigma(u_1 g_1 u_2, g_2 u_3) = \sigma(g_1, u_2 g_2),\ g_1,\ g_2 \in G,\ u_1,\ u_2,\ u_3 \in U. \tag{2.6}$$

Note that this implies that for every $u \in U$ and $b \in \tilde{B}$ we have

$$b\ \mathbf{s}(u)\ b^{-1} = \mathbf{s}(\mathbf{p}(b)\ u\ \mathbf{p}(b)^{-1}) \tag{2.7}$$

and in particular that $\mathbf{s}(U)$ is normal in $\tilde{B}$.

We fix the decomposition

$$n = n_1 n_2 \text{ where } n_1 = \gcd(n, 2rc + r - 1)$$

that plays a role in the structure of (the center of) $\tilde{G}$. Let

$$Z = \{xe : x^{2rc+r-1} \in F^{\times n}\}.$$

Then $\tilde{Z}$ is the center of $\tilde{G}$ (and of $\tilde{B}$) [17, Proposition 0.1.1]. We make the following simple observation.

**Lemma 1.** *We have*

$$Z = \{xe : x \in F^{\times n_2}\}.$$

*Proof.* We need to show that $F^{\times n_2} = \{x \in F^\times : x^{2rc+r-1} \in F^{\times n}\}$. If $x$ is an $n_2$th power then, since $n_1$ divides $2rc+r-1$, it is clear that $x^{2rc+r-1}$ is an $n$th power. If $x^{2rc+r-1} = y^n$ for some $y \in F^\times$ then $x^{\frac{2rc+r-1}{n_1}} y^{-n_2}$ is an $n_1$th root of unity. Since $F$ contains a primitive $n$th root of unity, there exists $\zeta \in F^\times$ such that $x^{\frac{2rc+r-1}{n_1}} = (\zeta y)^{n_2}$. Note that $\gcd(n_2, \frac{2rc+r-1}{n_1}) = 1$ and therefore $x$ is also an $n_2$th power. $\square$

For the rest of this work we assume that $|n|_F = 1$. Under this assumption we have

$$(u_1, u_2) = 1,\ u_1,\ u_2 \in \mathcal{O}^\times. \tag{2.8}$$

The group $K$ also splits in $\tilde{G}$. There is a map $\kappa : K \to \mu_n(F)$ such that $g \mapsto \kappa^*(g) = \langle g, \kappa(g)\rangle$ is a group homomorphism from $K$ to $\tilde{G}$. We denote its image by $K^*$. The splitting $\kappa^*$ is not unique, but its germ at the identity is. We shall fix $\kappa$ such that $\kappa^*$ is what Kazhdan-Patterson refer to as the canonical lift of $K$ to $\tilde{G}$. It is characterized by the property that

$$\mathbf{s}_{|A\cap K} = \kappa^*_{|A\cap K},\ \mathbf{s}_{|\mathfrak{W}} = \kappa^*_{|\mathfrak{W}} \text{ and } \mathbf{s}_{|U\cap K} = \kappa^*_{|U\cap K} \tag{2.9}$$

[17, Proposition 0.1.3]. The topology of $\tilde{G}$ as a locally compact group is determined by this embedding. For every subgroup $K_0$ of $K$ denote by

$$K_0^* = \kappa^*(K_0)$$

its image in $K^*$. Note that the Iwasawa decomposition of $G$ gives $\tilde{G} = \mathbf{s}(U)\tilde{A}K^*$.

2.2. **Spherical Whittaker functions.** Let $\epsilon : \mu_n(F) \to \mu_n(\mathbb{C})$ be an isomorphism, fixed once and for all.

**Definition 1.** Let $Q$ be a subgroup of $G$. A function $f : \tilde{Q} \to \mathbb{C}$ is called $\epsilon$-*genuine* if

$$f(\iota(\zeta)\, g) = \epsilon(\zeta) f(g),\ g \in \tilde{Q},\ \zeta \in \mu_n(F).$$

Consider the $\epsilon$-genuine spherical Hecke algebra

$$\mathcal{H}^\epsilon(\tilde{G}, K^*) = \{f : \tilde{G} \to \mathbb{C} : \operatorname{supp}(f) \text{ is compact and}$$
$$f(\iota(\zeta)k_1 g k_2) = \epsilon(\zeta) f(g),\ k_1,\ k_2 \in K^*,\ g \in \tilde{G},\ \zeta \in \mu_n(F)\}.$$

The Hecke algebra $\mathcal{H}^\epsilon(\tilde{G}, K^*)$ acts on the space $C^{\infty,\epsilon}(\tilde{G}/K^*)$ of right $K^*$-invariant, $\epsilon$-genuine functions on $\tilde{G}$ by the convolution

$$f * \phi(x) = \int_G f(\mathbf{s}(g))\phi(\mathbf{s}(g)^{-1}x)\ dg \tag{2.10}$$

where $f \in \mathcal{H}^\epsilon(\tilde{G}, K^*)$, $\phi \in C^{\infty,\epsilon}(\tilde{G}/K^*)$ and $x \in \tilde{G}$. Note that the function $g \mapsto f(g)\phi(g^{-1}x)$ on $\tilde{G}$ is $\iota(\mu_n(F))$-invariant and that the integration is over $G \simeq \iota(\mu_n(F))\backslash\tilde{G}$.

**Definition 2.** An $\epsilon$-genuine *spherical Whittaker function* on $\tilde{G}$ is an element $W \in C^{\infty,\epsilon}(\tilde{G}/K^*)$ so that

$$W(\mathbf{s}(u)g) = \psi_U(u)W(g),\ \ u \in U,\ g \in \tilde{G}$$

and $W$ is a common eigenfunction of $\mathcal{H}^\epsilon(\tilde{G}, K^*)$.

The spherical Whittaker functions on $\tilde{G}$ are the main objects of study of this work. Our main tool is the Casselman-Shalika method that is based on the theory of unramified principal series representations and that we now recall.

## 3. The unramified Principal series of $\tilde{G}$

The unramified principal series representations of $\tilde{G}$ were introduced in [17, §1.1]. We recall the construction of Kazhdan and Patterson. Consider the subgroups $A^n \subseteq A_* \subseteq A$ defined by

$$A^n = \{a = \mathrm{diag}(a_1, \dots, a_r) \in A : a_i \in F^{\times n}\}$$

and

$$A_* = A_\circ Z \text{ where } A_\circ = \{a = \mathrm{diag}(a_1, \dots, a_r) \in A : v(a_i) \equiv 0(n)\}.$$

The group $\tilde{A}_*$ is what Kazhdan-Patterson called the standard maximal abelian subgroup of $\tilde{A}$ (denoted by $\tilde{H}_*$ in [loc. cit.]). It is normalized by $\mathbf{s}(\mathfrak{W})$. In fact we have

$$\sigma(m, m') = 1,\, m,\, m' \in \mathbf{s}(\mathfrak{W})A_\circ.$$

This follows from the characterization of the block compatible 2-cocycle given in [1, (1.2), (1.4), (1.5)] and the fact that $\sigma$ is trivial on $A_\circ \times A_\circ$ (that follows from (2.8) and (2.5)). This implies that

$$\mathbf{s}(w)\mathbf{s}(a)\mathbf{s}(w)^{-1} = \mathbf{s}(waw^{-1}),\, w \in \mathfrak{W},\, a \in A_*. \tag{3.1}$$

For $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ we denote by $\chi_s$ the (non-genuine) character of $\tilde{B}$ defined by

$$\chi_s(\langle \mathrm{diag}(a_1, \dots, a_r)u, \zeta\rangle) = \prod_{i=1}^{r} |a_i|^{s_i},\, a_i \in F^\times,\, u \in U,\, \zeta \in \mu_n(F). \tag{3.2}$$

Any character of a subgroup $A'$ of $\tilde{A}$ will automatically be considered as a character of $A'\,\mathbf{s}(U)$ which is trivial on $\mathbf{s}(U)$. Set $B_* = A_*U$.

Let $\omega$ be an $\epsilon$-genuine character of $\tilde{A^n}\tilde{Z}$ and let $\omega'$ be a character of $\tilde{A}_*$ that extends $\omega$. Define the principal series representation associated to $\omega$ by

$$I(\omega') = \mathrm{ind}_{\tilde{B}_*}^{\tilde{G}}(\omega').$$

This is the representation of $\tilde{G}$ by right translations $(R(g)\varphi)(x) = \varphi(xg)$, $g$, $x \in G$ on the space of functions $\varphi : \tilde{G} \to \mathbb{C}$ that are right $K_0^*$-invariant for some open subgroup $K_0$ of $K$ and which satisfy

$$\varphi(bg) = (\chi_\rho\omega')(b)\varphi(g),\, b \in \tilde{B}_*,\, g \in \tilde{G} \text{ where } \rho = (\frac{r-1}{2}, \frac{r-3}{2}, \dots, \frac{1-r}{2}).$$

Although the realization of the representation $I(\omega')$ does depend on $\omega'$ its equivalence class is only dependent on $\omega$. The character $\omega$ of $\tilde{A^n}\tilde{Z}$ is called *unramified* if $a \mapsto \omega(\mathbf{s}(a^n))$ is an unramified character of $A$, i.e. if $\omega$ is trivial on $\tilde{A^n} \cap K^*$. It is called *normalized* if in addition $\omega_{|\tilde{Z}\cap K^*}$ is trivial. The representation $I(\omega')$ is then called a normalized, unramified, principal series representation. Every $\epsilon$-genuine, unramified character $\omega$ of $\tilde{A^n}\tilde{Z}$ can be twisted to a normalized one, i.e. there is a quasicharacter $\chi$ of $F^\times$ such that $\omega(\chi \circ \det \circ \mathbf{p})$ is a normalized unramified character of $\tilde{A}^n\tilde{Z}$ [17, Lemma 1.1.1]. It follows that if $\omega$ is unramified then the associated principal series representation $I(\omega')$ can be twisted to a normalized unramified one. If $\omega$ is an $\epsilon$-genuine, normalized unramified character of $\tilde{A}^n\tilde{Z}$ then there exists a unique extension $\omega'$ of $\omega$ to $\tilde{A}_*$ such that $\omega'_{|\tilde{A}_*\cap K^*} = 1$ [17, p. 60]. This is referred to as the *canonical extension.* The $K^*$-invariant subspace $I(\omega')^{K^*}$ of $I(\omega')$ is then one dimensional [17, Lemma 1.1.3]. Let $\varphi_K = \varphi_K(\omega') \in I(\omega')^{K^*}$ be normalized by $\varphi_K(\mathbf{s}(e) : \omega') = 1$. The normalized spherical section is also given by

$$\varphi_K(g : \omega') = \begin{cases} (\chi_\rho\omega')(b) & g = bk,\, b \in \tilde{B}_*,\, k \in K^* \\ 0 & g \notin \tilde{B}_* K^*. \end{cases}$$

3.1. **Parameterization.** Let $\omega$ be an $\epsilon$-genuine, normalized unramified character of $\tilde{A}^n\tilde{Z}$ and let $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ be such that

$$\omega(\mathbf{s}(a)) = \prod_{i=1}^{r} |a_i|^{s_i},\ a = \operatorname{diag}(a_1, \dots, a_r) \in A^n. \tag{3.3}$$

Note that the entries $s_i$ of $s$ are only determined by $\omega$ modulo $\frac{2\pi\sqrt{-1}}{n \log q}\mathbb{Z}$. If $s \in \mathbb{C}^r$ satisfies (3.3) we say that $s$ is an *exponent* of $\omega$.

**Lemma 2.** *Let $\omega$ be an $\epsilon$-genuine, normalized unramified character of $\tilde{A}^n\tilde{Z}$ and let $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ be an exponent of $\omega$. Then there exists a unique $\zeta \in \mu_{2n_1}(\mathbb{C})$ satisfying*

$$\zeta^{n_1} = \epsilon(\varrho)^{\frac{r}{2}(2rc+r-1)\frac{(n_1-1)n}{2}} \tag{3.4}$$

*such that*

$$\omega(\mathbf{s}(\varpi^{n_2}e)) = \zeta\, q^{-n_2(s_1+\dots+s_r)}. \tag{3.5}$$

*Every pair $(s, \zeta)$ with $s \in \mathbb{C}^r$ and $\zeta \in \mu_{2n_1}(\mathbb{C})$ satisfying* (3.4) *determines uniquely an $\epsilon$-genuine, normalized unramified character $\omega_{s,\zeta}$ of $\tilde{A}^n\tilde{Z}$ satisfying* (3.3) *and* (3.5)*. If $(t, \eta)$ is another such pair then $\omega_{s,\zeta} = \omega_{t,\eta}$ if and only if*

$$q^{-ns_i} = q^{-nt_i},\ i = 1, \dots, r \text{ and } \zeta\, q^{-n_2(s_1+\dots+s_r)} = \eta\, q^{-n_2(t_1+\dots+t_r)}.$$

*Proof.* It folllows from (2.5) and a simple induction that for every $x \in F^\times$ and integer $m \geq 0$ we have

$$\mathbf{s}(xe)^m = \iota(\varrho)^{\frac{r}{2}(2rc+r-1)v(x)\frac{(m-1)m}{2}}\mathbf{s}(x^m e). \tag{3.6}$$

Applying (3.6) to $x = \varpi^{n_2}$ and $m = n_1$ we get that

$$\mathbf{s}(\varpi^{n_2}e)^{n_1} = \iota(\varrho)^{\frac{r}{2}(2rc+r-1)\frac{(n_1-1)n}{2}}\mathbf{s}(\varpi^n e)$$

and therefore

$$\omega(\mathbf{s}(\varpi^{n_2}e))^{n_1} = \epsilon(\varrho)^{\frac{r}{2}(2rc+r-1)\frac{(n_1-1)n}{2}} q^{-n(s_1+\dots+s_r)}.$$

