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Unitary Periods and Jacquet's Relative Trace Formula

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To Steve Gelbart, with friendship

Abstract. The purpose of these notes is to survey some of the recent developments in the study of unitary periods of automorphic forms on GL_n over a quadratic extension. Jacquet characterized the functorial image of quadratic base change in terms of non vanishing of unitary periods. In a joint work with Lapid, we obtained a formula for the anisotropic unitary periods of certain cusp forms in terms of special values of Rankin-Selberg L-functions. The main tool to obtain both results is the relative trace formula of Jacquet. In this work we explain how it is used in the study of unitary periods. Furthermore, we generalize the fundamental lemma of Jacquet and refine our results on Bessel identities for principal series representations, by solving the transfer factor dichotomy.

1. Introduction

This manuscript is intended to survey some developments in the study of periods of automorphic forms in the context of GL_n over a quadratic extension. In this section, we shall state the main global result obtained in [**Jac05**] and the formula obtained in [**LO07**]. We then continue by introducing the necessary tools to explain the proofs. But first, we recall in a few words our general setting for the study of period integrals.

Let G be a reductive group defined over a number field F with adèle ring $\mathbb{A} = \mathbb{A}_F$. Let θ be an involution on G defined over F and set

$$X = \{g \in G : \theta(g) = g^{-1}\}.$$

The group G acts on the symmetric space X by the θ-twisted conjugation

$$(x, g) \mapsto \theta(g)^{-1}\, x\, g.$$

For every $x \in X(F)$ let H^x be the stabilizer of x in G. A cuspidal automorphic representation π of $G(\mathbb{A})$ is *distinguished* by H^x if there exists a cusp form ϕ in the space of π so that

$$\int_{H^x(F)\backslash H^x(\mathbb{A})} \phi(h)\ dh \neq 0.$$

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It is expected that there is (possibly a central extension of) an algebraic group G' related to (G, θ) and a functorial transfer (in the sense of Langlands functoriality) of automorphic forms from G' to G, so that distinction by some H^x characterizes the functorial image. Furthermore, in many cases, for distinguished representations, the value of the period integral is expected to be related to special values of L-functions.

In this work we consider one particular case of general rank, where G is the group GL_n over a quadratic extension and the involution is defined by $\theta(g) = {}^t\bar{g}^{-1}$ where $x \mapsto \bar{x}$ is the Galois action. Thus X is the space of Hermitian matrices in G and the action[1] of G on X is given by $(x, g) \mapsto {}^t\bar{g}xg$. For $x \in X(F)$, the stabilizer

$$H^x = \{g \in G : {}^t\bar{g}xg = x\}$$

is a unitary group. The group G' is GL_n over the base field and the relevant functorial transfer from G' to G is quadratic base change. Jacquet characterized the image of quadratic base change in terms of non vanishing of unitary periods.

Theorem 1.1. [**Jac05**, Theorem 4] *Let E/F be a quadratic extension of number fields. A cuspidal automorphic representation π of $GL_n(\mathbb{A}_E)$ is a base change from $GL_n(\mathbb{A}_F)$ if and only if it is distinguished by some unitary group.*

The formula obtained in [**LO07**] relates anisotropic unitary periods of certain (distinguished) cusp forms to special values of Rankin-Selberg L-functions. The setting is the following. Let F be a totally real number field of degree d and let E be a totally imaginary quadratic extension of F. We also denote by r the number of finite places of F that ramify in E. Let $G' = GL_n/F$ and let G be the restriction of scalars of GL_n from E to F. Let $\alpha = {}^t\bar{\alpha} \in G(F) = GL_n(E)$ be a Hermitian matrix which is either positive or negative definite in any real embedding of F. Consider the anisotropic unitary group

$$H = H^\alpha = \{g \ \in \mathbf{G} : {}^t\bar{g}\alpha\, g = \alpha\}.$$

Let $\omega = \omega_{E/F}$ be the idèle class character attached to E/F by class field theory and let $\theta = (\theta) \in G(\mathbb{A})$ be such that ${}^t\bar{\theta}_v\theta_v = \pm\alpha_v$ for every real place v of F and $\theta_v = e$ for every finite place v of F. Let π be an irreducible, everywhere unramified cuspidal representation of $G(\mathbb{A})$. Thus, it admits a $\mathbf{K}$-invariant, L^2-normalized automorphic form ϕ_0, where $\mathbf{K} = \prod_v K_v$ is the standard maximal compact subgroup of $G(\mathbb{A})$. Assume further that π is the base change from a cuspidal representation π' of $G'(\mathbb{A})$.

Theorem 1.2. [**LO07**, Theorem 1] *Under the above assumptions, we have*

$$\text{(1.1)} \quad \operatorname{vol}(H^e_{\mathbb{A}} \cap \mathbf{K})^{-2} \left| \int_{H^\alpha(F)\backslash H^\alpha(\mathbb{A})} \phi_0(h\theta^{-1})\ dh \right|^2 = 4^{1-r-nd} \left| \frac{\Delta_E}{\Delta_F} \right|^{\frac{n(n+1)}{2}} \frac{L(1, \pi' \times \tilde{\pi}' \otimes \omega)}{\operatorname{Res}_{s=1} L(s, \pi' \times \tilde{\pi}')}\ \mathcal{P}_\alpha(\pi').$$

Here Δ_F (resp. Δ_E) is the discriminant of F (resp. E). The Haar measure on $H^\alpha(\mathbb{A})$ is the pull-back of the one on $H^e(\mathbb{A})$ (via an inner twist). For the normalization of measure on $G_{\mathbb{A}}$, see §3.1. The term $\mathcal{P}_\alpha(\pi') = \prod_v \mathcal{P}_{\alpha_v}(\pi'_v)$ is a product, over the places v of F, of local factors and for almost all places v

[1] We use a right action in order to align ourselves with Jacquet's notation in [**Jac05**]. When recalling results from papers that use the left action (and sometimes a conjugate of θ rather than θ), we shall adjust the results accordingly.

we have $\mathcal{P}_{\alpha_v}(\pi'_v) = 1$. At an archimedean place v, thanks to the translate by θ, the term $\mathcal{P}_{\alpha_v}(\pi'_v)$ is independent of α_v and is given explicitly in (9.14) (e.g., if π'_v is unramified then $\mathcal{P}_{\alpha_v}(\pi'_v) = 1$). At a finite place v, the term $\mathcal{P}_{\alpha_v}(\pi'_v)$ is expressed explicitly in terms of the value at α_v of a local spherical function on the space of Hermitian matrices if v is inert and on G' if v is split. In the unramified case, the spherical functions were computed explicitly, by Macdonald in the split case and by Hironaka in the inert case. Hironaka's work is discussed in §6. The explicit expression for $\mathcal{P}_{\alpha_v}(\pi'_v)$ for unramified v is given in (9.12) (e.g., if $\alpha_v \in K_v$ then $\mathcal{P}_{\alpha_v}(\pi'_v) = 1$). For a ramified place v, the expression for $\mathcal{P}_{\alpha_v}(\pi'_v)$ in terms of Hironaka's spherical functions is given in (9.13). If $n = 2$, then Hironaka computed the spherical functions also for ramified quadratic extensions. Her results in [**Hir89, Hir90**] can make all the local terms in our formula (1.1) explicit in the case $n = 2$.

Remark 1.3. The anisotropic unitary period of ϕ_0 has a more arithmetic interpretation as a certain finite weighted sum of point evaluations. If, for example, $F = \mathbb{Q}$ and E is of class number one, then the sum is over classes in the genus class of α. This aspect and an interesting relation with a conjecture of Sarnak on the L^∞-norm of a cusp form is explained in [**LO07**], and we do not pursue it here any further.

Remark 1.4. The unitary period of an Eisenstein series induced from the Borel subgroup is expressed in [**Off07**, Theorem 1] as a finite sum of factorizable linear functionals with local factors expressed in [**Off07**, Corollary 1] in terms of Dirichlet L-functions. This formula contains information about classical and new types of representation numbers associated to Hermitian forms. These representation numbers are defined and studied in [**CO07**].

Remark 1.5. Our formula (1.1) indicates, as expected, that unitary periods of cusp forms should be factorizable, whereas the formula mentioned in Remark 1.4 indicates that the unitary period of an Eisenstein series should be expressed as a finite sum of factorizable linear functionals. This is reflected in the fact that a cuspidal representation of $G(\mathbb{A})$ in the image of quadratic base change is essentially (up to a twist by ω) the base change of a unique cuspidal representation on $G'(\mathbb{A})$ whereas an Eisenstein automorphic representation of $G(\mathbb{A})$, that is a base change, is the base change of several automorphic representations of $G'(\mathbb{A})$. The local factors of unitary periods are currently being studied further. They cannot be defined in purely local terms in the spirit of [**II**] for cases of local multiplicity one.

The rest of this manuscript is organized as follows. We begin in §2 with an informal presentation of the distributions involved in the trace formula comparison relevant to us. After introducing the notation in §3, we discuss each of the main ingredients necessary in order to explain the proofs of Theorem 1.1 and of Theorem 1.2. For the first theorem, the main local ingredient is Jacquet's study of matching of orbital integrals explained in §4, and the main global ingredient is Lapid's spectral expansion explained in §5. For the second theorem, in addition, we shall need Hironaka's explicit formulas for spherical functions on Hermitian matrices which we explain in §6 and certain local identities of Bessel distributions for principal series representation explained in §7. In fact, in §7, we refine the main results obtained in [**Off07**] by solving the transfer factor dichotomy raised also in [**Off06**]. The transfer factor dichotomy is explained in Remark 7.5. We then explain the proofs of the two theorems in §8 and §9, respectively. Finally, in §10 we generalize the fundamental lemma of Jacquet. The more general explicit matching of orbital integrals that we obtain was conjectured in [**Off06**].

