ORTHOGONAL PERIOD OF A $GL_3(\mathbb{Z})$ EISENSTEIN SERIES

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Abstract. We provide an explicit formula for the period integral of the unramified Eisenstein series on $GL_3(\mathbb{A}_\mathbb{Q})$ over the orthogonal subgroup associated with the identity matrix. The formula expresses the period integral as a finite sum of products of double Dirichlet series that are Fourier coefficients of Eisenstein series on the metaplectic double cover of GL_3.

1. Introduction

Let F be a number field, G a connected reductive group defined over F, and H a reductive F-subgroup of G. The period integral $P^H(\phi)$ of a cuspidal automorphic form on $G(\mathbb{A}_F)$ is defined by the absolutely convergent integral (cf. [AGR93, Proposition 1])

$$P^H(\phi) = \int_{(H(F)\backslash(H(\mathbb{A}_F)\cap G(\mathbb{A}_F)^1)} \phi(h)\, dh$$

where $G(\mathbb{A}_F)^1$ is the intersection of $\ker|\chi(\cdot)|_{\mathbb{A}_F}$ for all rational characters χ of G. For more general automorphic forms, the period integral $P^H(\phi)$ fails to converge but in many cases it is known how to regularize it [LR03]. Case study indicates that the value $P^H(\phi)$, when not zero, carries interesting arithmetic information.

Roughly speaking, in cases of local multiplicity one, i.e. when at every place v of F the space of H_v-invariant linear forms of an irreducible representation of G_v is one dimensional, the period integral P^H on an irreducible automorphic representation $\pi = \otimes_v \pi_v$ factorizes as a tensor product $P^H = \otimes_v P_v$ of H_v-invariant linear forms on π_v. This indicates a relation between $P^H(\phi)$ and automorphic L-functions. For example, the setting were $H = GL_n$ over F, E/F is a quadratic extension and G is the restriction of scalars from E to F of GL_n over E, is an example where local multiplicity one holds. In this case, the nonvanishing of the period $P^H(\phi)$ of a cusp form depicts the existence of a pole at $s = 1$ of the associated Asai L-function (cf. [Fli88, Section 1, Theorem]) and the (regularized) period $P^H(E(\varphi, \lambda))$

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of an Eisenstein series is related to special values of the Asai L-function (cf. [JLR99, Theorems 23 and 36]).

Remarkably, the period integral P^H is sometimes factorizable even though local multiplicity one fails. Consider now the case where G is defined as in the previous example, but H is the quasi split unitary group with respect to E/F. For cuspidal representations, non vanishing of P^H characterizes the image of quadratic base change from $G' = GL_n$ over F to G (cf. [Jac05] and [Jac]). Furthermore, although for "most" irreducible representations of G_v, the space of H_v-invariant linear forms has dimension 2^{n-1}, on a cuspidal representation the period P^H is factorizable (cf. [Jac01]). This factorization is best understood through the *relative trace formula* (RTF) of Jacquet. Roughly speaking, the RTF is a distribution on $G(\mathbb{A}_F)$ with a spectral expansion ranging over the H-*distinguished* spectrum, i.e. the part of the automorphic spectrum of $G(\mathbb{A}_F)$ where P^H is non vanishing. In the case at hand the RTF for (G, H) is compared with the Kuznetsov trace formula for $G' = GL_n$ over F. If π is a cuspidal representation of $G(\mathbb{A}_F)$ and it is the base change of π', a cuspidal representation of $G'(\mathbb{A}_F)$, then the contribution of π to the RTF is compared with the contribution of π' to the Kuznetsov trace formula. The multiplicity one of Whittaker functionals for G' allows the factorization of the contribution of π', hence that of the contribution of π and finally of P^H on π. The value $P^H(\phi)$ (or rather its absolute value squared) for a cusp form is related to special values of Rankin-Selberg L-functions (cf. [LO07]). Essential to the factorization of P^H in this case is the fact that (up to a quadratic twist) π' base-changing to π is unique. In some sense, the local factors π'_v of π' pick a one dimensional subspace of H_v-invariant linear forms on π_v and with the appropriate normalization, these give the local factors of P^H. For π an Eisenstein automorphic representation in the image of base change, π' is no longer unique (but the base-change fiber is finite). This is the reason that the (regularized) period $P^H(E(\varphi, \lambda))$ of an Eisenstein series can be expressed as a finite sum of factorizable linear forms. In effect this was carried out using a *stabilization* process (stabilizing the open double cosets in $P\backslash G/H$ over the algebraic closure of F where the Eisenstein series is induced from the parabolic subgroup P) for Eisenstein series induced from the Borel subgroup (cf. [LR00] for $n = 3$ and [Off07] for general n) and is work in progress for more general Eisenstein series.

Consider now the case where $G = GL_n$ over F and H is an orthogonal subgroup. Using his RTF formalism and evidence from the $n = 2$ case, Jacquet conjectured that in this setting *the role of* G' is played by the metaplectic double cover of G [Jac91]. For this G' local multiplicity one of Whitaker functionals fails. This leads us to expect that the period integral $P^H(\phi)$ of a cusp form is not factorizable. To date, the arithmetic interpretation of the period at hand is a mystery, even precise conjectures are yet to be made. This brings us, finally, to the subject matter of this note. Often, studying the period integral of an Eisenstein series is more approachable then that of a cusp form and yet may help to predict expectations for the cuspidal case (this was the case for $G = GL_2$ and H an anisotropic torus, where the classical formula of Maass for the period of an Eisenstein series in terms of the zeta function of an imaginary quadratic field significantly predates the analogous formula of Waldspurger for the absolute value squared of the period of a cusp form). In this work we provide a very explicit formula for the period integral $P^H(E(\varphi, \lambda))$ in the special case that $n = 3$, H is the orthogonal group associated to the identity matrix and $E(\varphi, \lambda)$ is the unramified Eisenstein series induced from the Borel subgroup. The formula we obtain expresses the period integral as a *finite* sum of products of certain double Dirichlet series. This formula, given in Theorem 6.1, is our main result. The double Dirichlet series that appear are related to the Fourier coefficients of Eisenstein series on $G'(\mathbb{A}_F)$ (cf. [BBFH07]). This fits perfectly into Jacquet's formalism and it is our hope that the formula in this very special case can shed a light on the arithmetic information carried by orthogonal periods in the general context.