Thus, $\zeta = \omega(\mathbf{s}(\varpi^{n_2}e))\, q^{n_2(s_1+\dots+s_r)}$ indeed satisfies (3.4). Clearly, (3.5) defines $\zeta$ uniquely. This proves the first statement of the lemma. Fix a pair $(s, \zeta)$ as in the statement of the lemma. By Lemma 1 a character $\omega_{s,\zeta}$ as desired, if it exists, is determined by its restriction to $\mathbf{s}(A^n)$ and its value on $\mathbf{s}(\varpi^{n_2}e)$. Therefore the character is uniquely determined by $(s,\zeta)$. We now show that such a character $\omega_{s,\zeta}$ exists. It follows from Lemma 1 that for any $a \in \tilde{Z}\tilde{A}^n$ there exists $u \in \mathcal{O}^{\times n_2}$ and $k \in \mathbb{Z}$ such that $\mathbf{p}(a) \in u\,\varpi^{n_2k}A^n$. Assume that

$$\iota(\zeta_1)\,\mathbf{s}(u_1e)\,\mathbf{s}(\varpi^{n_2}e)^{k_1}\,\mathbf{s}(a_1) = \iota(\zeta_2)\,\mathbf{s}(u_2e)\,\mathbf{s}(\varpi^{n_2}e)^{k_2}\,\mathbf{s}(a_2) \in \tilde{A}^n\tilde{Z} \tag{3.7}$$

with $\zeta_i \in \mu_n(F)$, $k_i \in \mathbb{Z}$, $u_i \in \mathcal{O}^{\times n_2}$ and $a_i = \operatorname{diag}(a_1^i,\dots,a_r^i) \in A^n$, $i = 1,2$. To show that $\omega_{s,\zeta}$ can be well-defined we need to show that

$$\epsilon(\zeta_1)\,\zeta^{k_1}q^{-n_2k_1(s_1+\cdots+s_r)}\prod_{j=1}^{r}\left|a_j^1\right|^{s_j} = \epsilon(\zeta_2)\,\zeta^{k_2}q^{-n_2k_2(s_1+\cdots+s_r)}\prod_{j=1}^{r}\left|a_j^2\right|^{s_j}. \tag{3.8}$$

Applying the projection $\mathbf{p}$ to both sides of (3.7) we have

$$\varpi^{n_2k_1}u_1a_1 = \varpi^{n_2k_2}u_2a_2. \tag{3.9}$$

Comparing valuations of each entry we see that

$$n_2k_1 + v(a_j^1) = n_2k_2 + v(a_j^2),\ j = 1,\dots,r.$$

This implies in particular that

$$q^{-n_2k_1(s_1+\cdots+s_r)}\prod_{j=1}^{r}\left|a_j^1\right|^{s_j} = q^{-n_2k_2(s_1+\cdots+s_r)}\prod_{j=1}^{r}\left|a_j^2\right|^{s_j} \tag{3.10}$$

and that

$$k_1 \equiv k_2 \mod n_1. \tag{3.11}$$

Without loss of generality we may assume that $k_2 \geq k_1$. From (3.9) and (3.11) we see that $u_1^{-1}u_2 \in \mathcal{O}^{\times n}$ and

$$\iota(\zeta_1\zeta_2^{-1}) = \mathbf{s}(\varpi^{n_2}e)^{k_2-k_1}\mathbf{s}(u_1^{-1}u_2e)\mathbf{s}(a_1^{-1}a_2) = \mathbf{s}(\varpi^{n_2}e)^{k_2-k_1}\mathbf{s}(u_1^{-1}u_2a_1^{-1}a_2).$$

Applying (3.6) again it follows that

$$\zeta_1\zeta_2^{-1} = \varrho^{\frac{r}{2}(2rc+r-1)n_2\frac{(k_2-k_1-1)(k_2-k_1)}{2}}.$$

On the other hand

$$\zeta^{k_2-k_1} = \epsilon(\varrho)^{\frac{r}{2}(2rc+r-1)\frac{(n_1-1)n}{2}\frac{k_2-k_1}{n_1}} = \epsilon(\varrho)^{\frac{r}{2}(2rc+r-1)n_2\frac{(n_1-1)(k_2-k_1)}{2}}.$$

Note that

$$n_2\frac{(k_2-k_1-1)(k_2-k_1)}{2} = n_2\frac{(n_1-1)(k_2-k_1)}{2} + nn_1\frac{\frac{k_2-k_1}{n_1}\left(\frac{k_2-k_1}{n_1}-1\right)}{2}$$

and since $\varrho^n = 1$ we get that

$$\zeta^{k_2-k_1} = \epsilon(\zeta_1\zeta_2^{-1}). \tag{3.12}$$

From (3.10) and (3.12) we get (3.8) and the existence of $\omega_{s,\zeta}$. The last equivalence condition of the lemma is now straightforward from the requirements (3.3) and (3.5) . $\square$

For the rest of this section we fix $s \in \mathbb{C}^r$ and $\zeta \in \mu_{2n_1}(\mathbb{C})$ satisfying (3.4), set $\omega = \omega_{s,\zeta}$ and let $\omega'$ be its canonical extension to $\tilde{A}_*$.

3.2. **Intertwining operators.** For $w \in \mathfrak{W}$ let $w\omega$ be the character of $\tilde{A}^n \tilde{Z}$ defined by

$$(w\omega)(a) = \omega(\mathbf{s}(w)^{-1} a \mathbf{s}(w))$$

and similarly, denote by $w\omega'$ the canonical extension of $w\omega$ to $\tilde{A}_*$. An important role in the study of the principal series representations is played by the intertwining operators $T_w : I(\omega') \to I(w\omega')$. They were studied in [17, §1.2] and we now recall some of their properties. If $s$ satisfies

$$\operatorname{Re} s_i > \operatorname{Re} s_{i+1} \text{ for all } i \text{ such that } (i, i+1) \in \Phi^-(w^{-1}) \tag{3.13}$$

then $T_w \varphi$ is defined for $\varphi \in I(\omega')$ by the absolutely convergent integral

$$T_w \varphi(g) = \int_{U_w} \varphi(\mathbf{s}(w_0 u) g) \, du. \tag{3.14}$$

For general $s$ the value of this integral can be regularized as follows. Using the decomposition $\tilde{G} = \tilde{B} K^*$, the character $\chi_s$ of $\tilde{B}$ (see (3.2)) can be extended uniquely to a right $K^*$-invariant function on $\tilde{G}$, that we denote by $\tilde{\chi}_s$. For $\varphi \in I(\omega')$ we define the holomorphic section $\varphi_t \in I(\chi_{t|\tilde{B}_*} \omega')$ by $\varphi_t = \tilde{\chi}_t \, \varphi$, $t = (t_1, \dots, t_r) \in \mathbb{C}^r$. Let

$$L(x) = (1 - q^{-x})^{-1}, \; x \in \mathbb{C}$$

be the local zeta function of $F$ and for $\alpha = (i, j) \in \Phi$ let $L_\alpha(s) = L(s_i - s_j)$. The function

$$t \mapsto \left( \prod_{\alpha \in \Phi^-(w^{-1})} L_\alpha(n(t+s))^{-1} \right) T_w \varphi_t$$

defined for all $t$ such that $s + t$ satisfies (3.13) is in fact a polynomial in $q^{\pm t_1}, \dots, q^{\pm t_r}$. For $s$ such that $\prod_{\alpha \in \Phi^-(w^{-1})} L_\alpha(ns)^{-1} \neq 0$ this allows us to define $T_w \varphi$ by

$$T_w \varphi = \left( \prod_{\alpha \in \Phi^-(w^{-1})} L_\alpha(ns) \right) \left[ \left( \prod_{\alpha \in \Phi^-(w^{-1})} L_\alpha(n(t+s))^{-1} \right) T_w \varphi_t \right]_{|t=0.} \tag{3.15}$$

It gives the regularization of the integral (3.14) and we symbolically denote it by

$$T_w \varphi(g) = \int_{U_w}^{*} \varphi(\mathbf{s}(w_0 u) g) \, du.$$

We call $s$ or $\omega$ *regular* if for all $w \in \mathfrak{W} \setminus \{e\}$ we have $\omega \neq w\omega$, i.e. if for all $i \neq j$ we have

$$s_i - s_j \notin \frac{2\pi\sqrt{-1}}{n \log q} \mathbb{Z}.$$

Thus, for every regular $\omega$ and every $w \in \mathfrak{W}$, $T_w$ is defined on $I(\omega')$ by (3.15). For the normalized $K^*$-invariant element we have

$$T_w \varphi_K(\omega') = c_w(s) \, \varphi_K(w\omega') \tag{3.16}$$

where

$$c_w(s) = \prod_{(i,j)\in\Phi^-(w^{-1})} \frac{L(n(s_i - s_j))}{L(n(s_i - s_j) + 1)}.$$

For every $w_1$, $w_2 \in \mathfrak{W}$ we have the following equality of intertwining operators from $I(\omega')$ to $I(w_1 w_2 \omega')$

$$T_{w_1} T_{w_2} = \frac{c_{w_1}(w_2 s) c_{w_2}(s)}{c_{w_1 w_2}(s)} T_{w_1 w_2}. \tag{3.17}$$

Recall further that

$$\frac{c_{w_1}(w_2 s) c_{w_2}(s)}{c_{w_1 w_2}(s)} = 1 \text{ whenever } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2). \tag{3.18}$$

3.3. **The Iwahori invariant subspace.** Let $\mathcal{I}$ denote the Iwahori subgroup of $K$ compatible with $B$. It is the group of all matrices in $K$ with upper triangular projection to $GL_r(\mathcal{O}/\mathfrak{p})$. The Casselman-Shalika method is based on an explicit expansion of $\varphi_K$ in terms of a carefully chosen basis, convenient for computations, of the Iwahori invariant subspace $I(\omega')^{\mathcal{I}^*}$ of $I(\omega')$. Next, we select the basis adopting the approach of Y. Sakellaridis (cf. [27]). As it turns out, this generalizes the basis used in [9, 10] (see Remark 1 below).

For every $a \in \tilde{A}$ let $\zeta_a : \tilde{A} \to \mu_n(F)$ be the homomorphism defined by

$$\iota(\zeta_a(b)) = aba^{-1}b^{-1},\ b \in \tilde{A}.$$

Since $\tilde{A}_*$ is a maximal abelian subgroup of $\tilde{A}$ it follows that $\zeta_a$ is trivial on $\tilde{A}_*$ if and only if $a \in \tilde{A}_*$. Note that $\zeta_a$ is trivial on $\tilde{Z}\tilde{A}^n$ for all $a \in \tilde{A}$. Since $\tilde{A}_* = \tilde{Z}\tilde{A}^n(\tilde{A} \cap K^*)$ we get that $\zeta_a$ is trivial on $\tilde{A} \cap K^*$ if and only if $a \in \tilde{A}_*$.

**Lemma 3.** *For every $\varphi \in I(\omega')^{\mathcal{I}^*}$ the support of $\varphi$ is contained in $\tilde{B}_* K^*$.*

*Proof.* The group $\tilde{G}$ has a disjoint decomposition

$$\tilde{G} = \bigsqcup_{a\in\tilde{A}_*\backslash\tilde{A}} \bigsqcup_{w\in\mathfrak{W}} \tilde{B}_* a\, \mathbf{s}(w)\mathcal{I}^* \tag{3.19}$$

and therefore $\varphi \in I(\omega')^{\mathcal{I}^*}$ is determined by its values $\varphi(a\,\mathbf{s}(w))$ for $w \in \mathfrak{W}$ and $a \in \tilde{A}$. The decomposition (3.19) gives in particular

$$\tilde{B}_* K^* = \bigsqcup_{w\in\mathfrak{W}} \tilde{B}_* \mathbf{s}(w)\mathcal{I}^*.$$

Fix $a \in \tilde{A}$ and let $a_0 \in A \cap K \subseteq \mathcal{I}$. It follows from (2.9) that $\kappa^*(a_0) = \mathbf{s}(a_0)$. Therefore, on the one hand we have

$$\varphi(a\mathbf{s}(w)\mathbf{s}(a_0)) = \varphi(a\mathbf{s}(w)),$$

and on the other hand by (3.1) we have

$$a\mathbf{s}(w)\mathbf{s}(a_0) = a\mathbf{s}(w a_0 w^{-1})\mathbf{s}(w) = \iota(\zeta_a(\mathbf{s}(w a_0 w^{-1})))\mathbf{s}(w a_0 w^{-1}) a\mathbf{s}(w).$$

Consequently

$$\varphi(a\mathbf{s}(w)\mathbf{s}(a_0)) = \epsilon(\zeta_a(w a_0 w^{-1}))\varphi(a\mathbf{s}(w)).$$

It follows that if $\varphi(a\mathbf{s}(w)) \neq 0$ for some $w \in \mathfrak{W}$ then $\zeta_a$ is trivial on $\tilde{A}_* \cap K^*$ and therefore that $a \in \tilde{A}_*$. The lemma follows. $\square$

It is easy to verify that for every $w \in \mathfrak{W}$ there exists a unique element $\varphi_w = \varphi_w(\omega') \in I(\omega')^{\mathcal{I}^*}$ that is supported on $\tilde{B}_*\mathbf{s}(w)\mathcal{I}^*$ and such that $\varphi_w(\mathbf{s}(w)) = 1$. It follows from Lemma 3 that the set

$$\mathfrak{B}_1 = \{\varphi_w : w \in \mathfrak{W}\}$$

is a basis of $I(\omega')^{\mathcal{I}^*}$ and we have

$$\varphi_K = \sum_{w \in \mathfrak{W}} \varphi_w.$$

This expansion of $\varphi_K$ is, however, not very useful for our purpose. We will soon see that the set

$$\mathfrak{B}_2 = \{T_w \varphi_{w_0}(w^{-1}\omega') : w \in \mathfrak{W}\}$$

is a basis of $I(\omega')^{\mathcal{I}^*}$; this is in fact the basis we are after. In order to see that $\mathfrak{B}_2$ is indeed a basis we will show that the transition matrix that expresses the set $\mathfrak{B}_2$ in terms of the basis $\mathfrak{B}_1$ is upper uni-triangular and in particular invertible. For this purpose we need to recall an elementary property of products of Bruhat cells in $G$. For $w \in \mathfrak{W}$ let $C(w) = BwB$.