For the new results of this work we provide complete proofs. Our discussion of proofs for all other results is less formal and of a more descriptive nature. We hope that this attitude will help the reader who is less familiar with the material to approach the subject.

This article is dedicated to Steve Gelbart. The author wishes to thank him for two most enjoyable years at the Weizmann Institute and for all his help and support.

2. The relative trace formula – an informal discussion

We go back to our general global setting where E/F is any quadratic extension of number fields and we keep the notation introduced in §1. Recall that X is the space of Hermitian matrices in G. An important tool in the study of period integrals of automorphic forms is the *relative trace formula* of Jacquet (RTF). For the case at hand, this is a distribution on the space $X(\mathbb{A})$. In order to obtain information about unitary periods, the RTF at hand is compared with the so called *Kuznetzov trace formula* (KTF) on $G'(\mathbb{A})$. We now describe the two distributions.

Let U' (resp. U) be the subgroup of upper triangular unipotent matrices in G' (resp. G). Let ψ' be a non trivial additive character on $F\backslash\mathbb{A}$ and let $\psi(x) = \psi'(x+\bar{x})$, $x \in \mathbb{A}_E$. Denote by $\psi'_{U'}$ the character of $U'(\mathbb{A})$ defined by

$$\psi'_{U'}(u) = \psi'(u_{1,2} + \cdots + u_{n-1,n})$$

and denote by ψ_U the character of $U(\mathbb{A})$ defined similarly with respect to ψ. The comparison of distributions (between the RTF and the KTF) amounts to an identity of the form

$$\begin{aligned}(2.1)\qquad & \int_{U(F)\backslash U(\mathbb{A})} \Big(\sum_{x\in X(F)} \Psi({}^t\bar{u}xu)\Big)\, \psi_U(u)\, du \\ &= \int_{(U'(F)\times U'(F))\backslash(U'(\mathbb{A})\times U'(\mathbb{A}))} \Big(\sum_{g\in G'(F)} \Phi({}^tu_1gu_2)\Big)\, \psi'_{U'}(u_1u_2)\, du_1\, du_2\end{aligned}$$

for suitably matching functions $\Psi \in C_c^\infty(X(\mathbb{A}))$ and $\Phi \in C_c^\infty(G'(\mathbb{A}))$.

The group U acts on X by $(x,u) \mapsto {}^t\bar{u}xu$. We call an element $x \in X(F)$ *relevant* if ψ_U is trivial on the stabilizer $\mathrm{Stab}_{U(\mathbb{A})}(x)$ of x in $U(\mathbb{A})$. Similarly, the group $U'\times U'$ acts on G' by $(g,u_1,u_2) \mapsto {}^tu_1gu_2$, and $g \in G'(F)$ is called relevant if $\psi'_{U'}$ is trivial on $\mathrm{Stab}_{U'(\mathbb{A})\times U'(\mathbb{A})}(g)$. Only relevant orbits contribute to the integrals in (2.1). The comparison in (2.1) is based on a natural bijection between the relevant orbits in $X(F)$ and in $G'(F)$. Indeed, a complete common set of representatives for the relevant orbits consists of elements of the form $w_{M'}a$ where $w_{M'}$ is the longest Weyl element of a standard parabolic subgroup M' of G' and a lies in the center

$T'_{M'}$ of $M'(F)$. Thus both sides admit a geometric expansion and (2.1) becomes

$$\sum_{M'} v_{M'} \sum_{a\in T'_{M'}} \int_{\mathrm{Stab}_{U(\mathbb{A})}(w_{M'})\backslash U(\mathbb{A})} \Psi({}^t\bar{u} w_{M'} a u)\ \psi_U(u)\ du \tag{2.2}$$

$$= \sum_{M'} v'_{M'} \sum_{a\in T'_{M'}} \int_{\mathrm{Stab}_{U'(\mathbb{A})\times U'(\mathbb{A})}(w_{M'})\backslash (U'(\mathbb{A})\times U'(\mathbb{A}))} \Phi({}^t u_1 w_{M'} a u_2))\ \psi'_{U'}(u_1 u_2)\ du_1\ du_2$$

where

$$v_{M'} = \mathrm{vol}(\mathrm{Stab}_{U(F)}(w_{M'})\backslash \mathrm{Stab}_{U(\mathbb{A})}(w_{M'}))$$

and

$$v'_{M'} = \mathrm{vol}(\mathrm{Stab}_{U'(F)\times U'(F)}(w_{M'})\backslash \mathrm{Stab}_{U'(\mathbb{A})\times U'(\mathbb{A})}(w_{M'})).$$

The functions Ψ and Φ have matching orbital integrals if each summand on the left hand side of (2.2) equals the corresponding summand on the right hand side. Since the orbital integrals are decomposable distributions, the matching of orbital integrals reduces to a local linear condition at all places. The more matching functions we can find, the more useful the identity (2.1) becomes for applications. In §4, we overview Jacquet's results concerning local matching of orbital integrals at the finite places.

Also crucial for applications is a spectral expansion for the distributions in each side of (2.1). For the right hand side, a fine spectral expansion can be given without much difficulty, as no convergence issues occur. For the left hand side, Lapid obtains in [**Lap06**] the fine spectral expansion for the RTF (see §5). This is the analogue of Arthur's result in [**Art82**] for the Arthur-Selberg trace formula. Lapid further proves the absolute convergence of the spectral expansion. The results of Jacquet and Lapid combined allow us to compare the contribution of the discrete spectrum to each side of (2.1). Roughly speaking, Jacquet obtains local matching for enough pairs of functions in order to apply a standard argument of linear independence of characters. The outcome is that for every cuspidal automorphic representation π of $G(\mathbb{A})$ and (available) pairs of matching functions, we have

$$\tilde{B}^{\psi}_{\pi}(\Psi) = \sum_{\pi'} B^{\psi'}_{\pi'}(\Phi) \tag{2.3}$$

where the sum is over all cuspidal representations π' of $G'(\mathbb{A})$ that base change to π, the *relative Bessel distribution* $\tilde{B}_{\pi} = \tilde{B}^{\psi}_{\pi}$ is the contribution of π to the RTF, and the Bessel distribution $B_{\pi'} = B^{\psi'}_{\pi'}$ is the contribution of π' to the KTF. For $\Phi \in C_c^{\infty}(G'(\mathbb{A}))$, the Bessel distribution is defined by

$$B_{\pi'}(\Phi) = \sum_{\phi'\in \mathrm{ob}(\pi')} \int_{U'(F)\backslash U'(\mathbb{A})} (\pi(\Phi)\phi')(u w_0)\psi'_{U'}(u)\ du \overline{\int_{U'(F)\backslash U'(\mathbb{A})} \phi'(u)\psi'_{U'}(u)\ du}$$

where the sum is over an orthonormal basis of π' and w_0 is the longest Weyl element in G'. This is independent of the choice of basis. For $\Psi \in C_c^{\infty}(X(\mathbb{A}))$ the relative Bessel distribution is defined by

$$\tilde{B}_{\pi}(\Psi) = \sum_{\phi\in \mathrm{ob}(\pi)} \int_{G(F)\backslash G(\mathbb{A})} \phi(g)\Big(\sum_{x\in X(F)} \Psi({}^t\bar{g} x g)\Big)\ dg \overline{\int_{U(F)\backslash U(\mathbb{A})} \phi(u)\psi_U(u)\ du}.$$

It can be expressed as a sum of distributions on $G(\mathbb{A})$:

$$\tilde{B}_\pi(\Psi) = \sum_{\{\xi\}} B_\pi^\xi(f_\xi)$$

where $\{\xi\}$ is a set of representatives for the $G(F)$-orbits in $X(F)$ and $f^\xi \in C_c^\infty(G(\mathbb{A}))$ is such that

$$\Psi({}^t\bar{g}\xi g) = \int_{H^\xi(\mathbb{A})} f^\xi(hg)\ dh.$$

The relative Bessel distribution on $G(\mathbb{A})$ is defined by

$$B_\pi^\xi(f) = \sum_{\phi \in \mathrm{ob}(\pi)} \int_{H^\xi(F)\backslash H^\xi(\mathbb{A})} (\pi(f)\phi)(h)\ dh \overline{\int_{U(F)\backslash U(\mathbb{A})} \phi(u)\psi_U(u)\ du}.$$

We then see that if the distribution $\tilde{B}_\pi$ is not identically zero, then π is distinguished by some unitary group.

If π is an irreducible, cuspidal representation of $G(\mathbb{A})$ which is the base change of the cuspidal representation π' of $G'(\mathbb{A})$, then π' and its quadratic twist $\pi' \otimes \omega$ are not equivalent and the sum on the right hand side of (2.3) is precisely over π' and $\pi' \otimes \omega$. In this case we can choose matching functions so that only one summand occurs and is indeed non zero. This way Jacquet obtains Theorem 1.1. For more details see §8.