We conclude this introduction with a description of the computation of Maass alluded to above. Let $E(z, s)$ be the real analytic Eisenstein series on $SL_2(\mathbb{Z})$. A classical result of Maass relates a weighted sum of $E(z, s)$ over CM points of discriminant $d < 0$ with the ζ function of the imaginary extension $\mathbb{Q}(\sqrt{d})$. This can be reinterpreted as relating an orthogonal period of the Eisenstein series with a Fourier coefficient of a half-intgeral weight automorphic form. Indeed, the ζ function of $\mathbb{Q}(\sqrt{d})$ shows up in the Fourier expansion of a half-integral weight Eisenstein series.

Let $z = x + iy$ with $x, y \in \mathbb{R}, y > 0$ be an element of the complex upper halfplane. Let Γ_∞ be the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices of the form $\left(\begin{smallmatrix} \pm 1 & * \\ 0 & \pm 1 \end{smallmatrix}\right)$. The weight zero real analytic Eisenstein series for $SL_2(\mathbb{Z})$ is defined by the absolutely convergent series

$$E(z, s) = \sum_{\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^s \tag{1.1}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and by analytic continuation for $s \in \mathbb{C}, s \neq 1$. Similarly, the Eisenstein series of weight $\frac{1}{2}$ for $\Gamma_0(4)$ is defined by

$$\tilde{E}(z, s) = \sum_{\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in (\Gamma_\infty \cap \Gamma_0(4)) \backslash \Gamma_0(4)} \epsilon_d^{-1} \left(\frac{c}{d}\right) \frac{\mathrm{Im}(\gamma z)^s}{\sqrt{cz + d}}, \tag{1.2}$$

where

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

The Fourier expansion of the half integral weight Eisenstein series was first computed by Maass [Maa38]. To describe the expansion, first define

$$K_m(s, y) = \int_{-\infty}^{\infty} \frac{e^{2\pi i m x}}{(x^2 + y^2)^s (x + iy)^{1/2}} dx.$$

Then

$$\tilde{E}(z,s) = y^s + c_0(s) y^s \frac{\zeta(4s-1)}{\zeta(4s)} + \sum_{m \neq 0} b_m(s) K_m(s,y) e^{2\pi i m x}, \tag{1.3}$$

where, for m squarefree,

$$b_m(s) = c_m(s) \frac{L(2s,\chi_m)}{\zeta(4s+1)}.$$

In the above equations, $c_m(s)$ is a quotient of Dirichlet polynomials in 2^{-s} and χ_m is the real primitive character corresponding to the extension $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$. See Propositions 1.3 and 1.4 of Goldfeld-Hoffstein [GH85] for precise formulas.

On the other hand, quadratic Dirichlet L-functions also arise as sums of the nonmetaplectic Eisenstein series over CM points. Let $z = x + iy$ in the upper half plane be an element of an imaginary quadratic field K of discriminant d_K. Let A be the ideal class in the ring of integers of K corresponding to $\mathbb{Z} + z\mathbb{Z}$. Let $q(m,n)$ be the binary quadratic form

$$q(m,n) = \frac{\sqrt{|d_K|}}{2\ \mathrm{Im}(z)} \mathbf{N}(mz+n) = \frac{\sqrt{|d_K|}}{2\ \mathrm{Im} z} |mz+n|^2$$

and ζ_q the Epstein zeta function

$$\zeta_q(s) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{q(m,n)^s}. \tag{1.4}$$

Then

$$\zeta_K(s, A^{-1}) = \frac{1}{w_K} \zeta_q(s) \tag{1.5}$$

where w_K is the number of roots of unity in K. These zeta functions can be expressed in terms of the nonmetaplectic Eisenstein series:

$$\zeta_K(s, A^{-1}) = \frac{1}{w_K} \zeta_q(s) = \frac{2^{1+s}}{w_K\, |d_K|^{s/2}} \zeta(2s) E(z,s) \tag{1.6}$$

By virtue of the bijective correspondences between ideal classes in the ring of integers of K, binary quadratic forms and CM points in the upper halfplane, we arrive at the identity

$$\zeta_K(s) = \frac{1}{w_K} \sum_q \zeta_q(s) = \frac{2^{1+s}}{w_K\, |d_K|^{s/2}} \zeta(2s) \sum_z E(z,s), \tag{1.7}$$

where the sum in the middle is over equivalence classes of integral binary quadratic forms of discriminant d_K and the rightmost sum is over $SL_2(\mathbb{Z})$ inequivalent CM points of discriminant d_K. Writing the zeta function of K as $\zeta_K(s) = \zeta(s) L(s, \chi_{d_K})$ gives the relation between Fourier coefficients of metaplectic Eisenstein series and sums of nonmetaplectic Eisenstein series.

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2. Adelic versus classical periods

Let $G = GL_n$ over $\mathbb{Q}$ and let $X = \{g \in G : {}^t g = g\}$ be the algebraic subset of symmetric matrices. Let $K = \prod_v K_v$ be the standard maximal compact subgroup of $G(\mathbb{A}_\mathbb{Q})$ where the product is over all places v of $\mathbb{Q}$, $K_p = G(\mathbb{Z}_p)$ for every prime number p and $K_\infty = O(n) = \{g \in G(\mathbb{R}) : g\,{}^t g = I_n\}$.

2.1. **The genus class.**

For x, $y \in X(\mathbb{Q})$ we say that x and y are in the same *class* and write $x \sim y$ if there exists $g \in G(\mathbb{Z})$ such that $y = g\,x\,{}^t g$ and we say that x and y are in the same *genus class* and write $x \approx y$ if for every place v of $\mathbb{Q}$ there exists $g \in K_v$ such that $y = g\,x\,{}^t g$. Of course classes refine genus classes. If $x \in X(\mathbb{Q})$ is positive definite, it is well known that there are finitely many classes in the genus class of x.

2.2. **An anisotropic orthogonal period as a sum over the genus.**

Fix once and for all $x \in X(\mathbb{Q})$ positive definite and let

$$H = \{g \in G : g\,x\,{}^t g = x\}$$

be the orthogonal group associated with x. Thus, H is anisotropic and the orthogonal period integral

$$P^H(\phi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})} \phi(h)\ dh$$

is well defined and absolutely convergent for any say continuous function ϕ on $H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})$.