**Lemma 4.** *For every $w_1$, $w_2 \in \mathfrak{W}$ we have*

$$C(w_1)C(w_2) \subseteq \left( \bigcup_{\ell(w) < \ell(w_1) + \ell(w_2)} C(w) \right) \cup C(w_1 w_2). \tag{3.20}$$

*Proof.* If $\ell(w_1) = 1$ then $C(w_1)C(w_2) \subseteq C(w_1w_2) \cup C(w_2)$ by the Tits system formalism (see for example [16, §28.3]) and in particular (3.20) holds. The lemma follows by a simple induction on $\ell(w_1)$. $\square$

**Lemma 5.** *The set $\mathfrak{B}_2$ is a basis of $I(\omega')^{\mathcal{I}^*}$.*

*Proof.* We will show that

$$T_{ww_0}\varphi_{w_0} - \varphi_w \in \bigoplus_{\ell(w') > \ell(w)} \mathbb{C}\, \varphi_{w'}. \tag{3.21}$$

This puts $\mathfrak{B}_2$ in upper uni-triangular relation with $\mathfrak{B}_1$ and therefore $\mathfrak{B}_2$ is indeed a basis. By Lemma 3, for every $\varphi \in I(\omega')^{\mathcal{I}^*}$ we have

$$\varphi = \sum_{w \in \mathfrak{W}} \varphi(\mathbf{s}(w))\varphi_w.$$

Using the change of variables $w \mapsto ww_0$, to get (3.21) it is therefore enough to show two things:

(1) $T_w\varphi_{w_0}(\mathbf{s}(ww_0)) = 1$, and
(2) if $w' \in \mathfrak{W} \setminus \{ww_0\}$ is such that $T_w\varphi_{w_0}(\mathbf{s}(w')) \neq 0$ then $\ell(w') > \ell(ww_0)$.

We first show the second statement. Recall that

$$\mathrm{supp}(\varphi_{w_0}) = \tilde{B}_*\mathbf{s}(w_0)\mathcal{I}^* \subseteq \widetilde{C(w_0)}.$$

If $\mathbf{s}(w^{-1}u)\mathbf{s}(w')$ lies in $\widetilde{C(w_0)}$ for some $u \in U_w$ then in particular $w^{-1}uw' \in C(w_0)$, i.e. $C(w^{-1})C(w')$ contains the open cell $C(w_0)$. It follows from Lemma 4 that either $w' = ww_0$ or $\ell(w_0) < \ell(w) + \ell(w')$, i.e. $\ell(w') > \ell(w_0) - \ell(w) = \ell(ww_0)$. In the region of convergence of the integral (3.14) it follows that if $T_w\varphi_{w_0}(\mathbf{s}(w')) \neq 0$ then either $w' = ww_0$ or $\ell(w') > \ell(ww_0)$ and by meromorphic continuation the same holds in general. It is left to show that $T_w\varphi_{w_0}(\mathbf{s}(ww_0)) = 1$. Again, using meromorphic continuation it is enough to show this in the region of convergence. Note that if $u \in U_w$ is such that $w^{-1}uww_0 \in Bw_0\mathcal{I}$ then there exists $b \in B$ and $u_0 \in U \cap \mathcal{I}$ such that $w^{-1}uww_0 = bw_0u_0$ and since $w^{-1}U_ww \subseteq \overline{U}$ we get that $b = (w^{-1}uw)(w_0(u_0)^{-1}w_0^{-1}) \in B \cap \overline{U} = \{e\}$. In particular $u \in U_w \cap K$. This implies that

$$T_w\varphi_{w_0}(\mathbf{s}(ww_0)) = \int_{U_w \cap K} \varphi_{w_0}(\mathbf{s}(w^{-1}u)\mathbf{s}(ww_0))\, du. \tag{3.22}$$

But if $u \in U_w \cap K$ and $w_1 = ww_0$ then

$$\mathbf{s}(w^{-1}u)\mathbf{s}(ww_0) = \mathbf{s}(w_0)\mathbf{s}(w_1)^{-1}\mathbf{s}(u)\mathbf{s}(w_1) = \mathbf{s}(w_0)\mathbf{s}(w_1^{-1}uw_1) \in \mathbf{s}(w_0)\mathcal{I}^*.$$

Here the first equality follows from (2.9) and the second holds since in addition $w_1^{-1}uw_1 \subseteq U \cap K$. Therefore by (2.6)

$$\sigma(w_1^{-1}u, w_1) = \sigma(w_1^{-1}uw_1w_1^{-1}, w_1) = \sigma(w_1^{-1}, w_1) = 1.$$

This implies that the integral in (3.22) is over the constant function 1. From our normalization of measures we indeed have

$$T_w\varphi_{w_0}(\mathbf{s}(ww_0)) = 1.$$

This shows (3.21) and completes the proof of the lemma. □

As an almost straightforward consequence we now have

**Lemma 6.** *The expansion of $\varphi_K$ in terms of the basis $\mathfrak{B}_2$ is given by*

$$\varphi_K(\omega') = \sum_{w \in \mathfrak{W}} \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}\, T_w\varphi_{w_0}(w^{-1}\omega').$$

*Proof.* By Lemma 5 there are constants $\alpha_w(s)$ such that

$$\varphi_K = \sum_{w \in \mathfrak{W}} \alpha_w(s)\, T_w\varphi_{w_0}. \tag{3.23}$$

It follows from the property (3.21) and the proof of Lemma 5 that $T_w\varphi_{w_0}(e)$ equals 1 if $w = w_0$ and 0 otherwise. Therefore evaluating (3.23) at $e$ we get that

$$\alpha_{w_0}(s) = 1. \tag{3.24}$$

Now apply an intertwining operator $T_{w'}$ to (3.23). On the one hand

$$T_{w'}\varphi_K(\omega') = c_{w'}(s)\varphi_K(w'\omega') = c_{w'}(s) \sum_{w \in \mathfrak{W}} \alpha_w(w's)T_w\varphi_{w_0}(w^{-1}w'\omega').$$

On the other hand by (3.17) we get that

$$T_{w'}\varphi_K(\omega') = \sum_{w\in\mathfrak{W}} \alpha_w(s)\, T_{w'}T_w\varphi_{w_0}(w^{-1}\omega') = c_{w'}(s) \sum_{w\in\mathfrak{W}} \frac{c_w(w^{-1}s)}{c_{w'w}(w^{-1}s)} \alpha_w(s)\, T_{w'w}\varphi_{w_0}(w^{-1}\omega')$$
$$= c_{w'}(s) \sum_{w\in\mathfrak{W}} \frac{c_{(w')^{-1}w}(w^{-1}w's)}{c_w(w^{-1}w's)} \alpha_{(w')^{-1}w}(s)\, T_w\varphi_{w_0}(w^{-1}w'\omega').$$

Now comparing the coefficient for $w = w_0$ for the two expansions of $T_{w'}\varphi_K(\omega')$ and taking (3.24) into consideration we obtain that

$$\frac{c_{(w')^{-1}w_0}(w_0^{-1}w's)}{c_{w_0}(w_0^{-1}w's)} \alpha_{(w')^{-1}w_0}(s) = 1,$$

i.e. that $\alpha_w(s) = \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}$. The lemma follows. □

*Remark* 1. The proof of Lemma 5 together with (3.17) and (3.18) also shows that for $w,\, w' \in \mathfrak{W}$ we have

$$T_{w'}T_w\varphi_{w_0}(e) = \begin{cases} 1 & w'w = w_0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, in the case $n = 1$, $\mathfrak{B}_2$ is the basis used by Casselman and Shalika reordered.

As we shall see, this basis proves useful for computation of the Whittaker spherical functions. In Section 10 we further demonstrate the utility of the above expansion of $\varphi_K$ by computing the zonal spherical functions for $\tilde{G}$.

## 4. Spherical Whittaker functions

Whittaker functionals on the principal series representations of $\tilde{G}$ were defined and studied in [17, §3]. We recall the relevant results, introduce the associated spherical Whittaker functions and compute them in terms of the Kazhdan-Patterson functional equations.

For the rest of this work fix a root of unity $\zeta_0$ satisfying (3.4). We often suppress from our notation the dependence of objects on $\zeta_0$. Let $s \in \mathbb{C}^r$, $\omega = \omega_{s,\zeta_0}$ the $\epsilon$-genuine, unramified, normalized character of $\tilde{A}^n\tilde{Z}$ associated to $(s, \zeta_0)$ by Lemma 2 and let $\omega'$ be its canonical extension to $\tilde{A}_*$. Denote by $\mathrm{Wh}(\omega')$ the space of Whittaker functionals on $I(\omega')$, i.e. the space of all $\mathcal{W} \in I(\omega')^*$ such that $\mathcal{W}(R(\mathbf{s}(u))\varphi) = \psi_U(u)\, \mathcal{W}(\varphi)$, $u \in U$ and $\varphi \in I(\omega')$. From [17, Lemma 1.3.2] we have that

$$\dim_{\mathbb{C}} \mathrm{Wh}(\omega') = \left|\tilde{A}_*\backslash\tilde{A}\right| = n_2\, n^{r-1}.$$

For every $a \in \tilde{A}$ Kazhdan and Patterson associated a Whittaker functional $\mathcal{W}_a \in \mathrm{Wh}(\omega')$ defined by

$$\mathcal{W}_a(\varphi) = \mathcal{W}_a(\varphi : \omega') = \int_U^* \varphi(a\, \mathbf{s}(w_0u))\psi_U(u)^{-1}\, du,\ \varphi \in I(\omega').$$

As is the case with the intertwining operators, the integral is absolutely convergent whenever $\operatorname{Re} s_1 > \cdots > \operatorname{Re} s_r$ and for general $s$ we write $\int_U^*$ for the regularized integral that provides the meromorphic continuation. It is easy to see that

$$\mathcal{W}_{a'a}(\omega') = (\chi_\rho \omega')(a')\,\mathcal{W}_a(\omega'),\ a' \in \tilde{A}_*,\ a \in \tilde{A}. \tag{4.1}$$

Furthermore, it follows from [17, Lemma 1.3.1] that for any set of representatives $\Gamma$ for $\tilde{A}_* \backslash \tilde{A}$ the set $\mathfrak{B}(\Gamma) = \{\mathcal{W}_\gamma(\omega') : \gamma \in \Gamma\}$ is a basis for $\mathrm{Wh}(\omega')$. If $\Gamma'$ is another set of representatives for $\tilde{A}_* \backslash \tilde{A}$ then the basis $\mathfrak{B}(\Gamma')$ is proportional to $\mathfrak{B}(\Gamma)$ and for every $w \in \mathfrak{W}$ there is a transition matrix $D_w(s) = D_w^{\Gamma,\Gamma'}(s) = (\tau_{\gamma,\gamma'}(w,s))_{\gamma\in\Gamma,\gamma'\in\Gamma'}$ between the $\left|\tilde{A}_* \backslash \tilde{A}\right|$-tuples $(\mathcal{W}_\gamma(w\omega') \circ T_w)_{\gamma\in\Gamma}$ and $(\mathcal{W}_{\gamma'}(\omega'))_{\gamma'\in\Gamma'}$ such that

$$(\mathcal{W}_\gamma(w\omega') \circ T_w)_{\gamma\in\Gamma} = D_w(s)(\mathcal{W}_{\gamma'}(\omega'))_{\gamma'\in\Gamma'}.$$

Note that, as the notation suggests, the matrix coefficient $\tau_{\gamma,\gamma'}(w,s)$ depends on $\gamma$ and $\gamma'$ but not on the sets $\Gamma$ and $\Gamma'$ in which they lie, the coefficient $\tau_{a,a'}(w,s)$ is now defined for any two elements $a,\ a' \in \tilde{A}$ (even if $a \neq a'$ but $\tilde{A}_* a = \tilde{A}_* a'$).