The identity (2.3) is also where we begin the computation of (1.1). More precisely, the relative Bessel distribution on the symmetric space $X(\mathbb{A})$ captures spectral information distinguished by any unitary group. Since we are only concerned with the period integral over H^α, it is enough to consider a test function Ψ on $X(\mathbb{A})$ which is supported on the $G(\mathbb{A})$-orbit of α, i.e., we set $f^\xi = 0$ for every representative $\xi \neq \alpha$. For a test function $f = f^\alpha$ on $G(\mathbb{A})$, we then say that f and Φ have matching orbital integrals if Ψ and Φ do. If the support of Φ is contained in $\ker(\omega \circ \det)$, then $B_{\pi'}(\Phi) = B_{\pi'\otimes\omega}(\Phi)$. Thus, for suitable matching functions f and Φ the formula (2.3) becomes

$$B_\pi^\alpha(f) = 2B_{\pi'}(\Phi). \tag{2.4}$$

This identity is the point of departure for (1.1). We may choose f to be a (certain translate of) a spherical Hecke function on $G(\mathbb{A})$ so that the left hand side of (2.4) is a unique summand over the spherical cusp form, which is a product of the anisotropic unitary period we wish to compute with a Fourier coefficient of ϕ_0 and the spherical Fourier transform of the Hecke function closely related to f. The distribution on the right hand side of (2.4) is factorizable thanks to results of Jacquet, up to an explicit global constant. To obtain the explicit formula for the period, it remains to compute the local factors at finitely many places. The matching function Φ, however, need not be a spherical Hecke function. Thus, to compute the local terms we use a local identity of Bessel distributions that relates $B_{\pi'_v}(\Phi_v)$, the local factor of $B_{\pi'}(\Phi)$ at v, to a local analogue of $B_\pi^\alpha(f)$ at v for matching functions f_v and Φ_v [**Off07**, Theorem 3]. Since f is (essentially) a spherical Hecke function, the local relative Bessel distribution can now be written as a unique summand, which we can express as a product of Hironaka's spherical function evaluated at α_v with a local Whittaker function and the spherical Fourier transform of the Hecke function related to f_v. Putting an absolute value squared on both sides, after some cancellation, we remain with the formula for the period integral in terms of Hironaka's spherical functions. As explained after the statement of Theorem 1.2, whenever applicable, we then use Hironaka's explicit formulas to make (1.1) explicit. This is of course, a very heuristic description of the line of proof. A more detailed description of the proof is in §9.

To summarize, the main ingredients for the proof of Theorem 1.1 are:

- local matching of orbital integrals [**Jac03b, Jac04, Jac05**];
- the fine spectral expansion of the relative trace formula [**Lap06**].

For the proof of Theorem 1.2, the additional ingredients are:

- explicit formulas for spherical functions on the p-adic space of invertible Hermitian matrices [**Hir99**];
- local identities of Bessel distributions for principal series representations [**Off07**].

3. Notation

We alternate between local and global settings throughout this work. We denote by bold letters such as $\mathbf{Y}$ an algebraic set defined over either a number field or a local field F and by the corresponding letter the set of rational points $Y = \mathbf{Y}(F)$. Globally, for every place v of F denote by $Y_v = \mathbf{Y}(F_v)$ the corresponding local space of F_v-rational points and let $Y_{\mathbb{A}} = \mathbf{Y}(\mathbb{A})$.

Denote by E/F a quadratic extension of either number fields in the global case or local fields of characteristic zero in the local case and let $x \mapsto \bar{x}$ denote the associated Galois action. Locally, we shall also allow the split case where $E = F \oplus F$. In that case the Galois action is $(x, y) \mapsto (y, x)$ for $x, y \in F$. In the global case denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adèles of F. Let $\mathrm{Nm}(x) = \mathrm{Nm}_{E/F}(x) = x\bar{x}$ be the norm map, $\mathrm{Tr}(x) = \mathrm{Tr}_{E/F}(x) = x + \bar{x}$ be the trace map and $\omega = \omega_{E/F}$ the quadratic character associated to E/F by class field theory. In the local split case ω is the trivial character. Denote by $\mathbf{E^1}$ the algebraic group defined over F by the kernel of Nm. Thus $E^1 = \{x \in E^\times : \mathrm{Nm}(x) = 1\}$. In the global case, for every place v of F we let $E_v = E \otimes_F F_v$. If v is split in E, then $E_v \simeq F_v \oplus F_v$; otherwise E_v/F_v is a quadratic extension of local fields. When F is a p-adic field, we denote by $\mathcal{O} = \mathcal{O}_F$ the ring of integers of F, by $\mathfrak{p} = \mathfrak{p}_F$ its maximal ideal, by $\varpi = \varpi_F$ a uniformizer in $\mathfrak{p}$ and by $q = q_F$ the cardinality of the residual field of F.

Let $\mathbf{G}'$ be the group GL_n regarded as an algebraic group defined over F, and let $\mathbf{G} = R_{E/F}(GL_n)$ be the restriction of scalars of GL_n from E to F. Thus $G = GL_n(E)$ whereas $G' = GL_n(F)$. We denote the quadratic base change transfer by bc. Thus, for an irreducible, cuspidal automorphic representation π' of $G'(\mathbb{A})$, $\mathrm{bc}(\pi')$ is the irreducible automorphic representation of $G(\mathbb{A})$ such that

$$L(s, \mathrm{bc}(\pi')) = L(s, \pi')L(s, \pi' \otimes \omega).$$

We denote by

$$\mathbf{X} = \{g \in \mathbf{G} : g = {}^t\bar{g}\}$$

the space of Hermitian matrices in $\mathbf{G}$ and consider it as a right $\mathbf{G}$-space with action

$$(x, g) \mapsto {}^t\bar{g}xg.$$

For every Hermitian matrix $x \in X$, let

$$\mathbf{H^x} = \{g \in \mathbf{G} : {}^t\bar{g}xg = x\}$$

be the associated unitary group.

Let ψ' be a non trivial character of F in the local case and of $F\backslash\mathbb{A}$ in the global case and let

$$\psi = \psi' \circ \mathrm{Tr}_{E/F}\,.$$

In the rest of this section we shall fix notation and conventions with respect to the group G and the character ψ. Similar notation and conventions for G' and ψ' will be appended with a prime.

In the local case we denote by K the standard maximal compact subgroup of G. Thus, $K' = GL_n(\mathcal{O})$ in the non-archimedean case, $K' = O(n)$ in the real case and $K' = U(n)$ in the complex case. In the global case we let $\mathbf{K} = \prod_v K_v$ denote the standard maximal compact subgroup of $G_{\mathbb{A}}$ where the product is over all places in F. Thus if, for example, v is a split place of F then $K_v = K'_v \times K'_v$. Let $\mathbf{B} = \mathbf{TU}$ be the subgroups of $\mathbf{G}$ so that B is the group of upper triangular matrices, T is the group of diagonal matrices and U is the group of upper triangular unipotent matrices in G. We denote by ψ_U the generic character of U in the local case and of $U\backslash U_{\mathbb{A}}$ in the global case defined by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

Let $\mathbf{Y}$ be an algebraic group defined over F. We denote by δ_Y the modulus function of the group $Y_{\mathbb{A}}$ in the global case and of Y in the local case. We denote by $X^*(Y)$ the lattice of F-rational characters on Y. Let $\mathfrak{a}_Y^* = X^*(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\mathfrak{a}_Y$ be its dual. We set $\mathfrak{a}_0 = \mathfrak{a}_T$ and $\mathfrak{a}_0^* = \mathfrak{a}_T^*$. We identify both $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ with $\mathbb{R}^n$. The natural matching between them denoted by $\langle \cdot, \cdot \rangle$ is then the standard inner product on $\mathbb{R}^n$ invariant under the Weyl group W of G with respect to T. Let $M \subseteq L$ be standard Levi subgroups of G. There is a natural embedding of $\mathfrak{a}_L$ into $\mathfrak{a}_M$. We denote by $\mathfrak{a}_M^L$ its orthogonal complement and use similar notation for the dual subspaces so that we also have $\mathfrak{a}_M^* = \mathfrak{a}_L^* \oplus (\mathfrak{a}_M^L)^*$. For every $\lambda \in \mathfrak{a}_0^*$, we denote by λ_M, λ^L (resp. λ_M^L) its orthogonal projection to the space $\mathfrak{a}_M^*$, $(\mathfrak{a}_0^L)^*$ (resp. $(\mathfrak{a}_M^L)^*$). For any real vector space $\mathfrak{a}$, we denote by $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. In the global case we denote by $Y_{\mathbb{A}}^1$ the intersection of $\ker|\chi|$ for all $\chi \in X^*(Y)$, where $|\chi| = \prod_v |\chi_v|_v$ is the associated character of $Y_{\mathbb{A}}$. For a standard Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, we denote by $\mathbf{T_M}$ the center of M and by A_M the split component of the center of $M_{\mathbb{A}}$. The height function $H : G(\mathbb{A}) \to \mathfrak{a}_0$ is defined by $e^{\langle H(g), \chi \rangle} = |\chi(t)|$ for $\chi \in X^*(T)$ via the Iwasawa decomposition $g = utk$, $u \in U_{\mathbb{A}}$, $t \in T_{\mathbb{A}}$, $k \in \mathbf{K}$. It defines an isomorphism $G_{\mathbb{A}}/G_{\mathbb{A}}^1 \simeq \mathfrak{a}_0$. More generally, the height function defines an isomorphism from A_M to $\mathfrak{a}_M$. In the local case, the height function $H : G \to \mathfrak{a}_0$ is defined similarly. Thus for every $g \in G(\mathbb{A})$, $H_w(g_w) = 0$ for almost all places w of E and $\sum_w H_w(g_w) = H(g)$. Note that with our conventions $H(g) = 2H'(g)$ whenever $g \in G'_{\mathbb{A}}$. We denote by $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}) \in \mathfrak{a}_0^*$ half the sum of the positive roots of G with respect to B. Thus $\delta_B = e^{\langle 2\rho, H(\cdot) \rangle}$. More generally, if $P = MU_P \subseteq Q = LU_Q$ are standard parabolic subgroups with their associated standard Levi decompositions, then

$$\delta_{P \cap L} = e^{\left\langle 2\rho_M^L, H(\cdot) \right\rangle}.$$

For any set Γ, we shall denote by $\mathbf{1}_\Gamma$ the characteristic function of Γ without specifying its domain.