Note that the imbedding of $G(\mathbb{R})$ in $G(\mathbb{A}_\mathbb{Q})$ in the "real coordinate" defines a bijection $G(\mathbb{Z})\backslash G(\mathbb{R})/K_\infty \simeq G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K$. Furthermore, the map $g \mapsto g\,{}^t g$ defines a bijection from $G(\mathbb{R})/K_\infty$ to the space $X^+(\mathbb{R})$ of positive definite symmetric matrices in $X(\mathbb{R})$. The resulting bijection

$$G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K \simeq G(\mathbb{Z})\backslash X^+(\mathbb{R}) \tag{2.1}$$

allows us to view any function $\phi(g)$ on $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K$ as a function (still denoted by) $\phi(x)$ on $G(\mathbb{Z})\backslash X^+(\mathbb{R})$.

By [Bor63, Proposition 2.3] there is a natural bijection between the double coset space $H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})/(H(\mathbb{A}_\mathbb{Q}) \cap K)$ and the set $\{y \in X_\mathbb{Q} : y \approx x\}/\sim$ of classes in the genus class of x. Let $g_\infty \in G(\mathbb{R})$ be such that $x = g_\infty\,{}^t g_\infty$ and let $g_0 \in G(\mathbb{A}_\mathbb{Q})$ have g_∞ in the infinite place and the identity matrix at all finite places. As in [CO07, Lemma 2.1], it can be deduced that for any function ϕ on $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K$ we have

$$(2.2)\quad \int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})} \phi(h\,g_0)\;dh = \operatorname{vol}(H(\mathbb{A}_\mathbb{Q})\cap g_0Kg_0^{-1}) \sum_{\{y\in X_\mathbb{Q}:y\approx x\}/\sim} \frac{\phi(y)}{\#\{g\in G(\mathbb{Z}):g\,y\,{}^tg=y\}}$$

where ϕ on the left and right hand sides correspond via (2.1). In short, the anisotropic orthogonal period associated with x of an automorphic form ϕ equals a finite weighted sum of point evaluations of ϕ over classes in the genus class of x.

2.3. **The unramified adelic Eisenstein series as a classical one.** Let $B = AU$ be the Borel subgroup of upper triangular matrices in G, where A is the subgroup of diagonal matrices and U is the subgroup of upper triangular unipotent matrices. For $\lambda = (\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ let

$$\varphi_\lambda(\operatorname{diag}(a_1,\dots,a_n)\,u\,k) = \prod_{i=1}^n |a_i|^{\lambda_i+\frac{n+1}{2}-i}$$

for $\operatorname{diag}(a_1,\dots,a_n)\in A(\mathbb{A}_\mathbb{Q})$, $u\in U(\mathbb{A}_\mathbb{Q})$ and $k\in K$. The unramified Eisenstein series $\mathcal{E}(g,\lambda)$ induced from B is defined by the meromorphic continuation of the series

$$\mathcal{E}(g,\lambda) = \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_\lambda(\gamma\,g).$$

Note that $\mathcal{E}(g,\lambda)$ is a function on $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K$. With the identification (2.1), for $x\in X^+(\mathbb{R})$ we have

$$(2.3)\qquad \mathcal{E}(x,\lambda) = \det x^{\frac{1}{2}(\lambda_1+\frac{n-1}{2})} \sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})} \prod_{i=1}^{n-1} d_{n-i}(\delta\,x\,{}^t\delta)^{\frac{1}{2}(\lambda_{i+1}-\lambda_i-1)}$$

where $d_i(x)$ denotes the determinant of the lower right $i\times i$ block of x.

Assume now that $n=3$. Arguing along the same lines as in [CO07, Section 4.2] we may write (2.3) as

$$(2.4)\qquad \begin{aligned}\mathcal{E}(x,\lambda_1,\lambda_2,\lambda_3) = \frac{1}{4}\,&\zeta(\lambda_2-\lambda_3+1)^{-1}\,\zeta(\lambda_1-\lambda_2+1)^{-1}\,(\det x)^{\frac{\lambda_2}{2}}\\ &\times \sum_{\substack{0\neq v,w\in\mathbb{Z}^3\\ v\perp w}} Q_{x,1}(v)^{\frac{1}{2}(\lambda_3-\lambda_2-1)}\,Q_{x,2}(w)^{\frac{1}{2}(\lambda_2-\lambda_1-1)}\end{aligned}$$

where $Q_{x,1}$ (resp. $Q_{x,2}$) is the quadratic form on $V=\mathbb{R}^3$ defined on the row vector $v\in V$ by $v\mapsto vxv^t$ (resp. $v\mapsto vx^{-1}v^t$). The genus class of the identity matrix $x=I_3$ consists of a unique class. Let $Q = Q_{I_3,1} = Q_{I_3,2}$. Combining (2.2) and (2.4) we see that when $x = I_3$ there exists a normalization of the Haar measure on $H(\mathbb{A}_{\mathbb{Q}})$ such that as a meromorphic function in $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ we have

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}})} \mathcal{E}(h,\lambda)\, dh = 4\,\mathcal{E}(I_3,\lambda) = \tag{2.5}$$

$$\zeta(\lambda_2-\lambda_3+1)^{-1}\,\zeta(\lambda_1-\lambda_2+1)^{-1} \sum_{\substack{0\neq v,w\in\mathbb{Z}^3\\ v\perp w}} Q(v)^{\frac{1}{2}(\lambda_3-\lambda_2-1)}\,Q(w)^{\frac{1}{2}(\lambda_2-\lambda_1-1)}.$$

Introduce the new variables $s_2 = (\lambda_2 - \lambda_3 + 1)/2, s_1 = (\lambda_1 - \lambda_2 + 1)/2$ and write the right hand side of (2.5) as

$$E(I_3; s_1, s_2) := \zeta(2s_1)^{-1}\zeta(2s_2)^{-1} \sum_{\substack{0\neq v,w\in\mathbb{Z}^3\\ v\perp w}} Q(v)^{-s_2}Q(w)^{-s_1}. \tag{2.6}$$

The rest of this work is devoted to the explicit computation of (2.6) which is given in Theorem 6.1.