In Section 5 we provide the Kazhdan-Patterson formula for $\tau_{a,a'}(w,s)$. In this section we compute the spherical Whittaker functions in terms of $D_w(s)$ by the Casselman-Shalika method. For every $a \in \tilde{A}$ define the $\epsilon$-genuine, spherical Whittaker function $W_a$ by

$$W_a(g) = W_a(g : \omega') = \mathcal{W}_a(R(g)\varphi_K : \omega'),\ g \in \tilde{G}.$$

Expanding $\varphi_K$ according to Lemma 6 we obtain

$$\begin{aligned}(W_\gamma(g:\omega'))_{\gamma\in\Gamma} &= \sum_w \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}\,(\mathcal{W}_\gamma(R(g)T_w\varphi_{w_0}(w^{-1}\omega') : \omega'))_{\gamma\in\Gamma} \\ &= \sum_w \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}\,(\mathcal{W}_\gamma(\omega') \circ T_w(R(g)\varphi_{w_0}(w^{-1}\omega')))_{\gamma\in\Gamma} \\ &= \sum_w \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}\,D_w(w^{-1}s)(\mathcal{W}_{\gamma'}(R(g)\varphi_{w_0} : w^{-1}\omega'))_{\gamma'\in\Gamma'}.\end{aligned} \tag{4.2}$$

Expanding (4.2) further we get that

$$W_a(g) = \sum_w \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)} \sum_{\gamma'\in\Gamma'} \tau_{a,\gamma'}(w,w^{-1}s)\mathcal{W}_{\gamma'}(R(g)\varphi_{w_0} : w^{-1}\omega'). \tag{4.3}$$

Since

$$W_a(\mathbf{s}(u)gk) = \psi_U(u)W_a(g),\ u \in U,\ g \in \tilde{G},\ k \in K^* \tag{4.4}$$

and $\tilde{G} = \mathbf{s}(U)\tilde{A}K^*$ it is enough to compute $W_a$ on $\tilde{A}$. Set

$$A^- = \{\mathrm{diag}(a_1,\dots,a_r) \in A : v(a_1) \geq v(a_2) \cdots \geq v(a_r)\}.$$

In the following lemma we compute the term $\mathcal{W}_a(R(b)\varphi_{w_0} : \omega')$ for every $a,\ b \in \tilde{A}$. As a consequence we get that $W_a|_{\tilde{A}}$ is supported on $\widetilde{A^-}$ and that for $g \in \widetilde{A^-}$ there is a unique

$\gamma'$ such that the inner summand in (4.3) associated to $\gamma'$ does not vanish. For $b \in \tilde{A}$ set

$$b^* = \mathbf{s}(w_0)b^{-1}\mathbf{s}(w_0)^{-1} \in \tilde{A}.$$

**Lemma 7.** *For $a, b \in \tilde{A}$ we have*

$$\mathcal{W}_a(R(b)\varphi_{w_0} : \omega') = \begin{cases} \chi_{2\rho}(b)(\chi_\rho\omega')(a(b^*)^{-1}) & b \in \widetilde{A^-} \text{ and } \tilde{A}_* a = \tilde{A}_* b^* \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We prove the lemma using the integral definition of $\mathcal{W}_a$ in the range of convergence. The lemma follows in the general case by meromorphic continuation. Note that for any $u \in U$ we have $a\,\mathbf{s}(w_0 u)\,b \in \tilde{B}_*\mathbf{s}(w_0)\mathcal{I}^*$ if and only if $a\,\mathbf{s}(w_0)\,b\,\mathbf{s}(w_0)^{-1} \in \tilde{A}_*$ and $b^{-1}ub \in U \cap K$. It further follows from (2.7) that $b\,\mathbf{s}(U)\,b^{-1} = \mathbf{s}(U)$. Since $\tilde{B}_*\mathbf{s}(w_0)\mathcal{I}^*$ is the support of $\varphi_{w_0}$ it follows that $\mathcal{W}_a(R(b)\varphi_{w_0} : \omega') = 0$ unless $a(b^*)^{-1} \in \tilde{A}_*$. If this is the case we have

$$\mathcal{W}_a(R(b)\varphi_{w_0} : \omega') = \int_{b(U\cap K)b^{-1}} \varphi_{w_0}(a\,\mathbf{s}(w_0 u)\,b)\psi_U(u)^{-1}\,du.$$

After a change of variable $u \mapsto bub^{-1}$, we apply (2.7) to obtain

$$\begin{aligned} \mathcal{W}_a(R(b)\varphi_{w_0}) &= \chi_{2\rho}(b)\int_{U\cap K} \varphi_{w_0}(a\,\mathbf{s}(w_0)\,b\,\mathbf{s}(u))\psi_U(\mathbf{p}(b)u\mathbf{p}(b)^{-1})^{-1}\,du \\ &= \chi_{2\rho}(b)\,\varphi_{w_0}(a\,\mathbf{s}(w_0)\,b)\int_{U\cap K} \psi_U(\mathbf{p}(b)u\mathbf{p}(b)^{-1})^{-1}\,du. \end{aligned}$$

The character $u \mapsto \psi_U(\mathbf{p}(b)u\mathbf{p}(b)^{-1})$ is trivial on $U \cap K$ if and only if $b \in \widetilde{A^-}$. It now further follows that $\mathcal{W}_a(R(b)\varphi_{w_0}) = 0$ unless $b \in \widetilde{A^-}$. If $b \in \widetilde{A^-}$ and $\tilde{A}_* a = \tilde{A}_* b^*$ then

$$\varphi_{w_0}(a\,\mathbf{s}(w_0)\,b) = \varphi_{w_0}(a(b^*)^{-1}\cdot\mathbf{s}(w_0)) = (\chi_\rho\omega')(a(b^*)^{-1})$$

since by definition $\varphi_{w_0}(\mathbf{s}(w_0)) = 1$. The lemma follows. □

We conclude this section with

**Theorem 1.** *For $a$, $b \in \tilde{A}$ we have $W_a(b : \omega') = 0$ unless $b \in \widetilde{A^-}$. In this case*

$$W_a(b : \omega') = \chi_{2\rho}(b)\sum_{w\in\mathfrak{W}} \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}\,\tau_{a,b^*}(w, w^{-1}s). \tag{4.5}$$

*Proof.* Apply Lemma 7 to (4.3) and choose $\Gamma'$ such that $b^* \in \Gamma'$. □

In the following section we review the functional equations of Kazhdan and Patterson which give a formula for the coefficients $\tau_{a,b^*}$ (see Proposition 1 below). Then in Sections 6, 7 and 8 we present analogous formulas for different sets of bases of spherical Whittaker functions.

## 5. The functional equation

The functional equation for the Whittaker functionals of normalized, unramified principal series representations was obtained by Kazhdan and Patterson [17, Lemma 1.3.3]. Minor corrections were pointed out in [1, Proposition 2.3] where they considered the special case $r = n - 1$. In particular, it was pointed out by Banks-Bump-Lieman that to justify the computation of the functional equation, it is essential to use, as we do, the block compatible 2-cocycle given by [2]. In this section we present the explicit description of the functional equations. That is, we provide formulas for the coefficients $\tau_{a,b}(w, s)$. The upshot is that (4.5) becomes an explicit formula for the spherical Whittaker functions.

Recall that $\Lambda = \mathbb{Z}^r$. Introduce the variables

$$y = (y_1, \dots, y_r) \text{ where } y_i = q^{-s_i}$$

with the $\mathfrak{W}$-action

$$wy = (y_{w^{-1}(1)}, \dots, y_{w^{-1}(r)}).$$

Set

$$y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_r^{\lambda_r} \text{ for } \lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda.$$

Note that $(wy)^\lambda = y^{w^{-1}\lambda}$. By abuse of notation, we further set $c_w(y) = c_w(s)$, $w \in \mathfrak{W}$. Thus,

$$c_w(y) = \prod_{\substack{i<j \\ w(i)>w(j)}} \frac{1 - q^{-1}(\frac{y_i}{y_j})^n}{1 - (\frac{y_i}{y_j})^n}.$$

Similarly, for $a$, $b \in \tilde{A}$ and $w \in \mathfrak{W}$ we write $\tau_{a,b}(w : y) = \tau_{a,b}(w, s)$. It follows from the equivariance (4.1) that for every $w \in \mathfrak{W}$, $a$, $a' \in \tilde{A}_*$ and $b$, $b' \in \tilde{A}$ we have

$$\tau_{ab,a'b'}(w : y) = \chi_\rho(a(a')^{-1})\,(w\omega')(a)\,\omega'(a')^{-1}\,\tau_{b,b'}(w : y). \tag{5.1}$$

For any $w_1$, $w_2 \in \mathfrak{W}$ and any sets of representatives $\Gamma$, $\Gamma'$, $\Gamma''$ for $\tilde{A}_*\backslash\tilde{A}$ the multiplicativity (3.17) of the intertwining operators implies the cocycle relation

$$D^{\Gamma,\Gamma''}_{w_1w_2}(s) = \frac{c_{w_1w_2}(s)}{c_{w_1}(w_2s)c_{w_2}(s)}\, D^{\Gamma,\Gamma'}_{w_1}(w_2s) D^{\Gamma',\Gamma''}_{w_2}(s). \tag{5.2}$$

It follows from (5.2) that

$$\tau_{a,b}(w_1w_2 : y) = \frac{c_{w_1w_2}(y)}{c_{w_1}(w_2y)c_{w_2}(y)} \sum_{\gamma\in\tilde{A}_*\backslash\tilde{A}} \tau_{a,\gamma}(w_1 : w_2y)\tau_{\gamma,b}(w_2 : y) \tag{5.3}$$

for any $a$, $b \in \tilde{A}$. Note that the summands are independent of coset representatives for $\gamma \in \tilde{A}_*\backslash\tilde{A}$. The property (5.3) reduces the computation of the coefficients $\tau_{a,b}(w : y)$ to that for $w$ a simple reflection. Define the sublattice $\Lambda_0$ of $\Lambda$ by

$$\Lambda_0 = \{(\lambda_1, \dots, \lambda_r) \in n_2\Lambda : \lambda_i - \lambda_{i+1} \equiv 0 \pmod n,\ i = 1, \dots, r-1\}.$$

For $a \in \tilde{A}$ such that $\mathbf{p}(a) = \mathrm{diag}(a_1, \dots, a_r)$ let

$$\mathfrak{f}(a) = (v(a_1), \dots, v(a_r)) \in \Lambda.$$

Then $a \in \tilde{A}_*$ if and only if $\mathfrak{f}(a) \in \Lambda_0$. For $k \in \mathbb{Z}/n\mathbb{Z}$ we denote by $\mathfrak{g}^\psi(k)$ the Gauss sum defined by

$$\mathfrak{g}^\psi(k) = \sum_{u \in \mathcal{O}^\times/1+\mathfrak{p}} \epsilon((u, \varpi^k))\psi(-\varpi^{-1}u).$$

For a simple reflection $w_\alpha$, $\alpha = (i, i+1) \in \Delta$ and $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda$ set

$$w_\alpha[\lambda] = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \dots, \lambda_r).$$

By composition, this defines an action $(w, \lambda) \mapsto w[\lambda]$ of $\mathfrak{W}$ on $\Lambda$ (the Coxeter relations are easily verified). We also define $\varpi^\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_r}) \in A$.

**Proposition 1** (Kazhdan-Patterson). *Let $\alpha = (i, i+1) \in \Delta$ and let $a$, $b \in \tilde{A}$. We have*

$$\tau_{a,b}(w_\alpha : y) = \tau^1_{a,b}(w_\alpha : y) + \tau^2_{a,b}(w_\alpha : y)$$

*where $\tau^i_{a,b}(w_\alpha : y)$ are defined by the following properties:*

$$\tau^i_{a_0a,b_0b}(w_\alpha : y) = \chi_\rho(a_0b_0^{-1})\,(w_\alpha\omega')(a_0)\,\omega'(b_0)^{-1}\,\tau^i_{a,b}(w_\alpha : y),\ \ a_0,\, b_0 \in \tilde{A}_*,\ i = 1, 2.$$

*For $\lambda$, $\lambda' \in \Lambda$ we have*

$$\tau^1_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda'})}(w_\alpha : y) = 0 \textit{ unless } \lambda - \lambda' \in \Lambda_0$$

$$\tau^2_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda'})}(w_\alpha : y) = 0 \textit{ unless } \lambda - w_\alpha[\lambda'] \in \Lambda_0,$$

$$\tau^1_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda})}(w_\alpha : y) = (1 - q^{-1})\,\frac{(y_{i+1}/y_i)^{n\lfloor\frac{\lambda_i - \lambda_{i+1}}{n}\rfloor}}{1 - (y_i/y_{i+1})^n},$$

*and*

$$\tau^2_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{w_\alpha[\lambda]})}(w_\alpha : y) = \epsilon(\varrho)^{\lambda_i\lambda_{i+1}}\,q^{\lambda_{i+1}-\lambda_i-2}\,\mathfrak{g}^\psi(\lambda_i - \lambda_{i+1} + 1).$$

## 6. A change of basis

In the previous two sections we gave an explicit formula for a particular basis of the space of spherical Whittaker functions by using the Casselman-Shalika method. In this section we translate these formulas into a new basis in order to facilitate the comparison with the local parts of Weyl group multiple Dirichlet series which is carried out in Section 9.

We identify the polynomial ring $\mathbb{C}[\Lambda]$ with $\mathbb{C}[y_1^{\pm 1}, \dots, y_r^{\pm 1}]$ and $\mathbb{C}[\Lambda_0]$ with the corresponding subring. Denote their respective fraction fields by $\mathbb{C}(\Lambda)$ and $\mathbb{C}(\Lambda_0)$. We note that by Proposition 1 and (5.3),

$$\tau_{a,b}(w : y) \in \mathbb{C}(\Lambda_0) \text{ for all } a,\, b \in \tilde{A},\ w \in \mathfrak{W}. \tag{6.1}$$

*Remark* 2. In place of the fraction fields $\mathbb{C}(\Lambda)$ and $\mathbb{C}(\Lambda_0)$ we could actually work over the smaller rings obtained by localizing $\mathbb{C}[\Lambda]$ and $\mathbb{C}[\Lambda_0]$ at the multiplicative set generated by

$$\left\{1 - \left(\frac{y_i}{y_j}\right)^n,\ 1 - q^{\pm 1}\left(\frac{y_i}{y_j}\right)^n : i < j\right\}.$$

However, this more refined information on the poles of the rational functions which arise will play no role in the sequel.