3.1. Measures. Our normalization of measures is the same as in [**LO07**]. We repeat our conventions here.

Discrete groups will be endowed with the counting measure. The measures on the local groups will be determined by the non trivial character ψ of E as follows. On E we put the measure $dx = d^{\psi}x$ which is self-dual with respect to ψ. If $\psi_a = \psi(a\ \cdot)$ for $a \in E^{\times}$, then $d^{\psi_a}x = |a|^{\frac{1}{2}}d^{\psi}x$. Set

$$\mathfrak{d}_E = \mathfrak{d}_E^{\psi} = \begin{cases} \mathrm{vol}(\mathcal{O}_E) & E \text{ non-archimedean,} \\ \mathrm{vol}([0,1]) & E \text{ real,} \\ \frac{1}{2}\,\mathrm{vol}(\{x+iy : 0 \le x, y \le 1\}) & E \text{ complex.} \end{cases}$$

If E is non-archimedean and ψ has conductor $\mathcal{O}_E$, then $\mathfrak{d}_E^{\psi} = 1$. The same is true if E is archimedean and $\psi(x) = e^{2\pi i \operatorname{Tr}_{E/\mathbb{R}} x}$. We have $\mathfrak{d}_E^{\psi_a} = |a|^{\frac{1}{2}}\mathfrak{d}_E^{\psi}$. Next, we put on U the measure $du = \otimes_{i<j} dx_{i,j}$. On $E^{\times}$ we take the measure $d^{\times}x = L(1, \mathbf{1}_{E^{\times}})\frac{dx}{|x|}$ where $L(1, \mathbf{1}_{E^{\times}})$ is the local L-factor of Tate. The measure dt on T will be determined by the isomorphism $T \simeq (E^{\times})^n$. On G we take the measure $dg = dt\ du\ dk$ with respect to the Iwasawa decomposition $G = TUK$ where dk is the measure on K with total mass 1. If E is p-adic and ψ has conductor $\mathcal{O}_E$, then the measure on G gives $\mathrm{vol}(K) = 1$.

Globally, for a character $\psi = \otimes_w \psi_w$ of $E\backslash\mathbb{A}_E$, we take on $\mathbb{A}_E$ the self-dual measure with respect to ψ. It is also given by $\otimes_w\ dx_w$ where $dx_w = d^{\psi_w}x_w$. This does not depend on the choice of ψ, and we have $\mathrm{vol}(E\backslash\mathbb{A}_E) = 1$. Similarly, $\mathfrak{d}_E := \prod_w \mathfrak{d}_{E_w}^{\psi_w}$ does not depend on ψ and in fact $\mathfrak{d}_E = |\Delta_E|^{-\frac{1}{2}}$ where Δ_E is the discriminant of E. On $\mathbb{A}_E^{\times}$ we put the measure $\otimes_w\ d^{\times}x_w$. On $\mathbb{A}_E^1$, the idèles of norm 1, we take the measure so that the measure induced on $\mathbb{A}_E^1\backslash\mathbb{A}_E^{\times}$ is the pull-back, under the isomorphism $|\cdot| : \mathbb{A}_E^1\backslash\mathbb{A}_E^{\times} \to \mathbb{R}_+$, of the standard multiplicative measure $\frac{dt}{t}$ on $\mathbb{R}_+$. Then $\mathrm{vol}(E^{\times}\backslash\mathbb{A}_E^1) = \lambda_{-1} = \mathrm{Res}_{s=1}\, L(s, \mathbf{1}_{E^{\times}})$ where $L(s, \mathbf{1}_{E^{\times}})$ is the completed Dedekind ζ-function for E. Similarly, on $G_{\mathbb{A}}$ we take $dg = \otimes_w\ dg_w$, which is also the measure determined by the Iwasawa decomposition $G_{\mathbb{A}} = T_{\mathbb{A}}U_{\mathbb{A}}\mathbf{K}$. We induce a measure on $G_{\mathbb{A}}^1$ by identifying $G_{\mathbb{A}}/G_{\mathbb{A}}^1$ with $\mathbb{R}_+$ via $|\det|$. In accordance with our conventions, the analogous measures with respect to F and ψ' have now been set as well.

On the unitary groups we choose measures consistently, i.e., for every $x \in X$ the Haar measure of the local unitary group H^x is the pull-back of the one on H^e via an inner twist. The global unitary groups are endowed with the product measure. Locally, the measure on X is given by the isomorphism $\bigsqcup H^{\xi}\backslash G \simeq X$ where the union is over representatives $\{\xi\}$ of the G-orbits in X. By our choice of consistent measures, this is independent of the choice of representatives. Globally, we take the product measure on $X_{\mathbb{A}}$ which is also given by the isomorphism $\bigsqcup H_{\mathbb{A}}^{\xi}\backslash G_{\mathbb{A}} \simeq X_{\mathbb{A}}$ where the union is now over representatives $\{\xi\}$ of the $G_{\mathbb{A}}$-orbits in $X_{\mathbb{A}}$ and $H_{\mathbb{A}}^{\xi} = \prod_v' H_v^{\xi_v}$ is given the product measure.

Locally, the measure on E^1 is defined by the relation

$$\int_{E^{\times}} f(z)\ dz = \int_{\mathrm{Nm}(E^{\times}) \subset F^{\times}} F(x)\ dx \quad \text{where} \quad F(\mathrm{Nm}\, t) = \int_{E^1} f(yt)\ dy.$$

Note then that in the non-archimedean case

$$\mathrm{vol}(E^1 \cap \mathcal{O}_E^\times) = \begin{cases} \frac{\mathfrak{d}_E}{\mathfrak{d}_F} & E/F \text{ is unramified (split or inert)} \\ 2\frac{\mathfrak{d}_E}{\mathfrak{d}_F} & \text{else.} \end{cases}$$

Globally, we take the product measure on $E^1_{\mathbb{A}}$. Whenever $\eta \in G$ and $x \in X$ are such that ${}^t\bar{\eta}x\eta \in T'$, we shall also consider the group $\mathbf{H}^{\mathbf{x}}_\eta = \mathbf{H}^{\mathbf{x}} \cap \eta\mathbf{B}\eta^{-1}$. It is not hard to see that $\eta^{-1}H^x_\eta\eta = \{\mathrm{diag}(a_1, \dots, a_n) : a_i \in E^1\}$. The measures on the groups H^x_η are then determined by their isomorphism with $(E^1)^n$.

3.2. Spherical Hecke algebras. In the local case denote by

$$\mathcal{H}_G(K) = C_c^\infty(K\backslash G/K)$$

the spherical Hecke algebra of G with respect to K. Multiplication is given by the convolution

$$f_1 * f_2(g) = \int_G f_1(y) f_2(gy^{-1})\, dy.$$

For $f \in \mathcal{H}_G(K)$, we denote by $f^\vee$ the function $f^\vee(g) = f(g^{-1})$. The spherical Fourier transform is defined by

$$\hat{f}(\lambda) = \int_G f(g) e^{\langle \lambda+\rho, H(g)\rangle}\, dg$$

for $f \in \mathcal{H}_G(K)$ and $\lambda \in \mathfrak{a}^*_{0,\mathbb{C}} \simeq \mathbb{C}^n$. If $\lambda_0 \in i\mathbb{R}^n$ and $\chi(t) = \chi_{\lambda_0}(t) = e^{\langle \lambda_0, H(t)\rangle}$ is the associated unramified unitary character of T, we also set

$$\hat{f}(\chi, \lambda) = \hat{f}(\lambda_0 + \lambda).$$

In the p-adic case the spherical Fourier transform defines an algebra isomorphism from $\mathcal{H}_G(K)$ to the algebra $\mathbb{C}[q_E^{\pm\lambda}]^W$ of Laurent polynomials in $q_E^\lambda = (q_E^{\lambda_1}, \dots, q_E^{\lambda_n})$ which are invariant under the action of the Weyl group W on λ.

Quadratic base change defines an injective endomorphism also denoted by

$$\mathrm{bc} : \mathcal{H}_G(K) \to \mathcal{H}_{G'}(K'),$$

characterized by the requirement that $\hat{f'} = \hat{f}$ whenever $f \in \mathcal{H}_G(K)$ and $f' = \mathrm{bc}(f)$. Note that since the spherical Fourier transform of $f^\vee$ at λ is $\hat{f}(-\lambda)$, we also have

$$\mathrm{bc}(f^\vee) = \mathrm{bc}(f)^\vee,\ f \in \mathcal{H}_G(K). \tag{3.1}$$

In the unramified p-adic case we have $q_E = q_F^2$. In this case, quadratic base change allows us to view $\mathcal{H}_{G'}(K')$ as a free $\mathcal{H}_G(K)$-module of rank 2^n.