3. The double Dirichlet series

We define the double Dirichlet series which arise in our evaluation of the $GL_3(\mathbb{Z})$ Eisenstein series at the identity. Let ψ_1, ψ_2 be two quadratic characters unramified away from 2. Then the double Dirichlet series $Z(s_1, s_2; \psi_1, \psi_2)$ is roughly of the form

$$\sum_d \frac{L(s_1,\chi_d)}{d^{s_2}}. \tag{3.1}$$

More precisely,

$$Z(s_1, s_2; \psi_1, \psi_2) = \sum_{\substack{d_1,d_2>0\\ odd}} \frac{\chi_{d_2'}(\hat{d}_1)}{d_1^{s_1} d_2^{s_2}} a(d_1, d_2)\psi_1(d_1)\psi_2(d_2), \tag{3.2}$$

where

- $d_2' = (-1)^{(d_2-1)/2} d_2$ and $\chi_{d_2'}$ is the Kronecker symbol associated to the squarefree part of d_2'
- $\hat{d}_1$ is the part of d_1 relatively prime to the squarefree part of d_2
- the coefficients $a(d_1, d_2)$ are multiplicative in both entries and are defined on prime powers by

$$a(p^k, p^l) = \begin{cases} \min(p^{k/2}, p^{l/2}) & \text{if } \min(k,l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

It can be shown that the functions $Z(s_1, s_2; \psi_1, \psi_2)$ appear in the Whittaker expansion of the metaplectic Eisenstein series on the double cover of $GL_3(\mathbb{R})$, see e.g [BBFH07]. As such these functions have an analytic continuation to $s_1, s_2 \in \mathbb{C}$ and satisfy a group of 6 functional equations.

We conclude this section by relating the heuristic definition (3.1) to the precise definition (3.2).

Theorem 3.1. *Let ψ_1, ψ_2 be quadratic characters ramified only at 2. Then*

$$Z(s,w;\psi_1,\psi_2) = \zeta_2(2w)\zeta_2(2s+2w-1) \sum_{\substack{d_2>0,\ odd \\ sqfree}} \frac{L_2(s,\chi_{d_2'}\psi_1)}{L_2(s+2w,\chi_{d_2'}\psi_1)} \frac{\psi_2(d_2)}{d_2^w},$$

where $L_2(s,\chi)$ denotes the Dirichlet L-function with the Euler factor at 2 removed.

Proof. See [CFH05]. □

4. Genus theory for binary quadratic forms

Our description of the genus characters follows the presentation in Section 3 of Bosma and Stevenhagen, [BS96]. Let D be a negative discriminant. Write $D = df^2$ where d is a fundamental discriminant. We will assume f is odd. Let $\mathrm{Cl}(D)$ be the group of $SL_2(\mathbb{Z})$ equivalence classes of primitive integral binary quadratic forms of discriminant D. We will denote the quadratic form $q(x,y) = ax^2 + bxy + cy^2$ by $[a,b,c]$. We call e a prime discriminant if $e = -4, 8, -8$ or $p' = (-1)^{(p-1)/2}p$ for an odd prime. Note that e is a fundamental discriminant. Write $D = D_1D_2$ where D_1 is an even fundamental discriminant and D_2 is an odd discriminant. Let D_0 be D_1 times the product of the prime discriminants dividing D_2.

For each odd prime p dividing D we define a character $\chi^{(p)}$ on $\mathrm{Cl}(D)$ by

$$\chi^{(p)}([a,b,c]) = \begin{cases} \chi_{p'}(a) & \text{if } (p,a) = 1 \\ \chi_{p'}(c) & \text{if } (p,c) = 1. \end{cases} \tag{4.1}$$

The primitivity of $[a,b,c]$ ensures that at least one of these two conditions will be satisfied. These characters generate a group $\mathcal{X}(D)$, called the group of genus class characters of $\mathrm{Cl}(D)$. The order of $\mathcal{X}(D)$ is $2^{\omega(D)-1}$, where $\omega(D)$ is the number of distinct prime divisors of D. For each squarefree odd number e_1 dividing D we define the genus class character

$$\chi_{e_1',e_2'} = \prod_{p|e_1} \chi^{(p)}$$

where $e_1'e_2' = D_0$. Then as e_1 ranges over the squarefree positive odd divisors of D, $\chi_{e_1',e_2'}$ will range over all the genus character exactly once (if D is even) or twice (if D is odd).

Two forms q_1 and q_2 are in the same genus if and only if $\chi(q_1) = \chi(q_2)$ for all $\chi \in \mathcal{X}(D)$. As in Section 2, we denote this by $q_1 \approx q_2$.

Using the identification between primitive integral binary quadratic forms of discriminant D and invertible ideal classes in the order $\mathcal{O}_D = \mathbb{Z}[(D + \sqrt{D})/2]$, we may define the genus characters on the group $Pic(\mathcal{O}_D)$. This allows us to associate to a genus class character χ the L-function

$$L_{\mathcal{O}_D}(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathbf{N}(\mathfrak{a})^s}$$

where the sum is over all invertible ideals of $\mathcal{O}_D$. In terms of the Epstein zeta function, we have

$$L_{\mathcal{O}_D}(s,\chi) = \frac{1}{\#\mathcal{O}_D^\times} \sum_{q\in \mathrm{Cl}(D)} \chi(q)\zeta_q(s). \tag{4.2}$$

Using the group of characters $\mathcal{X}(D)$, we may isolate individual genus classes on the right hand side of (4.2).

Proposition 4.1. *Let q_0 be a fixed form in $Cl(D)$. Then*

$$\sum_{q\approx q_0} \zeta_q(s) = \frac{\#\mathcal{O}_D^\times}{2^{\omega(D)-1}} \sum_{\chi\in\mathcal{X}(D)} \chi(q_0) L_{\mathcal{O}_D}(s,\chi).$$

Finally, the following proposition shows how to write an L-function associated to a genus class character in terms of ordinary Dirichlet L-functions.

Proposition 4.2. *Let e_1, e_2 be fundamental discriminants and let $D = e_1e_2f^2$. Then*

$$L_{\mathcal{O}_D}(s,\chi_{e_1,e_2}) = L(s,\chi_{e_1})L(s,\chi_{e_2}) \prod_{p^k||f} \mathcal{P}_k(p^{-s},\chi_{e_1}(p),\chi_{e_2}(p))$$

where $\mathcal{P}_k(p^{-s},\chi_{e_1}(p),\chi_{e_2}(p))$ is a Dirichlet polynomial defined by the generating series

$$F(u,X;\alpha,\beta) = \sum_{k\geq 0} \mathcal{P}_k(u,\alpha,\beta)X^k = \frac{(1-\alpha uX)(1-\beta uX)}{(1-X)(1-pu^2X)}. \tag{4.3}$$

Proof. See Remark 3 of Kaneko, [Kan05]. Actually, Kaneko considers only zeta functions of orders, not genus character L-functions as in the proposition, but the ideas are similar. □

5. The Gauss map

Let $V = \mathbb{Q}^3$ equipped with the quadratic form Q, $Q(x,y,z) = x^2+y^2+z^2$. We also let Q denote the associated bilinear form on $V \times V$. Let $L = \mathbb{Z}^3$ and let $L[n]$ be the set of vectors in L such that $Q(v) = n$. Let L_0 be the set of primitive integral vectors and let $L_0[n] = L_0 \cap L[n]$. Let

$$D = \begin{cases} -4n & \text{if } n \equiv 1 \text{ or } 2 \pmod 4 \\ -n & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{5.1}$$

(The case $n \equiv 0 \pmod 4$ will not occur in our computations below.)