Let $\Xi$ be the dual of the finite group $\mathcal{L} := \Lambda_0\backslash\Lambda$. Under the group isomorphism $a \mapsto \mathfrak{f}(a) : \tilde{A}_*\backslash\tilde{A} \to \Lambda_0\backslash\Lambda$ we may consider elements of $\Xi$ as characters of $\tilde{A}_*\backslash\tilde{A}$ as well. We also let $\Xi$ act on $\mathbb{C}[\Lambda]$ by

$$\xi\left(\sum a_\lambda y^\lambda\right) = \sum a_\lambda \xi(\lambda) y^\lambda, \quad \xi \in \Xi. \tag{6.2}$$

This extends naturally to an action of $\Xi$ on $\mathbb{C}(\Lambda)$. For $l \in \mathcal{L}$ we define

$$\mathbb{C}(\Lambda)_l = \{f \in \mathbb{C}(\Lambda) : \xi \cdot f = \xi(l) f, \text{for all } \xi \in \Xi\}.$$

Then

$$\mathbb{C}(\Lambda) = \bigoplus_{l\in\mathcal{L}} \mathbb{C}(\Lambda)_l.$$

Note further that $\mathbb{C}(\Lambda)_0 = \mathbb{C}(\Lambda_0)$,

$$f_1 f_2 \in \mathbb{C}(\Lambda)_{l_1+l_2}, \quad \text{for } f_1 \in \mathbb{C}(\Lambda)_{l_1},\ f_2 \in \mathbb{C}(\Lambda)_{l_2} \tag{6.3}$$

and

$$f(wy) \in \mathbb{C}(\Lambda)_{w^{-1}l}, \quad \text{for } f \in \mathbb{C}(\Lambda)_l,\ w \in \mathfrak{W}. \tag{6.4}$$

In Section 4 we gave a basis for the space of spherical Whittaker functions parametrized by (a set of representatives) for $\tilde{A}_*\backslash\tilde{A}$. In this section we will give a basis parametrized by elements of the dual group $\Xi$.

For the remainder of this section we fix a set $R$ of representatives for $\Lambda_0\backslash\Lambda$ and the corresponding set $\Gamma = \{\mathbf{s}(\varpi^\mu) : \mu \in R\}$ of representatives for $\tilde{A}_*\backslash\tilde{A}$. Let $\mathfrak{X}(\omega') = \mathrm{ind}_{\tilde{A}_*}^{\tilde{A}}(\chi_{-\rho}(\omega')^{-1})$. An element of $\mathfrak{X}(\omega')$ is uniquely determined by its values on $\Gamma$. Let $\phi(y) \in \mathfrak{X}(\omega')$ be defined by

$$\phi(\mathbf{s}(\varpi^\mu) : y) = q^{\rho\cdot\mu}\, y^{-\mu}, \text{ for } \mu \in R.$$

(The dot product on $\mathbb{R}^r$ here and henceforth is the standard one.) It follows that

$$\phi(a : y) \in \mathbb{C}(\Lambda)_{-\mathfrak{f}(a)},\ \phi(a : y)^{-1} \in \mathbb{C}(\Lambda)_{\mathfrak{f}(a)}, \text{ for } a \in \tilde{A}. \tag{6.5}$$

Note that $\xi\phi(y) \in \mathfrak{X}(\omega')$ for $\xi \in \Xi$ and $\{\xi\, \phi(y) : \xi \in \Xi\}$ forms a basis for $\mathfrak{X}(\omega')$. Let

$$W_\xi(g : y) = \sum_{a\in\tilde{A}_*\backslash\tilde{A}} \xi(a)\phi(a : y)\, W_a(g : y) \tag{6.6}$$

where $W_a(g : y) = W_a(g, \omega')$. Then $\{W_\xi : \xi \in \Xi\}$ is a basis of spherical Whittaker functions.

**Theorem 2.** *We have $W_\xi(b : y) = 0$ unless $b \in \tilde{A}^-$. For $b \in \tilde{A}^-$,*

$$W_\xi(b : y) = \chi_{2\rho}(b) \sum_{w\in\mathfrak{W}} \frac{c_{w_0}(wy)}{c_{w^{-1}}(wy)} \sum_{a\in\tilde{A}_*\backslash\tilde{A}} \xi(a)\phi(a : y)\tau_{a,b^*}(w^{-1} : wy). \tag{6.7}$$

*We can alternatively write this as*

$$W_\xi(b : y) = \chi_{2\rho}(b) \sum_{w\in\mathfrak{W}} \frac{c_{w_0}(wy)}{c_{w^{-1}}(wy)} \sum_{\mu\in R} \xi(\mu) q^{\rho\cdot\mu}\, y^{-\mu} \tau_{\mathbf{s}(\varpi^\mu),b^*}(w^{-1} : wy), \quad b \in \tilde{A}^-.$$

*Proof.* The Theorem is obtained by applying Theorem 1 to (6.6) with $g = b$ and changing the order of summation. ☐

6.1. **Reinterpretation in terms of a Weyl group action.** We reinterpret Theorem 2 in terms of an action of the Weyl group $\mathfrak{W}$ on the field $\mathbb{C}(\Lambda)$. The purpose of this is two-fold: first, the group action will facilitate the construction of a symmetric basis in Section 7, and second, the action defined here will be related to that defined by Chinta and Gunnells [13] (see Section 9) in order to show that the spherical Whittaker functions of this paper arise as the local parts of Weyl group multiple Dirichlet series.

For $a$, $b \in \tilde{A}$ and $w \in \mathfrak{W}$ let

$$\tilde{\tau}_{a,b}(w : y) = \frac{1}{c_w(y)} \phi(a : wy)\, \phi(b : y)^{-1}\, \tau_{a,b}(w : y). \tag{6.8}$$

It follows from (5.1) that $\tilde{\tau}_{a_0 a, b_0 b}(w : y) = \tilde{\tau}_{a,b}(w : y)$, $a_0$, $b_0 \in \tilde{A}_*$. Introduce the normalized matrix

$$\tilde{D}_w(y) = (\tilde{\tau}_{a,b}(w : y))_{a,\, b \in \tilde{A}_* \backslash \tilde{A}}.$$

As a consequence of (5.2) we have

$$\tilde{D}_{w_1 w_2}(y) = \tilde{D}_{w_1}(w_2 y) \tilde{D}_{w_2}(y), \quad w_1,\, w_2 \in \mathfrak{W}. \tag{6.9}$$

It further follows from (6.1), (6.3), (6.4) and (6.5) that

$$\tilde{\tau}_{a,b}(w : y) \in \mathbb{C}(\Lambda)_{\mathfrak{f}(b) - w^{-1}\mathfrak{f}(a)}. \tag{6.10}$$

For $\lambda \in R$, $f \in \mathbb{C}(\Lambda)_\lambda$ and $w \in \mathfrak{W}$ we define

$$(f|w)(y) = \sum_{a \in \Gamma} \tilde{\tau}_{a, \mathbf{s}(\varpi^{-\lambda})}(w^{-1} : wy) f(wy). \tag{6.11}$$

Extend the definition of $(f|w)$ to all $f \in \mathbb{C}(\Lambda)$ by linearity.

**Proposition 2.** *The map $(f, w) \mapsto f|w$ defines an action of the group $\mathfrak{W}$ on $\mathbb{C}(\Lambda)$.*

*Proof.* We need to show that

$$(f|w_1 w_2) = (f|w_1)|w_2$$

for all $w_1, w_2 \in \mathfrak{W}$. Without loss of generality we may assume that $f \in \mathbb{C}(\Lambda)_\lambda$. To ease the notation in the proof, we will write $\tilde{\tau}_{\mathfrak{f}(a), \mathfrak{f}(b)}(w : y) = \tilde{\tau}_{a,b}(w : y)$, $a$, $b \in \tilde{A}$. Then

$$(f|w_1)(y) = \sum_{\mu' \in R} \tilde{\tau}_{\mu', -\lambda}(w_1^{-1} : w_1 y) f(w_1 y).$$

It follows from (6.3), (6.4) and (6.10) that the $\mu'$ term in the above sum lies in $\mathbb{C}(\Lambda)_{-\mu'}$. Therefore, letting $w_2$ act on this term and then summing over all $\mu'$ gives

$$((f|w_1)|w_2)(y) = \left[ \sum_{\mu' \in R} \sum_{\mu'' \in R} \tilde{\tau}_{\mu'', \mu'}(w_2^{-1} : w_2 y) \tilde{\tau}_{\mu', -\lambda}(w_1^{-1} : w_1 w_2 y) \right] f(w_1 w_2 y).$$

Interchange the order of summation and use (6.9) in the form

$$\tilde{D}_{w_2^{-1} w_1^{-1}}(w_1 w_2 y) = \tilde{D}_{w_2^{-1}}(w_2 y) \tilde{D}_{w_1^{-1}}(w_1 w_2 y)$$

to rewrite the equation above as

$$((f|w_1)|w_2)(y) = \left[\sum_{\mu''\in R} \tilde{\tau}_{\mu'',-\lambda}(w_2^{-1}w_1^{-1} : w_1w_2y)\right] f(w_1w_2y).$$

But this is exactly $(f|w_1w_2)(y)$. □

Let $\xi_0 \in \Xi$ be the trivial character. It follows from (6.7) that

$$W_{\xi_0}(b : y) = \chi_{2\rho}(b) \sum_{w\in\mathfrak{W}} c_{w_0}(wy) \sum_{a\in\tilde{A}_*\backslash\tilde{A}} \phi(b^* : wy)\tilde{\tau}_{a,b^*}(w^{-1} : wy), \quad b \in \tilde{A}^-.$$

Taking (6.5) and the definition (6.11) of the group action into account we get that

$$W_{\xi_0}(b : y) = \chi_{2\rho}(b) \sum_{w\in\mathfrak{W}} c_{w_0}(wy)[\phi(b^* : \cdot)|\, w](y), \quad b \in \tilde{A}^-. \tag{6.12}$$

Recall that a spherical Whittaker function is determined by its values on $\tilde{U}\backslash\tilde{G}/K^*$. We now determine a complete set of representatives $\{d_\mu \in \tilde{A} : \mu \in \Lambda\}$ in such a way that $\phi(d_\mu^* : y)$ is a monomial function that we explicate for each $\mu$. Given $\mu \in \Lambda$ write $-\mu = \nu + n\lambda_1 + kn_2\mathbb{I}_r$, where $\nu \in R, \lambda_1 \in \Lambda$ and $0 \le k < n_1$. Since $n\lambda_1 + kn_2\mathbb{I}_r \in \Lambda_0$, the $\lambda_1, \nu$ and $k$ are all uniquely determined by $\mu$. When necessary we think of them as functions of $\mu$ (dependent of course on our choice of $R$) and denote $\lambda_1 = \lambda_1(\mu), \nu = \nu(\mu), k = k(\mu)$. Define $d_\mu$ to be the element of $\tilde{A}$ such that

$$d_\mu^* = \mathbf{s}(\varpi^{n_2}e)^k\mathbf{s}(\varpi^{n\lambda_1})\mathbf{s}(\varpi^\nu).$$

Following the definitions we have

$$\phi(d_\mu^* : y) = \zeta_0^k q^{-\rho\cdot\mu}y^\mu. \tag{6.13}$$

Note that $\mathfrak{f}(d_\lambda) = w_0\lambda$. Hence, indeed, $\{d_\lambda : \lambda \in \Lambda\}$ is a set of representatives for $\tilde{U}\backslash\tilde{G}/K^*$ and $d_\lambda \in \widetilde{A^-}$ if and only if $\lambda \in \Lambda^- = \{(\lambda_1, \dots, \lambda_r) \in \Lambda : \lambda_1 \le \lambda_2 \le \dots \le \lambda_r\}$. Henceforth, for $\mu \in \Lambda$ and $w \in \mathfrak{W}$ we denote $(f|\, w)(y)$ by $(y^\mu|\, w)$ when $f(y) = y^\mu$. (In Section 9 we also use a similar convention with respect to another action of $\mathfrak{W}$ without further mention.)

**Corollary 1.** *Let $\mu \in \Lambda^-$. Then*

$$W_{\xi_0}(d_\mu : y) = \zeta_0^{k(\mu)}\, q^{\rho\cdot\mu} \sum_{w\in\mathfrak{W}} c_{w_0}(wy)(y^\mu|\, w).$$

*In particular, $[W_{\xi_0}(g : \cdot)|\, w](y) = W_{\xi_0}(g : y)$ for all $w \in \mathfrak{W}$ and $g \in G$.*

*Proof.* Note that $\chi_{2\rho}(d_\mu) = q^{2\rho\cdot\mu}$. The Corollary follows from (6.12) and (6.13). □

For every $\mu \in \Lambda$ we have $\mu = \nu(-\mu) + n\lambda_1(-\mu) + k(-\mu)n_2\mathbb{I}_r$. There exists therefore $m(\mu) \in \{0, 1\}$ such that

$$\mathbf{s}(\varpi^\mu) = \iota(\varrho)^{m(\mu)}\, \mathbf{s}(\varpi^{n_2}e)^{k(-\mu)}\, \mathbf{s}(\varpi^{n\lambda_1(-\mu)})\, \mathbf{s}(\varpi^{\nu(-\mu)}).$$

(In fact, $m(\mu)$ can be computed explicitly using (2.5) and (3.6).) For later use we observe that

$$\phi(\mathbf{s}(\varpi^\mu) : y) = \epsilon(\varrho)^{m(\mu)} \zeta_0^{k(-\mu)} q^{\rho\cdot\mu} y^{-\mu}. \tag{6.14}$$

## 7. A symmetric basis of Whittaker functions

The purpose of this section is to construct a basis of spherical Whittaker functions so that, as in the $n = 1$ case, their values on $\tilde{G}$ are symmetric rational functions in the complex parameter $y$. In light of the functional equation satisfied by Eisenstein series (e.g. [24, Theorem IV.1.10]), such a basis should be of interest for the global theory.