In the p-adic case, we denote by $\mathcal{H}_X(K)$ the space of compactly supported K-invariant functions on X. It is an $\mathcal{H}_G(K)$-module with respect to the convolution

$$f * \Psi(x) = \int_G f(g^{-1}) \Psi({}^t\bar{g}xg)\, dg.$$

In §6, we review Hironaka's theory of a spherical Fourier transform on $\mathcal{H}_X(K)$. If E/F is unramified, then it is an isomorphism of $\mathcal{H}_X(K)$ with $\mathbb{C}[q_F^{\pm\lambda}]^W$ that identifies $\mathcal{H}_X(K)$ with $\mathcal{H}_{G'}(K')$ as free $\mathcal{H}_G(K)$-modules of rank 2^n.

3.3. Induced representations, Whittaker functionals and Eisenstein series. Let $P = MU_P$ be a standard parabolic subgroup of G with its standard Levi decomposition. For $\lambda \in \mathfrak{a}_{M,\mathbb{C}}$ and for a unitary representation π of M in the local case and of $M_{\mathbb{A}}$ in the global case, we denote by $I_P^G(\pi, \lambda)$, the representation parabolically induced from $\pi \otimes e^{\langle \lambda, H(\cdot) \rangle}$. We identify the representation spaces for all λ with the space $I_P^G(\pi)$ of sections satisfying

$$\varphi(umg) = e^{\langle \rho_M, H(m) \rangle} \pi(m)\varphi(g).$$

We then set $\varphi_\lambda(g) = e^{\langle \lambda, H(g) \rangle}\ \varphi(g)$. For the representation $I_P^G(\pi, \lambda)$, the action is then given by

$$(I_P^G(g, \pi, \lambda)\varphi)(y) = e^{-\langle \lambda, H(y) \rangle} \varphi_\lambda(yg).$$

Let χ be a unitary character of T in the local case or of $T\backslash T_{\mathbb{A}}$ in the global case. We also denote $I_B^G(\chi, \lambda)$ by $I(\chi, \lambda)$. The inner product on $I(\chi)$ will be given by

$$(\varphi_1, \varphi_2) = \int \varphi_1(g)\overline{\varphi}_2(g)\ dg \tag{3.2}$$

where the integral is over $B\backslash G$ in the local case and over $B_{\mathbb{A}}\backslash G_{\mathbb{A}}$ in the global case.

For an automorphic form ϕ on $G_{\mathbb{A}}$, its ψth Fourier coefficient is defined by

$$\mathcal{W}(\phi) = \mathcal{W}^\psi(\phi) = \int\limits_{U\backslash U_{\mathbb{A}}} \phi(u)\psi_U(u)\ du$$

and we denote by $\overline{\mathcal{W}}(\phi)$ its complex conjugate. We denote by w_0 the longest Weyl element in W. If $\varphi \in I(\chi)$ is a section in a principal series representation, the associated Whittaker function is given by the Jacquet integral

$$W(g, \varphi, \lambda) = W^\psi(g, \varphi, \lambda) = \int\limits_{U} \varphi_\lambda(w_0 ug)\psi_U(u)\ du$$

and again, we denote by $\overline{W}(g, \varphi, \lambda)$ its complex conjugate. We also set $\mathcal{W}(\varphi, \lambda) = W(e, \varphi, \lambda)$.

For a cuspidal representation π of $M_{\mathbb{A}}$, a parabolic subgroup Q containing P and $\varphi \in I_P^G(\pi)$, let $E^Q(\varphi, \lambda)$ be the Eisenstein series defined as the meromorphic continuation of the series

$$E^Q(g, \varphi, \lambda) = \sum_{\gamma \in P\backslash Q} \varphi_\lambda(\gamma g).$$

When $Q = G$ we shall often omit the superscript Q. For a unitary character χ of $T\backslash T_{\mathbb{A}}$ and $\varphi \in I(\chi)$, we have $\mathcal{W}(E(\varphi, \lambda)) = \mathcal{W}(\varphi, \lambda)$.

3.4. Bessel distributions. The distributions that occur in the spectral expansion of the RTF as well as in the KTF and related distributions on the local spaces are all of the type presented in [**JLR04**, §4.1] as generalized Bessel distributions. We recall here the definitions and set the notation.

Let (π_i, V_i), $i = 1, 2$ be a pair of admissible smooth representations of G with a G-invariant pairing $(\cdot, \cdot)$ which is linear in the first variable and conjugate linear in the second. For any continuous linear forms l_i on V_i, $i = 1, 2$, the Bessel distribution is defined by

$$\mathfrak{B}_{V_1, V_2}^{l_1, l_2, (\cdot,\cdot)}(f) = \mathfrak{B}^{l_1, l_2, (\cdot,\cdot)}(f) = \overline{l_2}[l_1 \circ \pi_1(f)]$$

for any $f \in C_c^\infty(G)$. Here we view $l_1 \circ \pi_1(f)$ as an element of the dual $V_1^\vee$ of V_1 and $\overline{l_2}$ as a linear form on $V_1^\vee$ through the pairing $(\cdot,\cdot)$. In particular, if π is unitary with an invariant inner product $(\cdot,\cdot)$, then

$$\mathfrak{B}_{V,V}^{l_1,l_2,(\cdot,\cdot)}(f) = \sum_{\varphi \in \mathrm{ob}(\pi)} l_1(\pi(f)\varphi)\overline{l_2(\varphi)}$$

for any continuous linear forms l_i on V where $\mathrm{ob}(\pi)$ is any choice of an orthonormal basis for V. Also if $\pi_1 = I(\chi,\lambda)$ and $\pi_2 = I(\chi,-\bar{\lambda})$ are principal series representations (note that with our convention they have the same representation space $I(\chi)$ with inner product (3.2)) then

$$\mathfrak{B}_{V_1,V_2}^{l_1,l_2,(\cdot,\cdot)}(f) = \sum_{\varphi \in \mathrm{ob}(I(\chi))} l_1(\pi(f)\varphi)\overline{l_2(\varphi)}$$

for any continuous linear forms l_i on $I(\chi)$. These generalized Bessel distributions, associated to principal series representations, will occur frequently in this work.

Remark 3.1. Strictly speaking, the generalized Bessel distributions are defined as above for K-finite functions and can be extended to the space of smooth functions of compact support (cf. [**JLR04**, §4.1]).

4. Orbital integrals

The orbital integrals that we consider have been studied extensively in a long series of papers [**JY90, JY92, Jac92, Ye93, JY96, Ye98, Jac98, JY99, Jac02, Jac03a, Jac03b, Jac04**] to list a few, culminating in the remarkable work of Jacquet [**Jac05**], where he obtains an explicit identity between orbital integrals that constitutes the *fundamental lemma* for the RTF for spherical Hecke functions.

We introduce the orbital integrals in the local case. The group $U' \times U'$ acts on G' by

$$(g,u_1,u_2) \mapsto {}^t u_1 g u_2,\ u_1,\ u_2 \in U',\ g \in G'. \tag{4.1}$$

An element $g \in G$ (or its orbit) is called *relevant* if the function $(u_1,u_2) \mapsto \psi_U(u_1u_2)$ is trivial on the stabilizer Stab_g of g in $U' \times U'$. For a function $\Phi \in C_c^\infty(G')$ and a relevant $g \in G'$, let

$$\Omega[\Phi,\psi' : g] = \int_{\mathrm{Stab}_g \backslash U' \times U'} \Phi({}^t u_1 g u_2)\psi'_{U'}(u_1u_2)\ du_1\ du_2. \tag{4.2}$$

Similarly, the group U acts on X by

$$(x,u) \mapsto {}^t \bar{u} x u,\ u \in U,\ x \in X, \tag{4.3}$$

and x (or its orbit) is called relevant if ψ_U is trivial on the stabilizer Stab_x of x. For $\Psi \in C_c^\infty(X)$ and a relevant $x \in X$, let

$$\Omega[\Psi,\psi,E/F : x] = \int_{\mathrm{Stab}_x \backslash U} \Psi({}^t \bar{u} x u)\psi_U(u)\ du. \tag{4.4}$$

The *matching of orbital integrals* is based on a natural bijection between the relevant $(U' \times U')$-orbits on G' and the relevant U-orbits on G. Indeed, as explained in §2, a complete common set of representatives for the relevant orbits consists of elements of the form $w_{M'}a$ where $w_{M'}$ is the longest Weyl element of a standard parabolic subgroup M' of G' and a lies in the center $T'_{M'}$ of M'. The orbital integrals are not constant on the orbits but we have

$$\psi'_{U'}(u_1u_2)\Omega[\Phi,\psi' : {}^tu_1gu_2] = \Omega[\Phi,\psi' : g]$$

and

$$\psi_U(u)\Omega[\Psi,\psi,E/F : {}^t\bar{u}xu] = \Omega[\Psi,\psi,E/F : x].$$

It is therefore enough to study the orbital integrals on the representatives $w_{M'}a$. In a sense that we shall soon explain, the orbital integrals are determined by their values on representatives of orbits of maximal dimension, i.e when $M' = T'$. In this case $w_{M'} = e$ and $a = \mathrm{diag}(a_1,\dots,a_n)$ is any element of T'. Let $\gamma(a)$ be the transfer factor defined by

$$\gamma(a) = \omega(a_1)\omega^2(a_2)\cdots\omega^n(a_n).$$

Note that it differs from Jacquet's transfer factor in [**Jac05**] by a factor of $\omega^n(\det a)$, i.e., it is the same transfer factor if n is even and differs by a factor of $\omega(\det a)$ if n is odd. We say that Φ and Ψ have matching orbital integrals for ψ' and write $\Phi \overset{\psi'}{\longleftrightarrow} \Psi$ if

$$\Omega[\Phi,\psi' : a] = \gamma(a)\ \Omega[\Psi,\psi,E/F : a],\ a\in T'. \tag{4.5}$$

In the rest of this section we restrict ourselves to the non-archimedean case. We begin by stating two density results for the orbital integrals associated with diagonal elements.