We have a map from $L_0[n]$ to equivalence classes of primitive binary quadratic forms of discriminant D defined as follows. Let $v \in L_0[n]$. Let W be the orthogonal complement of v (with respect to Q) and let M be a maximal Q-integral sublattice in W. Explicitly, we take $M = L \cap W$ if $n \equiv 0, 1 \mod 4$ and $M = \frac{1}{2}L \cap W$ if $n \equiv 3 \mod 4$. Let u, w be an integral basis for M. The restriction of Q to the two dimensional subspace W is a binary quadratic form, which we'll denote by q. With respect to an integral basis u, w of M, the Gram matrix of this restriction is

$$\begin{pmatrix} Q(u,u) & Q(u,v) \\ Q(u,v) & Q(v,v) \end{pmatrix}. \tag{5.2}$$

We call the map $\mathcal{G} : L_0[n] \to \mathrm{Cl}(D)$ defined by $\mathcal{G}(v) = Q|_{v^\perp}$ the Gauss map. We now describe the image of this map more explicitly for fixed n.

We begin with three observations.

(1) By the Hasse-Minkowski principle, if $q \in \mathrm{Cl}(D)$ is in the image of $\mathcal{G}$, then every form in the genus of q is also in the image.
(2) If $q_1 \approx q_2$ are two forms in the image of $\mathcal{G}$, then by Siegel's mass formula, the fiber over both forms has the same cardinality.
(3) If q_1 and q_2 are two forms in the image, then q_1 and q_2 are in the same genus.

These three facts follow because the ternary quadratic form Q is the only form in its genus. We refer the reader to Theorems 1 and 2 of the survey paper of Shimura [Shi06] for further details. More explicity we have the following theorem.

Theorem 5.1. *Let n be a positive integer which is not divisible by 4, D as in (5.1) above and let $q \in Cl(D)$ be a form in the image of $\mathcal{G}$. For any genus character χ_{e_1,e_2} of $Cl(D)$ with e_1 odd, we have*

$$\chi_{e_1,e_2}(q) = \begin{cases} \chi_{-8}(|e_1|) & \text{if } n \equiv 3 \pmod 4 \\ \chi_{-4}(|e_1|) & \text{if } n \equiv 1,2 \pmod 4. \end{cases}$$

Moreover

$$\#\mathcal{G}^{-1}(\{q\}) = \frac{24 \cdot 2^{\omega(n)}}{\#\mathcal{O}_D^\times} = \begin{cases} 48/\#\mathcal{O}_D^\times \cdot 2^{\omega(D)-1} & \text{if } n \equiv 3 \pmod 4 \\ 24/\#\mathcal{O}_D^\times \cdot 2^{\omega(D)-1} & \text{if } n \equiv 1,2 \pmod 4. \end{cases}$$

This theorem was first proven by Gauss [Gau86]. We again refer the reader to [Shi06] for a more modern presentation.

6. Proof of the main theorem

We will evaluate the minimal parabolic $GL_3(\mathbb{Z})$ Eisenstein series at the identity matrix. We recall

$$\zeta(2s_1)\zeta(2s_2)E(I,s_1,s_2) = \sum_{\substack{0\neq v\in L \\ 0\neq w\in L\cap v^\perp}} Q(v)^{-s_2}Q(w)^{-s_1}. \tag{6.1}$$

Our goal is the following theorem.

Theorem 6.1. *The Eisenstein series $E(I, s_1, s_2)$ can be expressed as a linear combination of products of the double Dirichlet series $Z(\psi_1, \psi_2) :=$*

$Z(s_1, s_2; \psi_1, \psi_2)$, where ψ_1, ψ_2 range over the characters ramified only at 2. Explicitly,

(6.2)

$$\begin{aligned}
&\zeta_2(2s_1)\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)E(I_3,s_1,s_2)/12\\
&\qquad = Z(1,\chi_{-4})Z(\chi_{-4},1) + Z(1,1)Z(\chi_{-4},\chi_{-4})\\
&\qquad\qquad + 2^{-s_1}Z(1,\chi_{-8})Z(\chi_{-4},1) + 2^{-s_1}Z(1,\chi_{-8})Z(\chi_{-4},\chi_{-4})\\
&\qquad + 2^{-s_2}Z(1,\chi_{-4})Z(\chi_{-8},1) + 2^{-s_2}Z(1,\chi_{-4})Z(\chi_{8},1)\\
&\qquad\qquad + 2^{-s_2}Z(1,1)Z(\chi_{-8},\chi_{-4}) - 2^{-s_2}Z(1,1)Z(\chi_{8},\chi_{-4})\\
&\qquad\qquad + 2^{-s_1-s_2}Z(1,\chi_{-8})Z(\chi_{-8},1) + 2^{-s_1-s_2}Z(1,\chi_{-8})Z(\chi_{8},1)\\
&\qquad\qquad + 2^{-s_1-s_2}Z(1,\chi_{8})Z(\chi_{-8},\chi_{-4}) - 2^{-s_1-s_2}Z(1,\chi_{8})Z(\chi_{8},\chi_{-4})\\
&\qquad + 2^{-s_2}Z(1,1)Z(1,\chi_{-8}) - 2^{-s_2}Z(1,\chi_{-4})Z(1,\chi_{8}).
\end{aligned}$$

Proof. Begin by breaking up the sum in (6.1) into congruence classes of $Q(v)$ mod 4. Because multiplication by 2 gives a bijection between $L(n)$ and $L(4n)$, we have

$$\sum_{\substack{0\neq v\in L\\ Q(v)\equiv 0 \bmod 4\\ 0\neq w\in L\cap v^\perp}} Q(v)^{-s_2}Q(w)^{-s_1} = 4^{-s_2}\zeta(2s_1)\zeta(2s_2)E(I,s_1,s_2). \tag{6.3}$$

Therefore

$$\begin{aligned}
(1-4^{-s_2})\zeta(2s_1)\zeta(2s_2)E(I,s_1,s_2) = &\sum_{\substack{0\neq v\in L\\ Q(v)\not\equiv 0 \bmod 4\\ w\in L\cap v^\perp}} Q(v)^{-s_2}Q(w)^{-s_1}\\
=&\zeta_2(2s_2)\sum_{\substack{v_0\in L_0\\ w\in L\cap v_0^\perp}} Q(v_0)^{-s_2}Q(w)^{-s_1}.
\end{aligned}$$