Let $f \mapsto \bar{f}$ for $f \in \mathbb{C}(\Lambda)$ denote complex conjugation (i.e. $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$ and $\bar{y}_i = y_i$, $i = 1, \dots, r$). We further denote by $X^\star = {}^t\overline{X}$ the conjugate-transpose of any matrix $X$ with entries in $\mathbb{C}(\Lambda)$. Finally, we set

$$\hat{y} = (\hat{y}_1, \dots, \hat{y}_r) \text{ where } \hat{y}_i = \frac{y_1 y_2 \cdots y_r}{y_i}.$$

We continue to let

$$W_{\xi_0}(g : y) = \sum_{a \in \Gamma} \phi(a : y)\, W_a(g : y)$$

be the spherical Whittaker function associated to the trivial character $\xi_0$, as defined in the previous section. For $g \in \tilde{G}$ let $v_g(y)$ be the column vector

$$v_g(y) = (\phi(b : y) W_b(g : y))_{b \in \Gamma}$$

and let $V_g$ be the spherical Whittaker function whose value at $g' \in \tilde{G}$ is given by

$$V_g(g' : y) = v_g(\hat{y})^\star \cdot v_{g'}(y).$$

Clearly $V_{g_0}(g : \cdot) \in \mathbb{C}(\Lambda)$ for every $g_0$, $g \in \tilde{G}$. Since $\{\phi(a : y)\, W_a(\cdot : y) : a \in \Gamma\}$ forms a basis of spherical Whittaker functions, there is a subset $I = \{g_a : a \in \Gamma\} \subseteq \tilde{G}$ of $|\Gamma|$ elements such that $\{v_{g_a} : a \in \Gamma\}$ are linearly independent over $\mathbb{C}(\Lambda)$. The set $\{V_{g_a}(\cdot : y) : a \in \Gamma\}$ is then a basis of spherical Whittaker functions. The next theorem asserts that this basis consists of symmetric functions in $y$.

**Theorem 3.** *For every $g_0$, $g \in \tilde{G}$ and $w \in \mathfrak{W}$ we have*

$$V_{g_0}(g : wy) = V_{g_0}(g : y).$$

The rest of this section is dedicated to the proof of Theorem 3. It requires a certain symmetry satisfied by the matrix $\tilde{D}_w(y)$ that we pursue first.

The following proposition is merely rewriting Proposition 1 in terms of the normalized coefficients taking (6.14) into consideration.

**Proposition 3.** *Let $\alpha = (i, i+1) \in \Delta$ and let $a$, $b \in \tilde{A}$. We have*

$$\tilde{\tau}_{a,b}(w_\alpha : y) = \tilde{\tau}^1_{a,b}(w_\alpha : y) + \tilde{\tau}^2_{a,b}(w_\alpha : y)$$

*where $\tilde{\tau}^i_{a,b}(w_\alpha : y)$, $i = 1, 2$ are characterized by the following properties:*

$$\tilde{\tau}^i_{a_0 a, b_0 b}(w_\alpha : y) = \tilde{\tau}^i_{a,b}(w_\alpha : y),\ a_0,\, b_0 \in \tilde{A}_*,\ i = 1, 2. \tag{7.1}$$

*For* $\lambda,\ \lambda' \in \Lambda$ *we have*

$$\tilde{\tau}^1_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda'})}(w_\alpha : y) = 0 \text{ unless } \lambda - \lambda' \in \Lambda_0 \tag{7.2}$$

$$\tilde{\tau}^2_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda'})}(w_\alpha : y) = 0 \text{ unless } \lambda - w_\alpha[\lambda'] \in \Lambda_0, \tag{7.3}$$

$$\tilde{\tau}^1_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda})}(w_\alpha : y) = \frac{1-q^{-1}}{c_{w_\alpha}(y)}\,\frac{(y_{i+1}/y_i)^{n\lfloor\frac{\lambda_i-\lambda_{i+1}}{n}\rfloor}}{1-(y_i/y_{i+1})^n}\ (y_i/y_{i+1})^{\lambda_i-\lambda_{i+1}}, \tag{7.4}$$

*and*

$$\begin{aligned}&\tilde{\tau}^2_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{w_\alpha[\lambda]})}(w_\alpha : y)\\ &\quad= \epsilon(\varrho)^{m(w_\alpha[\lambda])-m(\lambda)+\lambda_i\lambda_{i+1}}\,\zeta_0^{k(-w_\alpha[\lambda])-k(-\lambda)}\,\mathfrak{g}^\psi(\lambda_i-\lambda_{i+1}+1)\,\frac{q^{-1}}{c_{w_\alpha}(y)}\,\frac{y_{i+1}}{y_i}.\end{aligned} \tag{7.5}$$

As a consequence of Proposition 3 we deduce the following symmetries of $\tilde{D}_w(y)$.

**Lemma 8.** *For any simple root* $\alpha = (i, i+1) \in \Delta$ *we have*

$$\tilde{D}_{w_\alpha}(\hat{y}) = \tilde{D}_{w_\alpha}(y)^{-1} \tag{7.6}$$

*and*

$$\tilde{D}_{w_\alpha}(y)^\star = \tilde{D}_{w_\alpha}(y). \tag{7.7}$$

*Proof.* It follows from Proposition 3 that all entries of $\tilde{D}_{w_\alpha}(y)$ are rational functions in $y_i/y_{i+1}$ independent of $y_j$ for $j \neq i,\ i+1$. If $y' = (y'_1,\dots,y'_r) = w_\alpha y$ and $y'' = (y''_1,\dots,y''_r) = \hat{y}$ then $y'_i/y'_{i+1} = y''_i/y''_{i+1}$ and therefore $\tilde{D}_{w_\alpha}(\hat{y}) = \tilde{D}_{w_\alpha}(w_\alpha y)$. But by (6.9) we have $\tilde{D}_{w_\alpha}(w_\alpha y) = \tilde{D}_{w_\alpha}(y)^{-1}$ and (7.6) follows. To show (7.7), by (7.1), (7.2) and (7.3) it is enough to show for $\lambda \in \Lambda$ that

$$\overline{\tilde{\tau}^1_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda})}(w_\alpha : y)} = \tilde{\tau}^1_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{\lambda})}(w_\alpha : y) \tag{7.8}$$

and

$$\overline{\tilde{\tau}^2_{\mathbf{s}(\varpi^{w_\alpha[\lambda]}),\mathbf{s}(\varpi^{\lambda})}(w_\alpha : y)} = \tilde{\tau}^2_{\mathbf{s}(\varpi^\lambda),\mathbf{s}(\varpi^{w_\alpha[\lambda]})}(w_\alpha : y). \tag{7.9}$$

The equality (7.8) is straightforward from (7.4). From (7.5) we get that

$$\begin{aligned}&\overline{\tilde{\tau}^2_{\mathbf{s}(\varpi^{w_\alpha[\lambda]}),\mathbf{s}(\varpi^{\lambda})}(w_\alpha : y)}\\ &\quad= (\overline{\epsilon(\varrho)})^{m(\lambda)-m(w_\alpha[\lambda])+(\lambda_{i+1}-1)(\lambda_i+1)}\,(\overline{\zeta_0})^{k(-\lambda)-k(-w_\alpha[\lambda])}\,\overline{\mathfrak{g}^\psi(\lambda_{i+1}-\lambda_i-1)}\,\frac{q^{-1}}{c_{w_\alpha}(y)}\,\frac{y_{i+1}}{y_i}.\end{aligned} \tag{7.10}$$

Recall that $\epsilon(\varrho) = \overline{\epsilon(\varrho)} \in \{\pm 1\}$ and $\zeta_0^{-1} = \overline{\zeta_0} \in \mu_{2n_1}(\mathbb{C})$. Furthermore, for $k \not\equiv 0 \mod n$, the Gauss sum satisfies $\mathfrak{g}^\psi(k)\,\mathfrak{g}^\psi(-k) = q\,\epsilon(\varrho)^k$ and $\left|\mathfrak{g}^\psi(k)\right| = q^{\frac{1}{2}}$ and therefore, for every $k \in \mathbb{Z}$ we have

$$\overline{\mathfrak{g}^\psi(k)} = \epsilon(\varrho)^k\ \mathfrak{g}^\psi(-k).$$

Thus, $\overline{\epsilon(\varrho)^{\lambda_{i+1}-\lambda_i-1}\,\mathfrak{g}^\psi(\lambda_{i+1}-\lambda_i-1)} = \mathfrak{g}^\psi(\lambda_i-\lambda_{i+1}+1)$ and (7.9) follows from (7.10). The rest of the lemma readily follows. ◻

**Corollary 2.** *For every $w \in \mathfrak{W}$ we have*

$$\tilde{D}_w(\hat{y})^\star = \tilde{D}_w(y)^{-1}.$$

*Proof.* If $\ell(w) = 1$ the identity follows from the two identities of Lemma 8. For general $w$ note that $w\hat{y} = \widehat{wy}$. Applying (6.9) the corollary therefore follows by induction on $\ell(w)$. $\square$

*Proof of Theorem 3.* To start let $g \in \tilde{A}$. By virtue of the fact that

$$W_{\xi_0}(g : y) = \sum_{b \in \Gamma} \phi(b : y)\, W_b(g : y) \tag{7.11}$$

is invariant under the action $|w$ for $w \in \mathfrak{W}$ (Corollary 1), (4.5), (6.1), (6.5) and the definition of the action $|w$ we get that

$$W_{\xi_0}(g : y) = \sum_{b \in \Gamma} \sum_{a \in \Gamma} \tilde{\tau}_{a,b}(w^{-1} : wy) \phi(b : wy) W_b(g : wy). \tag{7.12}$$

Equate the terms on the righthand sides of (7.11) and (7.12) which lie in $\mathbb{C}(\Lambda)_{-\lambda}$ to conclude that

$$\phi(\mathbf{s}(\varpi^\lambda) : y) W_{\mathbf{s}(\varpi^\lambda)}(g : y) = \sum_{b \in \Gamma} \tilde{\tau}_{\mathbf{s}(\varpi^\lambda),b}(w^{-1} : wy) \phi(b : wy) W_b(g : wy) \tag{7.13}$$

for all $\lambda \in R$. In matrix notation this is saying precisely that

$$v_g(y) = \tilde{D}_{w^{-1}}(wy) v_g(wy) \text{ or equivalently, } v_g(wy) = \tilde{D}_{w^{-1}}(wy)^{-1} v_g(y).$$

This was all assuming that $g \in \tilde{A}$, but by (4.4), this identity holds for all $g \in \tilde{G}$. Therefore, by Corollary 2 we have

$$\begin{aligned} V_{g_0}(g : wy) &= v_{g_0}(\widehat{wy})^\star\ v_g(wy) \\ &= v_{g_0}(\hat{y})^\star\ (\tilde{D}_{w^{-1}}(\widehat{wy})^\star)^{-1}\, \tilde{D}_{w^{-1}}(wy)^{-1}\, v_g(y) \\ &= v_{g_0}(\hat{y})^\star\ \tilde{D}_{w^{-1}}(wy)\, \tilde{D}_{w^{-1}}(wy)^{-1}\, v_g(y) \\ &= V_{g_0}(g : y) \end{aligned}$$

and the theorem follows. $\square$

## 8. Preparation for comparison

The purpose of this section is to give a more explicit description of the terms $(y^\lambda|w)$ which appear in Corollary 1. This description will be used in the following section to prove Theorem 4 which gives the precise relationship between the spherical Whittaker functions and the local parts of the type $A$ Weyl group multiple Dirichlet series defined in [13].

Assume in this section and the next that $-1$ is an $n$th root of unity in $F$ or equivalently that $\varrho = 1$. Note that in this case the property (3.4) simply says that $\zeta_0$ is an $n_1$th root of unity and in particular an $n$th root of unity. It follows from Lemma 2 that the variables $y_i$ are only determined by $\omega$ up to an $n$th root of unity and therefore that every $\epsilon$-genuine, normalized unramified character of $\tilde{A}^n\tilde{Z}$ is of the form $\omega_{s,1}$ for some $s \in \mathbb{C}^r$. We may and do therefore assume, through this and the next section, that $\zeta_0 = 1$. Under our assumption that $\varrho = 1$ we also have

$$\mathbf{s}(\varpi^{\lambda})\mathbf{s}(\varpi^{\lambda'}) = \mathbf{s}(\varpi^{\lambda+\lambda'}),\ \lambda,\ \lambda' \in \Lambda.$$

Thus, $d_\lambda^* = \mathbf{s}(\varpi^{-\lambda})$ for all $\lambda \in \Lambda$. It further follows from (6.14) that $\phi(\mathbf{s}(\varpi^{\lambda}) : y) = q^{\rho\cdot\lambda}\, y^{-\lambda}$ for every $\lambda \in \Lambda$.