THEOREM 4.1. [**Jac03a**, Théorème 1.1] *If $\Phi \in C_c^\infty(G')$ is such that $\Omega[\Phi,\psi' : a] = 0$ for all $a \in T'$, then $\Omega[\Phi,\psi' : g] = 0$ for every relevant $g \in G'$.*

THEOREM 4.2. [**Jac03a**, Théorème 2.1] *There exist transfer factors $\gamma(w_{M'}a,\psi')$ such that whenever $\Phi \overset{\psi'}{\longleftrightarrow} \Psi$ we also have*

$$\Omega[\Phi,\psi' : w_{M'}a] = \gamma(w_{M'}a,\psi')\ \Omega[\Psi,\psi,E/F : w_{M'}a],\ a\in T'_{M'}.$$

In fact, assuming that F is of characteristic zero and that E/F is unramified, with some mild restrictions on the residual characteristic, Theorems 4.1 and 4.2 were already proved in [**Jac98**]. But once he developed his machinery for the study of orbital integrals, Jacquet's proofs in [**Jac03a**] become much simpler.

For global applications, there are two major tasks in matching orbital integrals. The first, referred to as *smooth matching* is to show the existence of enough pairs $\Phi \longleftrightarrow \Psi$ of matching functions. The second and more difficult problem of explicitly matching a bi K'-invariant function on G' with a K-invariant function on X, is the *fundamental lemma* for the relative trace formula. Jacquet obtained the following results:

THEOREM 4.3 (Smooth matching [**Jac03b**]). *For every $\Phi \in C_c^\infty(G')$, there exists $\Psi \in C_c^\infty(X)$ and for every $\Psi \in C_c^\infty(X)$, there exists $\Phi \in C_c^\infty(G')$ so that $\Phi \overset{\psi'}{\longleftrightarrow} \Psi$.*

The space $C_c^\infty(X)$ is a $C_c^\infty(G)$-module under the convolution

$$f*\Psi(x) = \int_G f(g^{-1})\Psi({}^t\bar{g}xg)\ dg,\ f\in C_c^\infty(G),\ \Psi\in C_c^\infty(X). \tag{4.6}$$

Denote by $\Psi^{(0)}$ the characteristic function of $X\cap K$.

THEOREM 4.4 (The fundamental lemma of Jacquet [**Jac05**]). *Assume that F has odd residual characteristic, that ψ' has conductor $\mathcal{O}_F$ and that E/F is unramified. For every Hecke function $f \in \mathcal{H}_G(K)$, we have*

$$\mathrm{bc}(f) \overset{\psi'}{\longleftrightarrow} f * \Psi^{(0)}.$$

This theorem was first proved by Ngô in [**Ngô99a**] for local fields of positive characteristic (see also [**Ngô99b**] for the matching of Hecke unit elements in positive characteristic and [**Jac04**] in characteristic zero).

We now wish to explain the machinery developed by Jacquet and to very roughly explain the idea behind his proofs. It turns out to be more useful to linearize the problem and consider more general orbital integrals. The group $U' \times U'$ also acts on the linear space $M_n(F)$ of $n \times n$ matrices with entries in F by (4.1) and relevant orbits in $M_n(F)$ can be defined as before. For a Schwartz function $\Phi \in C_c^\infty(M_n(F))$ and a relevant $g \in M_n(F)$, we define the orbital integral $\Omega[\Phi, \psi' : g]$ by the formula (4.2). Similarly, let

$$H_n(E/F) = \{X \in M_n(E) : {}^t\bar{X} = X\}$$

be the linear space of $n \times n$ Hermitian matrices and let U act on $H_n(E/F)$ by (4.3). For a Schwartz function $\Psi \in C_c^\infty(H_n(E/F))$ and a relevant $x \in H_n(E/F)$, we define the orbital integral $\Omega[\Psi, \psi, E/F : x]$ by the formula (4.4). A diagonal matrix $a = \mathrm{diag}(a_1, \dots, a_n)$ with entries in F is relevant if and only if $a_1 \cdots a_{n-1} \neq 0$. In [**Jac03a**], the density results Theorem 4.1 and Theorem 4.2 are in fact proved for all Schwartz functions. It is therefore enough to consider only orbital integrals for relevant diagonal matrices. For a Schwartz function f either in $C_c^\infty(M_n(F))$ or in $C_c^\infty(H_n(E/F))$ and an integer k, denote by $f[k]$ the product of f with the characteristic function of $\{X : |\det X|_F = |\varpi^k|_F\}$. Since det is fixed on orbits, the definition of matching of orbital integrals can be generalized to $\Phi \in C_c^\infty(M_n(F))$ and $\Psi \in C_c^\infty(H_n(E/F))$ by writing

$$\Phi \overset{\psi'}{\longleftrightarrow} \Psi \quad \text{whenever} \quad \Phi[k] \overset{\psi'}{\longleftrightarrow} \Psi[k],\ k \in \mathbb{Z}. \tag{4.7}$$

Note that we cannot directly define matching by (4.5) since the transfer factor is not always defined on the relevant diagonal elements.

It will also be convenient to normalize the orbital integrals. For $a = \mathrm{diag}(a_1, \dots, a_n)$, let

$$\tilde{\Omega}[\Phi, \psi' : a] = |a_1^{n-1} a_2^{n-2} \cdots a_{n-1}|_F \Omega[\Phi, \psi' : a]$$

and

$$\tilde{\Omega}[\Psi, \psi, E/F : a] = \omega(a_1^{n-1} a_2^{n-2} \cdots a_{n-1}) |a_1^{n-1} a_2^{n-2} \cdots a_{n-1}|_F \Omega[\Psi, \psi', E/F : a].$$

Let

$$\mathcal{O}_n^{\psi'}(F) = \{a \mapsto \tilde{\Omega}[\Phi, \psi' : a] : \Phi \in C_c^\infty(M_n(F))\},$$

and let

$$\mathcal{O}_n^{\psi}(E/F) = \{a \mapsto \tilde{\Omega}[\Psi, \psi, E/F : a] : \Psi \in C_c^\infty(H_n(E/F))\}.$$

Since $\bar{\psi}_U(u) = \psi_U(\epsilon u \epsilon^{-1})$ for $\epsilon = \mathrm{diag}(1, -1, 1, -1, \dots)$ and similarly for $\psi'_{U'}$ and since $\epsilon^2 = e$, it is easy to see that $\mathcal{O}_n^{\psi'}(F) = \mathcal{O}_n^{\bar{\psi}'}(F)$ and that $\mathcal{O}_n^{\psi}(E/F) = \mathcal{O}_n^{\bar{\psi}}(E/F)$. Theorem 4.3 is a consequence of the equality

$$\mathcal{O}_n^{\psi'}(F) = \mathcal{O}_n^{\psi}(E/F). \tag{4.8}$$

Indeed, if $\tilde{\Omega}[\Phi, \psi' : a] = \tilde{\Omega}[\Psi, \psi, E/F : a]$ for all relevant a, then for all integers k we have

$$\Phi[k] \overset{\psi'}{\longleftrightarrow} (\omega^n \circ \det)\Psi[k].$$

If in the first place the function Φ is in $C_c^\infty(G')$ (resp. Ψ is in $C_c^\infty(X)$), then it equals the sum over finitely many k of $\Phi[k]$ (resp. $\Psi[k]$) and by linearity we get that Φ (resp. Ψ) matches a function in $C_c^\infty(X)$ (resp. $C_c^\infty(G')$).