The second line follows after writing $v\in L$ as cv_0 with $v_0\in L_0$ and c an odd positive integer. Note that we have dropped the condition $Q(v)\not\equiv 0 \bmod 4$ as it becomes redundant for $v_0\in L_0$. Thus

$$\begin{aligned}
&\zeta(2s_1)E(I,s_1,s_2) =\\
&\left(\sum_{\substack{v\in L_0\\ Q(v)\equiv 1 \bmod 4}} + \sum_{\substack{v\in L_0\\ Q(v)\equiv 2 \bmod 4}} + \sum_{\substack{v\in L_0\\ Q(v)\equiv 3 \bmod 4}}\right) Q(v)^{-s_2}Q(w)^{-s_1}
\end{aligned} \tag{6.4}$$

is equal to $S_1+S_2+S_3$, say. We treat each of these 3 sums separately. Begin with S_1:

$$\begin{aligned} S_1 &= \sum_{\substack{n>0 \\ n\equiv 1 \mod 4}} \frac{1}{n^{s_2}} \left(\sum_{v\in L_0[n]} \zeta_{\mathcal{G}(v)}(s_1) \right) \\ &= \sum_{\substack{n>0 \\ n\equiv 1 \mod 4}} \frac{1}{n^{s_2}} \left(\frac{24\cdot 2^{\omega(n)}}{\#\mathcal{O}_{-4n}^{\times}} \sum_{q\sim q_{0,n}} \zeta_q(s_1) \right) \end{aligned}$$

where $q_{0,n}$ is a form in $\mathrm{Cl}(-4n)$ satisfying $\chi_{e_1',e_2'}(q_{0,n}) = \chi_{-4}(e_1)$ for all squarefree odd divisors e_1 of n. This follows from Theorem 5.1. Since $\omega(n) = \omega(-4n)-1$, Proposition 4.1 now implies that

$$S_1 = 24 \sum_{\substack{n>0 \\ n\equiv 1 \mod 4}} \frac{1}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-4}(e_1) L_{\mathcal{O}_{-4n}}(s_1, \chi_{e_1',e_2'}). \right) \tag{6.5}$$

As in Section 4, e_2' is chosen to be the fundamental discriminant such that $e_1'e_2'$ is equal to the product of the prime discriminants dividing $-4n$. Reintroduce the integers $n\equiv 3 \mod 4$ in (6.5):

$$S_1/12 = \sum_{\substack{n>0 \\ n\equiv 1 \mod 4}} \frac{1+\chi_{-4}(n)}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-4}(e_1) L_{\mathcal{O}_{-4n}}(s_1, \chi_{e_1',e_2'}) \right)$$

$$\begin{aligned} &= \sum_{\substack{n>0 \\ odd}} \frac{1}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-4}(e_1) L_{\mathcal{O}_{-4n}}(s_1, \chi_{e_1',e_2'}) \right) \\ &\quad + \sum_{\substack{n>0 \\ odd}} \frac{\chi_{-4}(n)}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-4}(e_1) L_{\mathcal{O}_{-4n}}(s_1, \chi_{e_1',e_2'}) \right). \end{aligned} \tag{6.6}$$

Now write $n = e_1e_2f^2$ with e_1, e_2, f odd and reverse the order of summation in both sums in (6.6). For $\psi = 1$ or χ_{-4},

$$\sum_{\substack{n>0 \\ odd}} \frac{\psi(n)}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-4}(e_1) L_{\mathcal{O}_{-4n}}(s_1, \chi_{e_1',e_2'}) \right)$$

$$= \sum_{\substack{e_1,e_2>0 \\ odd,sqfree}} \frac{\psi(e_1e_2)\chi_{-4}(e_1)}{(e_1e_2)^{s_2}} \sum_{f>0,odd} \left(\frac{L_{\mathcal{O}_{-4e_1e_2f^2}}(s_1, \chi_{e_1',-4e_2'})}{f^{2s_2}} \right). \tag{6.7}$$

By virtue of Proposition 4.2, the inner sum in (6.7) is an Euler product which may be explicitly evaluated as

$$\sum_{f>0,odd} \frac{L_{\mathcal{O}_{-4e_1e_2f^2}}(s_1, \chi_{e_1'}, {}_{-4e_2'})}{f^{2s_2}}$$

$$= L(s_1, \chi_{e_1'})L(s_1, \chi_{-4e_2'}) \prod_{p\neq 2} \sum_{k=0}^{\infty} \frac{\mathcal{P}_k(p^{-s_1}, \chi_{e_1'}(p), \chi_{-4e_2'}(p))}{p^{-2kw}}$$

$$= L(s_1, \chi_{e_1'})L(s_1, \chi_{-4e_2'}) \frac{\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)}{L_2(s_1+2s_2, \chi_{e_1'})L_2(s_1+2s_2, \chi_{-4e_2'})}. \tag{6.8}$$

Thus, (6.6) becomes

(6.9)

$$\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)\times$$
$$\sum_{\psi=1,\chi_{-4}} \left(\sum_{\substack{e_1>0\\ odd}} \frac{\psi\chi_{-4}(e_1)}{e_1^{s_2}} \frac{L(s_1, \chi_{e_1'})}{L_2(s_1+2s_2, \chi_{e_1'})} \right) \left(\sum_{\substack{e_2>0\\ odd}} \frac{\psi(e_2)}{e_2^{s_2}} \frac{L(s_1, \chi_{-4e_2'})}{L_2(s_1+2s_2, \chi_{-4e_2'})} \right).$$

Comparing with Theorem 3.1, the second term in parentheses above is just

$$\frac{Z(s_1, s_2; \chi_{-4}, \psi)}{\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)}. \tag{6.10}$$

To write the first in terms of the double Dirichlet series of Section 3.1, we remove the Euler factor at 2 from the L function which appear in the numerator:

$$L(s_1, \chi_{e_1'}) = L_2(s_1, \chi_{e_1'}) \left(1 + \frac{\chi_{e_1'}(2)}{2^{s_1}}\right) \left(1 - \frac{1}{4^{s_1}}\right)^{-1}.$$

Now $\chi_{e_1'}(2) = \chi_8(e)$, so the first term in parentheses in (6.9) is

(6.11)

$$\frac{(1-\frac{1}{4^{s_1}})^{-1}}{\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)} \left[Z(s_1, s_2; 1, \psi\chi_{-4}) + 2^{-s_1} Z(s_1, s_2; 1, \psi\chi_{-8}) \right].$$

Putting (6.10),(6.11) into (6.9) completes our evaluation of S_1.