For $w \in \mathfrak{W}$ define

$$T_w(\lambda : y) = q^{\rho\cdot\lambda}\, c_{w_0}(wy)(y^{\lambda}|\ w). \tag{8.1}$$

By Corollary 1 we have

$$W_{\xi_0}(d_\lambda : y) = \sum_{w\in\mathfrak{W}} T_w(\lambda : y).$$

Expanding (8.1) and applying (6.13) we have,

$$T_w(\lambda : y) = q^{\rho\cdot\lambda}\, c_{w_0}(wy)\, (wy)^{\lambda} \sum_{\mu\in R} \tilde{\tau}_{\mathbf{s}(\varpi^{-\mu}),\mathbf{s}(\varpi^{-\lambda})}(w^{-1} : wy).$$

Note that

$$T_e(\lambda : y) = q^{\rho\cdot\lambda}\, c_{w_0}(y)\, y^{\lambda}$$

and for a simple reflection $w_\alpha$ with $\alpha = (i, i+1)$, Proposition 3 implies that

$$\begin{aligned} T_{w_\alpha}(\lambda : y) &= q^{\rho\cdot\lambda}\, c_{w_0}(w_\alpha y)\, y^{w_\alpha\lambda} \times \left[\tilde{\tau}^1_{\mathbf{s}(\varpi^{-\lambda}),\mathbf{s}(\varpi^{-\lambda})}(w_\alpha : w_\alpha y) + \tilde{\tau}^2_{\mathbf{s}(\varpi^{w_\alpha[-\lambda]}),\mathbf{s}(\varpi^{-\lambda})}(w_\alpha : w_\alpha y)\right] \\ &= q^{\rho\cdot\lambda}\, \frac{c_{w_0}(w_\alpha y)}{c_{w_\alpha}(w_\alpha y)} y^{w_\alpha\lambda} \left[(1-q^{-1})\, \frac{(\frac{y_i}{y_{i+1}})^{n\lfloor\frac{\lambda_{i+1}-\lambda_i}{n}\rfloor-(\lambda_{i+1}-\lambda_i)}}{1-(\frac{y_{i+1}}{y_i})^n} + \right. \\ &\qquad \left. q^{-1}\, \mathfrak{g}^{\psi}(\lambda_i - \lambda_{i+1} - 1)\frac{y_i}{y_{i+1}}\right]. \end{aligned}$$

Note that

$$\frac{c_{w_0}(w_\alpha y)}{c_{w_\alpha}(w_\alpha y)} = \frac{c_{w_0}(y)}{c_{w_\alpha}(y)}$$

and recall that

$$c_{w_\alpha}(y) = \frac{1-q^{-1}(\frac{y_i}{y_{i+1}})^n}{1-(\frac{y_i}{y_{i+1}})^n} = -(\frac{y_{i+1}}{y_i})^n\, \frac{1-q^{-1}(\frac{y_i}{y_{i+1}})^n}{1-(\frac{y_{i+1}}{y_i})^n}.$$

We introduce the notation $(k)_n = k - n\lfloor\frac{k}{n}\rfloor$ for every $k \in \mathbb{Z}$. Thus $0 \le (k)_n < n$. In the following proposition we record the last expression we obtained for $T_{w_\alpha}(\lambda : y)$ using this notation.

**Proposition 4.** *For a simple reflection $w_\alpha$ with $\alpha = (i, i+1)$ we have*

$$T_{w_\alpha}(\lambda : y) = q^{\rho\cdot\lambda}\, c_{w_0}(w_\alpha y)(y^\lambda | w_\alpha)$$
$$= -q^{\rho\cdot\lambda} \frac{\left(\frac{y_i}{y_{i+1}}\right)^n c_{w_0}(y)}{1 - q^{-1}\left(\frac{y_i}{y_{i+1}}\right)^n}\, y^{w_\alpha\lambda} \left[(1-q^{-1})\left(\frac{y_i}{y_{i+1}}\right)^{-(\lambda_{i+1}-\lambda_i)_n}\right.$$
$$\left. - q^{-1}\,\mathfrak{g}^\psi(\lambda_i - \lambda_{i+1} - 1)\left(\frac{y_i}{y_{i+1}}\right)^{1-n}\left(1 - \left(\frac{y_i}{y_{i+1}}\right)^n\right)\right].$$

## 9. The $\mathfrak{p}$-part of a Weyl group multiple Dirichlet series

In this section we review the Chinta-Gunnells [13] construction of the local part of a Weyl group multiple Dirichlet series associated to the root system $A_{r-1}$. We continue to assume that $-1$ is an $n$th root of unity. We begin by defining a group action of the Weyl group $\mathfrak{W}$ on the field of rational functions. We denote this action by $\|$ in order to distinguish it from the action introduced in (6.11) above and the action of [13] introduced below. For $\lambda \in R$, $f \in \mathbb{C}(\Lambda)_\lambda$ and $w_\alpha$ the simple reflection associated to $\alpha = (i, i+1)$, we define

$$(f\|w_\alpha)(y) = \frac{f(w_\alpha y)}{1 - q^{-1}\left(\frac{y_i}{y_{i+1}}\right)^n}\left[(1-q^{-1})\left(\frac{y_i}{y_{i+1}}\right)^{-(\lambda_{i+1}-\lambda_i)_n}\right.$$
$$\left. -q^{-1}\mathfrak{g}^\psi(\lambda_i - \lambda_{i+1} - 1)\left(\frac{y_i}{y_{i+1}}\right)^{1-n}\left(1 - \left(\frac{y_i}{y_{i+1}}\right)^n\right)\right]. \tag{9.1}$$

Having defined $f\|w_\alpha$ for $f \in \mathbb{C}(\Lambda)_\lambda$, we extend the definition to $\mathbb{C}(\Lambda)$ by linearity. These definitions extend to give an action of the entire Weyl group $\mathfrak{W}$ on $\mathbb{C}(\Lambda)$, see [13, Theorem 3.2]. Actually, some changes of variable are necessary to relate the action defined here to that of [13]. In order to precisely describe the relation between the two actions, we denote the action of [13] with twisting parameter $\ell = (l_2, \ldots, l_r)$ by $|_{\ell,WMD}$. The action of [13] is actually only defined on the localization of the ring $\mathbb{C}[x_1, \ldots, x_{r-1}]$ at the multiplicative set generated by the polynomials $1 - x_i^n, 1 - q^{\pm 1}x_i^n$ for $i = 1, \ldots, r-1$, but is easily seen to extend to the field $\mathbb{C}(x_1, \ldots, x_{r-1})$. (We make the change of variables $x_i \mapsto q^{-1}x_i$ in order to eliminate some extraneous powers of $q$.) Let $F \in \mathbb{C}(x_1, \ldots, x_{r-1})$ be a rational function. Define $f \in \mathbb{C}(\Lambda)$ by $f(y_1, \ldots, y_r) = F(\frac{y_1}{y_2}, \ldots, \frac{y_{r-1}}{y_r})$. Then, letting $\lambda = (0, l_2, l_2 + l_3, \ldots, l_2 + \cdots + l_r)$,

$$(F|_{\ell,WMD}w)\left(\frac{y_1}{y_2}, \ldots, \frac{y_{r-1}}{y_r}\right) = y^{-\lambda}(y^\lambda f\|w)(y_1, \ldots, y_r). \tag{9.2}$$

To verify (9.2), it suffices to do so for $w$ a simple reflection acting on monomials. This follows from a direct comparison of (9.1) with Eq. (3.14) of [13]. To make the comparison, recall that we have made the substitutions $x_i \mapsto x_i/q$. Further, the Gauss sum we use here is the conjugate of that used in [13]. Then, using the equation $(a+1)_n = (a)_n + 1$ for $a$

not congruent to $-1$ mod $n$ and arguing separately in the two cases $d_i(\alpha) - 2k_i + l_i + 1$ is zero or nonzero mod $n$ (notation being of [ibid.]), we easily arrive at (9.2).

Let us write $c^{(2)}(y) = \prod_{i<j} \left(1 - \left(\frac{y_i}{y_j}\right)^n\right)$ and for $w$ in the Weyl group $\mathfrak{W}$, define $j(w, y) = c^{(2)}(y)/c^{(2)}(wy)$. It is proved in Section 3 of [13] that

$$N(y;\ell) = y^{-\lambda} c_{w_0}(y) \sum_{w \in \mathfrak{W}} j(w, y)(y^\lambda \| w)$$

is a polynomial in the $y_i/y_{i+1}$. These polynomials are used in [13] to construct Weyl group multiple Dirichlet series.

We now turn to the main result of this section — the comparison of $N(y;\ell)$ with the Whittaker function $W_{\xi_0}(d_\lambda : y)$.

**Theorem 4.** *Let $\lambda = (0, l_2, l_2 + l_3, \ldots, l_2 + \cdots + l_r)$ and $\ell = (l_2, \ldots, l_r)$ with $l_i$ non-negative integers. We have*

$$y^\lambda N(y;\ell) = q^{-\rho\cdot\lambda} W_{\xi_0}(d_\lambda : y).$$

*Proof.* By Theorem 2 we need to prove that

$$\sum_{w \in \mathfrak{W}} c_{w_0}(y) j(w, y)(y^\lambda \| w) = \sum_{w \in \mathfrak{W}} c_{w_0}(wy)(y^\lambda | w).$$

We will show that the sums match up term by term, that is, that

$$j(w, y)(y^\lambda \| w) = \frac{c_{w_0}(wy)}{c_{w_0}(y)}(y^\lambda | w) \tag{9.3}$$

for all $w$ in the Weyl group. Since both $f \mapsto j(w, y)(f \| w)(y)$ and $f \mapsto \frac{c_{w_0}(wy)}{c_{w_0}(y)}(f|w)(y)$ give actions of $\mathfrak{W}$ on $\mathbb{C}(\Lambda)$, it suffices to verify (9.3) for all simple reflections $w_\alpha$. Note that $j(w_\alpha, y) = -(\frac{y_i}{y_{i+1}})^n$ and therefore this follows easily by comparing Proposition 4 to the definition (9.1).

This completes the proof of the theorem. □

*Remark* 3. In the course of proving (9.3), we have in fact shown

$$j(w, y)(f \| w) = \frac{c_{w_0}(wy)}{c_{w_0}(y)}(f|w)(y)$$

for all $f \in \mathbb{C}(\Lambda)$ and all $w \in \mathfrak{W}$, because both actions are extended from monomials to $\mathbb{C}(\Lambda)$ in the same manner.

## 10. Zonal spherical functions of $\tilde{G}$

Let $C^{\infty,\epsilon}(K^*\backslash\tilde{G}/K^*)$ be the space of $\epsilon$-genuine bi $K^*$-invariant functions on $\tilde{G}$. The action of $\mathcal{H}^\epsilon(\tilde{G}, K^*)$ on $C^{\infty,\epsilon}(\tilde{G}/K^*)$ given by (2.10) restricts to an action on the subspace $C^{\infty,\epsilon}(K^*\backslash\tilde{G}/K^*)$.

**Definition 3.** A function $\Omega \in C^{\infty,\epsilon}(K^*\backslash\tilde{G}/K^*)$ is an $\epsilon$-genuine, zonal spherical function on $\tilde{G}$ if it is a common eigenfunction of $\mathcal{H}^\epsilon(\tilde{G}, K^*)$.

Recall from (3.2) that $\chi_{2\rho}$ is the modulus function of $\tilde{B}$. We wish to relate between two different ways to express a $\tilde{G}$-invariant linear form on the space of functions $f : \tilde{G} \to \mathbb{C}$ that satisfy

$$f(bg) = \chi_{2\rho}(b) f(g),\ b \in \tilde{B},\ g \in \tilde{G}. \tag{10.1}$$

Of course the space of such linear forms is one dimensional. Since $\tilde{B}\backslash\tilde{G} \simeq B\backslash G$ we can use the well known formula for $G$ (cf. [18]) to obtain that

$$f \mapsto \int_{K^*} f(k)\ dk = \frac{\prod_{i=1}^r L(i)}{L(1)^r} \int_U f(\mathbf{s}(w_0 u))\ du$$

is $\tilde{G}$-invariant.

For $s \in \mathbb{C}^r$ and $\zeta \in \mu_{2n_1}(\mathbb{C})$ satisfying (3.4) let $\omega = \omega_{s,\zeta}$ be the $\epsilon$-genuine, unramified, normalized character of $\tilde{A}^n \tilde{Z}$ associated to $(s, \zeta)$ by Lemma 2 and let $\omega'$ be its canonical extension. Note that $I((\omega')^{-1})$ is an $\epsilon^{-1}$-genuine, normalized unramified principal series representation contragradient to $I(\omega')$. Indeed, note that for any $\varphi \in I(\omega')$, $\tilde{\varphi} \in I((\omega')^{-1})$ the function

$$g \mapsto \sum_{\gamma \in \tilde{A}_* \backslash \tilde{A}} \chi_{-2\rho}(\gamma) \varphi(\gamma g) \tilde{\varphi}(\gamma g)$$

is well-defined (independent of a choice of representatives $\gamma$) and satisfies the equivariance condition (10.1). A $\tilde{G}$-invariant pairing is therefore given by

$$\langle \varphi, \tilde{\varphi} \rangle = \sum_{\gamma \in \tilde{A}_* \backslash \tilde{A}} \chi_{-2\rho}(\gamma) \int_{K^*} \varphi(\gamma k) \tilde{\varphi}(\gamma k)\ dk \tag{10.2}$$

$$= \frac{\prod_{i=1}^r L(i)}{L(1)^r} \sum_{\gamma \in \tilde{A}_* \backslash \tilde{A}} \chi_{-2\rho}(\gamma) \int_U \varphi(\gamma\, \mathbf{s}(w_0 u)) \tilde{\varphi}(\gamma\, \mathbf{s}(w_0 u))\ du. \tag{10.3}$$

Let $\Omega_s = \Omega^{\epsilon}_{s,\zeta}$ be the $\epsilon$-genuine, zonal spherical function on $\tilde{G}$ defined by

$$\Omega_s(g) = \left\langle R(g) \varphi_K(\omega'), \varphi_K({\omega'}^{-1}) \right\rangle.$$

It can be expressed as

$$\Omega_s(g) = \mathcal{V}_s(R(g) \varphi_K(\omega'))$$

where $\mathcal{V}_s \in (I(\omega'))^*$ is the linear form defined by

$$\mathcal{V}_s(\varphi) = \left\langle \varphi, \varphi_K({\omega'}^{-1}) \right\rangle.$$