The advantage of the linearized problem of smooth matching, is that we may use Fourier analysis on the spaces of Schwartz functions. We define the Fourier transform $\mathcal{F} = \mathcal{F}_{\psi'}$ as follows. For $\Phi \in C_c^\infty(M_n(F))$, let

$$\mathcal{F}(\Phi)(X) = \int\limits_{M_n(F)} \Phi(Y)\psi'(-\operatorname{Tr}(Yw_0Xw_0))\ dy,$$

and for $\Psi \in C_c^\infty(H_n(E/F))$, let

$$\mathcal{F}(\Psi)(X) = \int\limits_{H_n(E/F)} \Psi(Y)\psi'(-\operatorname{Tr}(Yw_0Xw_0))\ dy.$$

The Fourier inversion formula is the statement that $\mathcal{F}_{\bar{\psi}'} \circ \mathcal{F}_{\psi'} = \mathrm{Id}$. To make use of the Fourier transform, Jacquet introduced a transform on the spaces of normalized orbital integrals. For a function Ω on the set of relevant diagonal matrices, whenever well-defined, the *Jacquet transform* $\mathfrak{J} = \mathfrak{J}_{\psi'}$ is given by the iterated integral

$$\mathfrak{J}(\Omega)(a_1, \dots, a_n) = \int \Omega(b_1, \dots, b_n)\psi'\left(-\sum_{i=1}^{n} b_i a_{n+1-i} + \sum_{i=1}^{n-1} \frac{1}{b_i a_{n-i}}\right)\ db_n\ db_{n-1} \cdots\ db_1$$

over $b_i \in F$. Not without effort, Jacquet shows that the Jacquet transform is well defined on $\mathcal{O}_n^{\psi'}(F)$ and on $\mathcal{O}_n^{\psi}(E/F)$ and that the Jacquet and the Fourier transforms essentially intertwine with the operation of taking orbital integrals. More precisely, Theorems 1 and 2 of [**Jac03b**] state that the following diagrams commute:

$$\begin{array}{ccc} C_c^\infty(M_n(F)) & \xrightarrow{\tilde{\Omega}_{\psi'}} & \mathcal{O}_n^{\psi'}(F) \\ {\scriptstyle \mathcal{F}_{\psi'}}\downarrow & & {\scriptstyle \mathfrak{J}_{\psi'}}\downarrow \\ C_c^\infty(M_n(F)) & \xrightarrow{\tilde{\Omega}_{\bar{\psi}'}} & \mathcal{O}_n^{\psi'}(F) \end{array} \quad \text{and} \quad \begin{array}{ccc} C_c^\infty(H_n(E/F)) & \xrightarrow{\tilde{\Omega}_{\psi}} & \mathcal{O}_n^{\psi}(E/F) \\ {\scriptstyle \mathcal{F}_{\psi'}}\downarrow & & {\scriptstyle \mathfrak{J}_{\psi'}}\downarrow \\ C_c^\infty(H_n(E/F)) & \xrightarrow{c\ \tilde{\Omega}_{\bar{\psi}}} & \mathcal{O}_n^{\psi}(E/F) \end{array} \tag{4.9}$$

where

$$\tilde{\Omega}_{\psi'}(\Phi)(a) = \tilde{\Omega}[\Phi, \psi' : a],\ \tilde{\Omega}_{\psi}(\Psi)(a) = \tilde{\Omega}[\Psi, \psi, E/F : a],\ c = c(E/F, \psi')^{\frac{(n-1)n}{2}},$$

and $c(E/F, \psi')$ is the Weil constant defined by the identity

$$\int\limits_E \hat{\phi}(x)\psi'(ax\bar{x})\ dx = |a|_F^{-1}\omega(a)c(E/F, \psi) \int\limits_E \phi(x)\psi'(-a^{-1}x\bar{x})\ dx$$

for all $\phi \in C_c^\infty(E)$ and $a \in F^\times$, where the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = \int\limits_E \phi(y)\psi(-(xy))\ dy.$$

Applying this identity twice and the Fourier inversion formula, we see that

$$c(E/F,\psi')c(E/F,\bar{\psi}')=1.$$

Thus, the Fourier inversion formula and the commutative diagrams in (4.9) imply the inversion formula of the Jacquet transform on $\mathcal{O}_n^{\psi'}(F)$ and on $\mathcal{O}_n^{\psi}(E/F)$. Namely,

$$\mathfrak{J}_{\bar{\psi}'}\circ\mathfrak{J}_{\psi'}=\mathrm{Id}\,.$$

Applying Weil's formula for the integral of the Fourier transform of a Schwartz function on a vector space against a character of second order [**Wei64**], the proof amounts to an elementary yet complicated computation based on certain intermediate orbital integrals and an inductive argument. The inversion formula implies that

$$\Phi \overset{\psi'}{\longleftrightarrow} \Psi \text{ if and only if } \mathcal{F}_{\psi'}(\Phi) \overset{\bar{\psi}'}{\longleftrightarrow} c(E/F,\psi)^{\frac{(n-1)n}{2}}\mathcal{F}_{\psi'}(\Psi). \tag{4.10}$$

This equivalence is the main reason for linearizing the problem and introducing the Jacquet transform. Another useful and much more elementary formula is obtained in [**Jac03b**, Proposition 4]. For $\Phi\in C_c^\infty(M_n(F))$, the function $a\mapsto\Omega[\Phi,\psi':w_0a]$ on $F^\times$ is smooth and of compact support; furthermore, it satisfies the identity

$$\Omega[\Phi,\psi':w_0a]=|a|_F^{1-n^2}\int\limits_F \Omega\left[\mathcal{F}_{\psi'}(\Phi),\bar{\psi}':\begin{pmatrix} -w_{n-1}a & 0\\ 0 & b\end{pmatrix}\right]. \tag{4.11}$$

There is an analogue of (4.11) for $\Psi\in C_c^\infty(H_n(E/F))$. Theorems 4.1 and 4.2 follow from (4.10) and (4.11). Indeed, for the decomposable representatives, i.e., those of the form $w_{M'}a=\mathrm{diag}(w_1a_1,w_2a_2)$ where $n=n_1+n_2$ with $n_i>0$ and w_ia_i is one of our relevant representatives for $GL_{n_i}(F)$, $i=1,2$, both the vanishing stated in Theorem 4.1 and the existence of transfer factors as in Theorem 4.2 follow by induction using certain intermediate orbital integrals. Once this is granted both theorems are straightforward consequences of (4.10), (4.11) and its analogue for Ψ that take care of the non decomposable representatives. The equality (4.8) and therefore Theorem 4.3 also follow from (4.10) and (4.11) with the help of the intermediate orbital integrals. The proof requires some more explanation given in [**Jac03b**, §8].

Remark 4.5. In [**Off05**], the Jacquet transform is defined on another space of orbital integrals, the space of $n\times n$ symmetric matrices over F. It is proved that the analogue of the diagram on the right hand side of (4.9) commutes and therefore that a similar inversion formula holds for the Jacquet transform. An analogue of the simpler identity (4.11) is also provided.

Though extremely useful, the inversion formula for the Jacquet transform is still far from enough machinery in order to face the fundamental lemma. Again the problem is linearized. We assume from now on that E/F is unramified of odd residual characteristic and that ψ' has conductor $\mathcal{O}_F$. For a function $\Psi\in C_c^\infty(X)$, we denote by $\Psi_{\omega^n}\in C_c^\infty(X)$ the function defined by

$$\Psi_{\omega^n}(x)=\omega^n(\det x)\Psi(x),\ x\in X.$$

Note that this definition cannot be generalized to $C_c^\infty(H_n(E/F))$. Note further that although the transfer factor $\gamma(a)$ and the expression $\omega^n(\det a)$ may not be defined on all relevant diagonal elements a, their product (which is the transfer factor used by Jacquet) is always defined. For convenience, we shall therefore introduce an abuse of notation as follows. For $\Phi \in C_c^\infty(M_n(F))$ and $\Psi \in C_c^\infty(H_n(E/F))$, we shall write

$$\Phi \overset{\psi}{\longleftrightarrow} \Psi_{\omega^n} \quad \text{whenever} \quad \tilde{\Omega}[\Phi, \psi' : a] = \tilde{\Omega}[\Psi, \psi, E/F : a].$$

It is still true that

$$\Phi \overset{\psi}{\longleftrightarrow} \Psi_{\omega^n} \quad \text{if and only if} \quad \Phi[k] \overset{\psi}{\longleftrightarrow} (\Psi[k])_{\omega^n},\ k \in \mathbb{Z}. \tag{4.12}$$

Let Φ_0 be the characteristic function of $M_n(\mathcal{O}_F)$ and similarly let Ψ_0 be the characteristic function of the lattice $H_n(E/F) \cap M_n(\mathcal{O}_E)$. The linearized version of the fundamental lemma is the explicit matching [**Jac05**, Theorem 1]:

$$\Phi_0[\mathrm{bc}(f)] \overset{\psi'}{\longleftrightarrow} \Psi_0[f]_{\omega^n} \tag{4.13}$$

where for $\Phi \in C_c^\infty(M_n(F))$ and $f' \in C_c^\infty(G')$ we set

$$\Phi[f'](Y) = \int\limits_{G'} \Phi(Yg) f'(g)\, dg,\ Y \in M_n(F),$$

and for $\Psi \in C_c^\infty(H_n(E/F))$ and $f \in C_c^\infty(G)$ we set

$$\Psi[f](Y) = \int\limits_{G} \Psi({}^t\bar{g} Y g) f(g)\, dg,\ Y \in H_n(E/F).$$

Note that if $\Phi \in C_c^\infty(G')$, then

$$\Phi[f'] = f'^\vee * \Phi$$

and if $\Psi \in C_c^\infty(X)$, then

$$\Psi[f] = f^\vee * \Psi.$$

Keeping in mind (4.7), Theorem 4.4 follows from (4.13). In fact, the simple argument given in [**Jac05**, p. 613] provides more explicit pairs of matching functions than Jacquet admits to have given in his paper. For all $k \geq 0$, let $\Phi^{(k)} = \Phi_0[k]$ and $\Psi^{(k)} = \Psi_0[k]$. The functions $\Phi^{(k)}$ and $\Psi^{(k)}$ are the characteristic functions of the set of integral matrices in the corresponding spaces with determinant of valuation k. Assume that $f \in \mathcal{H}_G(K)$ is supported on $\{g \in G : |\det g|_E = |\varpi^m|_E\}$. Then $\mathrm{bc}(f)$ is supported on $\{g \in G' : |\det g|_F = |\varpi^{2m}|_F\}$. Note then that

$$\Phi_0[\mathrm{bc}(f)][k-2m] = \Phi^{(k)}[\mathrm{bc}(f)] = \mathrm{bc}(f)^\vee * \Phi^{(k)}$$

and

$$(\Psi_0[f][k-2m])_{\omega^n} = (f^\vee * \Psi^{(k)})_{\omega^n} = (-1)^{nk} f^\vee * \Psi^{(k)}$$

since $\omega^n \circ \det$ is the constant $(-1)^{nk}$ on the support of $f^\vee * \Psi^{(k)}$. Applying (3.1), (4.12) and linearity of the orbital integrals, we therefore get that (4.13) implies

$$\mathrm{bc}(f) * \Phi^{(k)} \overset{\psi'}{\longleftrightarrow} (-1)^{nk} f * \Psi^{(k)} \tag{4.14}$$

for all $f \in \mathcal{H}_G(K)$ and all $k \geq 0$. In particular, the case $k = 0$ is Theorem 4.4.