The evaluations of S_2 and S_3 are similar and will be omitted. We merely list the results below.

Proposition 6.2. *Abbreviate $Z(s_1, s_2; \psi_1, \psi_2)$ by $Z(\psi_1, \psi_2)$. Let*

$$S_i^* = \frac{S_i}{12}(1-4^{-s_1})\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)$$

for $i = 1,2,3$. We have

$$
\begin{aligned}
S_1^* &= Z(1,\chi_{-4})Z(\chi_{-4},1) + Z(1,1)Z(\chi_{-4},\chi_{-4}) \\
&\quad + 2^{-s_1}Z(1,\chi_{-8})Z(\chi_{-4},1) + 2^{-s_1}Z(1,\chi_{-8})Z(\chi_{-4},\chi_{-4}) \\
2^{s_2}S_2^* &= Z(1,\chi_{-4})Z(\chi_{-8},1) + Z(1,\chi_{-4})Z(\chi_8,1) \\
&\quad + Z(1,1)Z(\chi_{-8},\chi_{-4}) - Z(1,1)Z(\chi_8,\chi_{-4}) \\
&\quad + 2^{-s_1}Z(1,\chi_{-8})Z(\chi_{-8},1) + 2^{-s_1}Z(1,\chi_{-8})Z(\chi_8,1) \\
&\quad + 2^{-s_1}Z(1,\chi_8)Z(\chi_{-8},\chi_{-4}) - 2^{-s_1}Z(1,\chi_8)Z(\chi_8,\chi_{-4}) \\
2^{s_2}S_3^* &= Z(1,1)Z(1,\chi_{-8}) - Z(1,\chi_{-4})Z(1,\chi_8)
\end{aligned}
$$

Adding up $S_1 + S_2 + S_3$ completes the proof of the theorem □

7. Concluding remarks

7.1. **A two variable converse theorem.** Hamburger's converse theorem states that a Dirichlet series satisfying the same functional equation as the Riemann zeta function must be a constant multiple of the Riemann zeta function, [Ham21]. It is natural to ask for a two variable analogue of this result. We formulate such an analogue here.

Conjecture 7.1. *Let $D(s,w) = \sum_{m,n\geq 0} \frac{a(m,n)}{m^s n^w}$ be a double Dirichlet series in two complex variables which is absolutely convergent for $Re(s), Re(w) > 1$. Define*

$$D^*(s,w) = G(s,w)D(s,w)$$

where

$$G(s,w) = \zeta(2s)\zeta(2w)\zeta(2s+2w-1)\Gamma(s)\Gamma(w)\Gamma(s+w-\tfrac{1}{2}).$$

Suppose that

1. *$D^*(s,w)$ has a meromorphic continuation to $(s,w) \in \mathbb{C}^2$*
2. *$D^*(s,w)$ is invariant under $(s,w) \mapsto (1-s, s+w-\frac{1}{2})$ and $(s,w) \mapsto (s+w-\frac{1}{2}, 1-w)$*
3. *$D(s,w)$ satisfies the limits*

$$\lim_{s\to\infty} D(s,w) = 24\frac{\zeta(s)}{\zeta(2s)}L(s,\chi_{-4}) \ \text{and}\ \lim_{w\to\infty} D(s,w) = 24\frac{\zeta(w)}{\zeta(2w)}L(w,\chi_{-4})$$

Then $D(s,w) = E(I_3, s, w)$.

This conjecture would provide an alternate proof of our main result Theorem 6.1, since, with a little work, one can directly show that the double Dirichlet series on the right hand side of (6.2) satisfies the same conditions as the $D(s,w)$ of the conjecture after multiplying by 12 and clearing the zeta factors. This would have the following arithmetic consequence. Whereas we proved the main identity using Gauss's result (Theorem 5.1) on the image of $\mathcal{G}$, a independent proof of the main identity will give a result almost as strong as Theorem 5.1. In particular, the conjecture would give a new proof of Gauss's result on the number of representations of an integer as a sum of 3 squares.

7.2. **Siegel modular forms and double Dirichlet series.** Let $r(m,n)$ be the number of pairs of vectors $v, w \in \mathbb{Z}^3$ such that $Q(v) = n, Q(w) = m$ and v is orthogonal to w. Comparing with (6.1), we see that the double Dirichlet series

$$D(s,w) = \sum_{n,m \geq 1} \frac{r(m,n)}{m^s n^w}$$

is equal to $\zeta(2s)\zeta(2w)E(I,s,w)$. From the theory of Eisenstein series we know that $D(s,w)$ has a meromorphic continuation to $\mathbb{C}^2$ and satisfies a group of 6 functional equations. On the other hand $r(m,n)$ are the diagonal Fourier coefficients of a Siegel modular theta series θ of genus 2. Thus $D(s,w)$ can be obtained as an integral transform of θ. It is natural to ask if the analytic properties of $D(s,w)$ can be obtained from the automorphic properties of θ. If so, then presumably one can construct a double Dirichlet series with analytic continuation and functional equations by taking the same integral transform of any genus 2 Siegel modular form. We believe this warrants further investigation.

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8. Appendix

Some additional computations. Won't be in the final paper.

$$\begin{aligned} S_3 &= \sum_{\substack{n>0 \\ n\equiv 3 \bmod 4}} \frac{1}{n^{s_2}} \left(\sum_{v\in L_0[n]} \zeta_{\mathcal{G}(v)}(s_1) \right) \\ &= \sum_{\substack{n>0 \\ n\equiv 3 \bmod 4}} \frac{1}{n^{s_2}} \left(\frac{2^{-s_1} 24 \cdot 2^{\omega(n)}}{\#\mathcal{O}_{-n}^{\times}} \sum_{q\sim q_{0,n}} \zeta_q(s_1) \right) \end{aligned}$$

where $q_{0,n}$ is a form in $\mathrm{Cl}(-n)$ satisfying $\chi_{e_1',e_2'}(q_{0,n}) = \chi_{-8}(e_1)$ for all square-free odd divisors e_1 of n. This follows from Theorem 5.1. (The additional 2^{-s_1} is there because the sublattice $v^\perp \cap L$ is not maximal when $Q(v)$ is congruent to 3 mod 4.) Proposition 4.1 now implies that

$$S_3 = 24 \cdot 2^{1-s_1} \sum_{\substack{n>0 \\ n\equiv 3 \bmod 4}} \frac{1}{2n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-4}(e_1) L_{\mathcal{O}_{-n}}(s_1, \chi_{e_1',e_2'}). \right) \tag{8.1}$$