For every $w \in \mathfrak{W}$ the linear form $\mathcal{V}_s \circ T_w$ on $I(w^{-1}\omega')$ is then $K^*$-invariant. Since

$$\dim (I(\omega')^*)^{K^*} = \dim I({\omega'}^{-1})^{K^*} = 1,$$

there is a scalar $d_w(s)$ such that

$$\mathcal{V}_s \circ T_w = d_w(s)\ \mathcal{V}_{w^{-1}s}. \tag{10.4}$$

The scalar $d_w(s)$ can easily be computed by evaluating both sides of (10.4) at $\varphi_K(w^{-1}\omega')$. On the one hand, since $\varphi_K$ is supported on $\tilde{B}_*K^*$ and its value on $K^*$ equals 1 we get from the definition (10.2) of the inner product that

$$\mathcal{V}_{w^{-1}s}(\varphi_K(w^{-1}\omega')) = 1.$$

On the other hand applying (3.16) as well we get that

$$\mathcal{V}_s \circ T_w(\varphi_K(w^{-1}\omega')) = c_w(w^{-1}s)\ \mathcal{V}_s(\varphi_K(\omega')) = c_w(w^{-1}s).$$

Thus,

$$d_w(s) = c_w(w^{-1}s).$$

We now expand $\varphi_K$ as in Lemma 6. We obtain that

$$\Omega_s(g) = \sum_{w\in\mathfrak{W}} \frac{c_{w_0}(w^{-1}s)}{c_w(w^{-1}s)}\ \mathcal{V}_s \circ T_w(R(g)\varphi_{w_0}) = \sum_{w\in\mathfrak{W}} c_{w_0}(w^{-1}s)\mathcal{V}_{w^{-1}s}(R(g)\varphi_{w_0}). \tag{10.5}$$

We are now ready to compute the zonal spherical functions explicitly. The Cartan decomposition of $G$ implies that

$$\tilde{G} = K^*\tilde{A}K^*.$$

Note first that if $f$ is any $\epsilon$-genuine and bi $K^*$-invariant function on $\tilde{G}$ then $\mathrm{supp}(f) \subseteq K^*\tilde{A}_*K^*$. Indeed for any $a \in \tilde{A}$ and any $a_0 \in \tilde{A}\cap K^*$ we have

$$f(a) = f(aa_0) = f(\iota(\zeta_a(a_0))a_0a) = \epsilon(\zeta_a(a_0))f(a).$$

But, as we have already observed in Section 3.3, if $a \notin \tilde{A}_*$ then $\zeta_a$ is not trivial on $\tilde{A}\cap K^*$ and therefore we must have $f(a) = 0$. We therefore get that

$$\mathrm{supp}(\Omega_s) \subseteq K^*\tilde{A}_*K^*.$$

For $\lambda = (\lambda_1,\dots,\lambda_r) \in \Lambda$ let $\varpi^\lambda = \mathrm{diag}(\varpi^{\lambda_1},\dots,\varpi^{\lambda_r}) \in A$ and let

$$a_\lambda = \mathbf{s}(w_0)^{-1}\mathbf{s}(\varpi^\lambda)\mathbf{s}(w_0).$$

Let

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda_1 \geq \cdots \geq \lambda_r\}.$$

Note that for every $g \in K^*\tilde{A}_*K^*$ there is a unique integer $j$ such that $0 \leq j < n_1$ and a unique $\lambda \in \Lambda^+$ such that

$$g = \iota(\zeta_1)\,\mathbf{s}(\varpi^{n_2}e)^j\,k_1\,a_{n\lambda}^{-1}\,k_2$$

for some $\zeta_1 \in \mu_n(F)$, $k_1$, $k_2 \in K^*$. We then have

$$\Omega_s(g) = \epsilon(\zeta_1)\zeta^j q^{-jn_2(s_1+\cdots+s_r)}\Omega_s(a_{n\lambda}^{-1}).$$

Thus it is enough to compute $\Omega_s(a_{n\lambda}^{-1})$ for every $\lambda \in \Lambda^+$.

**Lemma 9.** *For $\lambda \in \Lambda^+$ we have*

$$\mathcal{V}_s(R(a_{n\lambda}^{-1})\varphi_{w_0}(\omega')) = \frac{\prod_{i=1}^r L(i)}{L(1)^r}q^{n\lambda\cdot(s-\rho)}.$$

*Proof.* For this computation it is more convenient to apply the expression (10.3) for the inner product. Since the inner product is $\tilde{G}$-invariant we have

$$\frac{L(1)^r}{\prod_{i=1}^r L(i)}\mathcal{V}_s(R(a_{n\lambda}^{-1})\varphi_{w_0}(\omega')) = \sum_{a\in \tilde{A}_*\backslash \tilde{A}} \int_U \varphi_{w_0}(a\,\mathbf{s}(w_0u):\omega')\varphi_K(a\,\mathbf{s}(w_0u)a_{n\lambda}:\omega'^{-1})\;du.$$

If $a\,\mathbf{s}(w_0u)\in \operatorname{supp}(\varphi_{w_0}) = \tilde{B}_*w_0\mathcal{I}^* = \tilde{B}_*w_0(\mathcal{I}^*\cap \mathbf{s}(U))$ then there exists $b\in \tilde{B}_*$ and $u_0\in \mathcal{I}\cap U$ such that $a\,\mathbf{s}(w_0u) = b\,\mathbf{s}(w_0u_0)$ and therefore $\mathbf{p}(b) = \mathbf{p}(a)w_0uu_0^{-1}w_0^{-1}\in A_*$. It follows in particular that $a\in \tilde{A}_*$. This implies that

$$\frac{L(1)^r}{\prod_{i=1}^r L(i)}\mathcal{V}_s(R(a_{n\lambda}^{-1})\varphi_{w_0}) = \int_U \varphi_{w_0}(\mathbf{s}(w_0u))\varphi_K(\mathbf{s}(w_0u)a_{n\lambda})\;du.$$

We have already observed that $\mathbf{s}(w_0u)\in \tilde{B}_*w_0\mathcal{I}^*$ if and only if $u\in U\cap K$ and therefore

$$\frac{L(1)^r}{\prod_{i=1}^r L(i)}\mathcal{V}_s(R(a_{n\lambda}^{-1})\varphi_{w_0}) = \int_{U\cap K}\varphi_K(\mathbf{s}(w_0u)a_{n\lambda})\;du.$$

By (2.7) we have

$$a_{n\lambda}^{-1}\mathbf{s}(u)a_{n\lambda} = \mathbf{s}(\mathbf{p}(a_{n\lambda})^{-1}u\mathbf{p}(a_{n\lambda}))$$

and since $n\lambda\in\Lambda^+$ and $u\in U\cap K$ we also have $\mathbf{p}(a_{n\lambda})^{-1}u\mathbf{p}(a_{n\lambda})\in U\cap K$. It follows that

$$\mathbf{s}(w_0u)a_{n\lambda}\in \mathbf{s}(w_0)a_{n\lambda}K^* = \mathbf{s}(\varpi^{n\lambda})K^*$$

and therefore

$$\frac{L(1)^r}{\prod_{i=1}^r L(i)}\mathcal{V}_s(R(a_{n\lambda}^{-1})\varphi_{w_0}^s) = \varphi_K(\mathbf{s}(\varpi^{n\lambda}):\omega'^{-1}) = (\chi_\rho\,\omega^{-1})(\mathbf{s}(\varpi^{n\lambda})).$$

The lemma follows. □

Now plugging Lemma 9 to the formula (10.5) we get for $\lambda\in\Lambda^+$

$$\Omega_s(a_{n\lambda}^{-1}) = \frac{\prod_{i=1}^r L(i)}{L(1)^r}\sum_{w\in\mathfrak{W}} c_{w_0}(w^{-1}s)q^{n\lambda\cdot(w^{-1}s-\rho)}.$$

This can be expressed as the $\lambda$th Hall-Littlewood polynomial with parameter $q^{-1}$ as follows. For $\lambda\in\Lambda^+$ the Hall-Littlewood polynomial $P_\lambda$ is a monic, symmetric Laurent polynomial in the variable $x=(x_1,\dots,x_r)$ and parameter $t$ defined by

$$P_\lambda(x_1,\dots,x_r;t) = \frac{(1-t)^r}{V_\lambda(t)}\sum_{w\in\mathfrak{W}} w\left(x^\lambda\prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}\right)$$

where $x^\lambda = x_1^{\lambda_1}\cdots x_r^{\lambda_r}$ and

$$V_\lambda(t) = \prod_i v_{N_i(\lambda)}(t)$$

where $N_i(\lambda) = \#\{j : 1\le j\le n,\ \lambda_j = i\}$ and $v_N(t) = \prod_{i=1}^N(1-t^i)$. What we have shown can be summarized as follows.

**Theorem 5.** *Let $g \in \tilde{G}$, $s \in \mathbb{C}^r$. Suppose $\zeta \in \mu_{2n_1}(\mathbb{C})$ satisfies* (3.4)*. If $g \notin K^* \tilde{A}_* K^*$ then $\Omega_{s,\zeta}(g) = 0$. If $g \in K^* \tilde{A}_* K^*$ write*

$$g = \iota(\zeta_1)\, \mathbf{s}(\varpi^{n_2} e)^j\, k_1\, a_{n\lambda}^{-1}\, k_2$$

*with $\zeta_1 \in \mu_n(F)$, $k_1$, $k_2 \in K^*$. Then*

$$\Omega^{\epsilon}_{s,\zeta}(g) = \epsilon(\zeta_1)\, \zeta^j\, q^{-n\,\rho\cdot\lambda}\, \frac{V_{n\lambda}(q^{-1})}{V_0(q^{-1})}\, Y^{-j}\, P_\lambda(X_1, \ldots, X_r; q^{-1})$$

*where $X_i = q^{ns_i}$, $i = 1, \ldots, r$ and $Y = q^{n_2(s_1+\cdots+s_r)}$.*

## References


[1] William Banks, Daniel Bump, and Daniel Lieman. Whittaker-Fourier coefficients of metaplectic Eisenstein series. *Compositio Math.*, 135(2):153–178, 2003.

[2] William D. Banks, Jason Levy, and Mark R. Sepanski. Block-compatible metaplectic cocycles. *J. Reine Angew. Math.*, 507:131–163, 1999.

[3] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $GL(n, F)$, where $F$ is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.

[4] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series I. In S. Friedberg, D. Bump, D. Goldfeld, and J. Hoffstein, editors, *Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory*, volume 75 of *Proc. Symp. Pure Math.*, pages 91–114, 2006.

[5] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series. III. Eisenstein series and twisted unstable $A_r$. *Ann. of Math. (2)*, 166(1):293–316, 2007.

[6] Ben Brubaker, Daniel Bump, and Solomon Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. of Math. (2)*, 173(2):1081–1120, 2011.

[7] Ben Brubaker, Daniel Bump, and Solomon Friedberg. *Weyl group multiple Dirichlet series: type A combinatorial theory*, volume 175 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2011.

[8] Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein. $p$-adic Whittaker functions on the metaplectic group. *Duke Math. J.*, 63(2):379–397, 1991.

[9] W. Casselman. The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[10] W. Casselman and J. Shalika. The unramified principal series of $p$-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[11] Gautam Chinta, Solomon Friedberg, and Paul E. Gunnells. On the $p$-parts of quadratic Weyl group multiple Dirichlet series. *J. Reine Angew. Math.*, 623:1–23, 2008.

[12] Gautam Chinta and Paul E. Gunnells. Weyl group multiple Dirichlet series constructed from quadratic characters. *Invent. Math.*, 167(2):327–353, 2007.

[13] Gautam Chinta and Paul E. Gunnells. Constructing Weyl group multiple Dirichlet series. *J. Amer. Math. Soc.*, 23(1):189–215, 2010.

[14] Yumiko Hironaka. Spherical functions and local densities on Hermitian forms. *J. Math. Soc. Japan*, 51(3):553–581, 1999.

[15] Yumiko Hironaka and Fumihiro Satō. Spherical functions and local densities of alternating forms. *Amer. J. Math.*, 110(3):473–512, 1988.

[16] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[17] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[18] R. P. Langlands. The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 143–148. Amer. Math. Soc., Providence, R.I., 1966.
[19] Hung Yean Loke and Gordan Savin. Modular forms on non-linear double covers of algebraic groups. *Trans. Amer. Math. Soc.*, 362(9):4901–4920, 2010.
[20] I. G. Macdonald. *Spherical functions on a group of p-adic type*. Ramanujan Institute, Centre for Advanced Study in Mathematics,University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.
[21] Peter J. McNamara. The metaplectic Casselman-Shalika formula. Preprint.
[22] Peter J. McNamara. Principal series representations of metaplectic groups over local fields. Preprint.
[23] Peter J. McNamara. Metaplectic Whittaker functions and crystal bases. *Duke Math. J.*, 156(1):1–31, 2011.
[24] C. Mœglin and J.-L. Waldspurger. *Spectral decomposition and Eisenstein series*, volume 113 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture].
[25] Omer Offen. Relative spherical functions on $\wp$-adic symmetric spaces (three cases). *Pacific J. Math.*, 215(1):97–149, 2004.
[26] I. Piatetski-Shapiro and S. Rallis. A new way to get Euler products. *J. Reine Angew. Math.*, 392:110–124, 1988.
[27] Yiannis Sakellaridis. Spherical functions on spherical varieties. *arXiv:0905.4244v1, submitted for publication.*
[28] Yiannis Sakellaridis. A Casselman-Shalika formula for the Shalika model of $\mathrm{GL}_n$. *Canad. J. Math.*, 58(5):1095–1120, 2006.
[29] Gordan Savin. On unramified representations of covering groups. *J. Reine Angew. Math.*, 566:111–134, 2004.
[30] Takuro Shintani. On an explicit formula for class-1 "Whittaker functions" on $GL_n$ over $P$-adic fields. *Proc. Japan Acad.*, 52(4):180–182, 1976.

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