The 1/2 is there because we get all the genus characters twice when we sum over divisors of n. As in Section 4, e_2' is chosen to be the fundamental discriminant such that $e_1'e_2'$ is equal to the product of the prime discriminants dividing $-n$. Reintroduce the integers $n \equiv 1 \mod 4$ in (8.1):

$$2^{s_1} S_3/12 = \sum_{\substack{n>0 \\ n\equiv 3 \mod 4}} \frac{1-\chi_{-4}(n)}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-8}(e_1) L_{\mathcal{O}_{-n}}(s_1, \chi_{e_1', e_2'}) \right)$$

$$(8.2) \qquad = \sum_{\substack{n>0 \\ odd}} \frac{1}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-8}(e_1) L_{\mathcal{O}_{-n}}(s_1, \chi_{e_1', e_2'}) \right) - \sum_{\substack{n>0 \\ odd}} \frac{\chi_{-4}(n)}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-8}(e_1) L_{\mathcal{O}_{-n}}(s_1, \chi_{e_1', e_2'}) \right).$$

Now write $n = e_1 e_2 f^2$ with e_1, e_2, f odd and reverse the order of summation in both sums in (8.1). For $\psi = 1$ or χ_{-4},

$$\sum_{n>0} \frac{\psi(n)}{n^{s_2}} \left(\sum_{\substack{e_1|n \\ sqfree}} \chi_{-8}(e_1) L_{\mathcal{O}_{-n}}(s_1, \chi_{e_1', e_2'}) \right)$$

$$(8.3) \qquad = \sum_{\substack{e_1,e_2>0 \\ odd, sqfree}} \frac{\psi(e_1 e_2)\chi_{-8}(e_1)}{(e_1 e_2)^{s_2}} \sum_{f>0, odd} \left(\frac{L_{\mathcal{O}_{e_1 e_2 f^2}}(s_1, \chi_{e_1', e_2'})}{f^{2s_2}} \right).$$

By Proposition 4.2, the inner sum in (8.3) is an Euler product which may be explicitly evaluated as

$$\sum_{f>0, odd} \frac{L_{\mathcal{O}_{e_1 e_2 f^2}}(s_1, \chi_{e_1', e_2'})}{f^{2s_2}} = L(s_1, \chi_{e_1'}) L(s_1, \chi_{e_2'}) \prod_{p\neq 2} \sum_{k=0}^{\infty} \frac{\mathcal{P}_k(p^{-s_1}, \chi_{e_1'}(p), \chi_{e_2'}(p))}{p^{-2kw}}$$

$$(8.4) \qquad = L(s_1, \chi_{e_1'}) L(s_1, \chi_{e_2'}) \frac{\zeta_2(2s_2)\zeta_2(2s_1+2s_2-1)}{L_2(s_1+2s_2, \chi_{e_1'}) L_2(s_1+2s_2, \chi_{e_2'})}.$$

Thus, (8.2) becomes

$$(8.5) \quad \zeta_2(2s_2)\zeta_2(2s_1+2s_2-1) \times \sum_{\psi=1,\chi_{-4}} \epsilon(\psi) \left(\sum_{\substack{e_1>0 \\ odd}} \frac{\psi\chi_{-8}(e_1)}{e_1^{s_2}} \frac{L(s_1, \chi_{e_1'})}{L_2(s_1+2s_2, \chi_{e_1'})} \right) \left(\sum_{\substack{e_2>0 \\ odd}} \frac{\psi(e_2)}{e_2^{s_2}} \frac{L(s_1, \chi_{e_2'})}{L_2(s_1+2s_2, \chi_{e_2'})} \right).$$

Remove the Euler factors at 2. We get $(1-4^{-s_1})^2$ times

$$[Z(1,-8)+2^{-s_1}Z(1,-4)][Z(1,1)+2^{-s_1}Z(1,8)]$$
$$-[Z(1,8)+2^{-s_1}Z(1,1)][Z(1,-4)+2^{-s_1}Z(1,-8)].$$

Expand.

For S_2 we have

$$2^{s_2}S_2/24=\sum_{\substack{n>0\\ odd}}\frac{1}{n^{s_2}}\left(\sum_{\substack{e_1|n\\ sqfree}}\chi_{-4}(e_1)L_{\mathcal{O}_{-8n}}(s_1,\chi_{e_1',\pm 8e_2'})\right), \tag{8.6}$$

where it is a $+8$ if $n\equiv 3$ mod 4 and -8 if $n\equiv 1$ mod 4. I guess we write this as $1/2$ times

$$\sum_{\substack{n>0\\ odd}}\frac{1}{n^{s_2}}\left(\sum_{\substack{e_1|n\\ sqfree}}\chi_{-4}(e_1)L_{\mathcal{O}_{-8n}}(s_1,\chi_{e_1',-8e_2'})\right)$$
$$+\sum_{\substack{n>0\\ odd}}\frac{\chi_{-4}(n)}{n^{s_2}}\left(\sum_{\substack{e_1|n\\ sqfree}}\chi_{-4}(e_1)L_{\mathcal{O}_{-8n}}(s_1,\chi_{e_1',-8e_2'})\right)$$
$$+\sum_{\substack{n>0\\ odd}}\frac{1}{n^{s_2}}\left(\sum_{\substack{e_1|n\\ sqfree}}\chi_{-4}(e_1)L_{\mathcal{O}_{-8n}}(s_1,\chi_{e_1',8e_2'})\right)$$
$$-\sum_{\substack{n>0\\ odd}}\frac{\chi_{-4}(n)}{n^{s_2}}\left(\sum_{\substack{e_1|n\\ sqfree}}\chi_{-4}(e_1)L_{\mathcal{O}_{-8n}}(s_1,\chi_{e_1',8e_2'})\right)$$

Write this in terms of double Dirichlet series: $(1-4^{-s_1})$ times

$$[Z(1,-4)+2^{-s_1}Z(1,-8)]Z(-8,1)+[Z(1,1)+2^{-s_1}Z(1,8)]Z(-8,-4)$$
$$+[Z(1,-4)+2^{-s_1}Z(1,-8)]Z(8,1)-[Z(1,1)+2^{-s_1}Z(1,8)]Z(8,-4).$$

Expand and you'll get the answer.