Remark 4.6. The matching (4.14) is more general than Theorem 4.4. In particular, the matching

$$\Phi^{(k)} \overset{\psi'}{\longleftrightarrow} (-1)^{nk} \Psi^{(k)}$$

when k is odd is for functions supported on matrices with determinant of odd valuation. One such pair of matching functions is enough in order to determine the transfer factor dichotomy explained in §7.

We now explain how (4.13) is proved. The technical heart of the proof is a certain *uncertainty principle* for the space $\mathcal{O}_n^{\psi'}(F)$. The standard uncertainty principal for a function $f \in C_c^\infty(F)$ and its Fourier transform $\mathcal{F}(f)$ can be formulated as follows. If the support of f lies in $\mathfrak{p}^k$ and the support of $\mathcal{F}(f)$ lies in $\mathfrak{p}^{-k}$, then f lies in the one dimensional space spanned by the characteristic function of $\mathfrak{p}^k$. Jacquet's generalization for the space of normalized orbital integrals is described as follows. Let

$$\Lambda_n = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : m_1 \ge \dots \ge m_n\}.$$

For $m = (m_1, \dots, m_n) \in \Lambda_n$, let $\tilde{m} = (-m_n, \dots, -m_1) \in \Lambda_n$ and let $m \preceq m'$ be the standard partial order on Λ_n defined by

$$m_1 + \dots + m_i \le m'_1 + \dots + m'_i,\ i = 1, \dots, n-1,\ m_1 + \dots + m_n = m'_1 + \dots + m'_n.$$

For $m \in \Lambda_n$ let $\mathcal{F}(m)$ be the space of functions $\Omega \in \mathcal{O}_n^{\psi'}(F)$ such that the support of Ω lies in the set of all relevant $a = \mathrm{diag}(a_1, \dots, a_n)$ such that

$$|a_1 \cdots a_i|_F \le |\varpi^{-(m_1+\dots+m_i)}|_F,\ i = 1, \dots, n, \tag{4.15}$$

and the support of $\mathfrak{J}(\Omega)$ lies in the set of all relevant a such that

$$|a_1 \cdots a_i|_F \le |\varpi^{m_n+\dots+m_{n+1-i}}|_F,\ i = 1, \dots, n. \tag{4.16}$$

Since $m \preceq m'$ implies that $\tilde{m} \preceq \tilde{m}'$, we then also have $\mathcal{F}(m) \subseteq \mathcal{F}(m')$. We also let

$$\Phi_m = \Phi_0(\cdot\ \varpi^m) \text{ and } \Psi_m = \Psi_0(\varpi^m\ \cdot\ \varpi^m)$$

where $\varpi^m = \mathrm{diag}(\varpi^{m_1}, \dots, \varpi^{m_n})$. Note that

$$\mathcal{F}(\Phi_m) = q^{n(m_1+\dots+m_n)}\Phi_{\tilde{m}} \text{ and } \mathcal{F}(\Psi_m) = q^{2n(m_1+\dots+m_n)}\Psi_{\tilde{m}}$$

where $q = q_F$. Since the diagrams in (4.9) commute, it is not hard to see that the function $a \mapsto \tilde{\Omega}[\Phi_m, \psi : a]$ lies in $\mathcal{F}(m)$ and that $a \mapsto \tilde{\Omega}[\Psi_m, \psi, E/F : a]$ lies in $\mathcal{F}(2m)$.

THEOREM 4.7 (The uncertainty principle ([**Jac05**], Proposition 4)). *The functions*

$$a \mapsto \tilde{\Omega}[\Phi_{m'}, \psi : a],\ m' \preceq m$$

form a basis of the space $\mathcal{F}(m)$.

We remark first that for $m = (0, \dots, 0)$ this statement was already proved in [**Jac04**], and it implies the matching (4.14) for $f = \mathbf{1}_K$. The proof of Theorem 4.7 is of combinatorial nature and is rather long. We shall not explain it here, but let us remark that in order to describe conditions such as (4.15) and (4.16) on the support of functions in $\mathcal{O}_n^{\psi'}(F)$, Jacquet introduces the terminology of box diagrams and proves a series of lemmas concerning the diagrams in [**Jac05**, §7-§9]. If $m_1 - m_2 \ge 2$, then the existence of $m' \precneqq m$ with $m'_i = m_i,\ i \ge 3$ (e.g., $m' = (m_1 - 1, m_2 + 1, m_3, \dots, m_n)$) simplifies the proof of Theorem 4.7 using induction. When $m_1 - m_2 \le 1$ the proof is more delicate and requires the full use of the machinery of box diagrams developed.

It follows from the uncertainty principle that there exist unique constants $\xi_m^{m'}$ for all $m' \preceq 2m$, such that

$$\sum_{m' \preceq 2m} \xi_m^{m'} q^{\langle 2m' - m, \rho \rangle} \Phi'_m \xleftrightarrow{\psi'} (\Psi_m)_{\omega^n}. \tag{4.17}$$

For every $m \in \Lambda_n$ let $\sigma_m \in \mathbb{C}[q^{\pm\lambda}]^W$ be the associated Schur polynomial. There exist constants $\theta_m^{m'}$, $m' \preceq 2m$ such that

$$\sigma_m(q_E^\lambda) = \sigma_m(q^{2\lambda}) = \sum_{m' \preceq 2m} \theta_m^{m'} \sigma_{m'}(q^\lambda).$$

Denote by $f_m^F \in \mathcal{H}_{G'}(K')$ the Hecke function such that $\hat{f}_m^F(\lambda) = q^{\langle\rho,m\rangle}\sigma_m(q^\lambda)$ and similarly let $f_m^E \in \mathcal{H}_G(K)$ be such that $\hat{f}_m^E(\lambda) = q^{2\langle\rho,m\rangle}\sigma_m(q^{2\lambda})$. Thus $f_m^F, m \in \Lambda_n$ is a basis for $\mathcal{H}_{G'}(K')$, $f_m^E, m \in \Lambda_n$ is a basis for $\mathcal{H}_G(K)$ and based on the Shintani, Casselman-Shalika formula for the spherical Whittaker function, Jacquet showed that

$$\mathrm{bc}(f_m^E) = \sum_{m' \preceq 2m} \theta_m^{m'} q^{\left\langle \rho, 2m - m' \right\rangle} f_{m'}^F.$$

By linearity, it is enough to prove (4.13) for $f = f_m^E$. Jacquet also showed in [**Jac05**, p. 628] that the orbital integral of $\Psi_0[f_m^E]$ is the same as the orbital integral of $q^{2\langle 2\rho,m\rangle}(\Psi_m)$ and that the orbital integral of $\Phi_0[f_m^F]$ is the same as the orbital integral of $q^{\langle 2\rho,m\rangle}\Phi_m$. It follows, that in order to prove (4.13) it is enough to show that for every $m \in \Lambda_n$, we have

$$\sum_{m' \preceq 2m} \theta_m^{m'} q^{\left\langle \rho, m' - 2m \right\rangle} \Phi_{m'} \overset{\psi'}{\longleftrightarrow} (\Psi_m)_{\omega^n}. \tag{4.18}$$

With this in mind, Jacquet defines the linear map $\beta : \mathcal{H}_G(K) \to \mathcal{H}_{G'}(K')$ by

$$\beta(f_m^E) = \sum_{m' \preceq 2m} \xi_m^{m'} q^{\left\langle \rho, 2m - m' \right\rangle} f_{m'}^F.$$

The constants $\xi_m^{m'}$ were defined to satisfy (4.17)

$$\sum_{m' \preceq 2m} \xi_m^{m'} q^{\left\langle \rho, m' - 2m \right\rangle} \Phi_{m'} \overset{\psi'}{\longleftrightarrow} (\Psi_m)_{\omega^n}.$$

Using the above arguments, we then see that

$$\Phi_0[\beta(f)] \overset{\psi'}{\longleftrightarrow} \Psi_0[f]_{\omega^n}$$

for every $f \in \mathcal{H}_G(K)$. To prove the fundamental lemma, it is therefore left to show that $\beta = \mathrm{bc}$ or what amounts to the same that $\xi_m^{m'} = \theta_m^{m'}$ whenever $m' \preceq 2m$. Computing the constants explicitly, Jacquet shows that $\xi_m^{m'} = \theta_m^{m'}$ whenever $m_1 - m_n \leq 1$. This amounts to saying that β agrees with bc on a set of generators for the Hecke algebra $\mathcal{H}_G(K)$. A global argument is then used in order to prove the identity on the entire Hecke algebra. Indeed, applying the map β at almost every inert place to a simple version of the relative trace formula Jacquet shows that β is an algebra homomorphism.

5. The fine spectral expansion of the relative trace formula

In this section E/F is an extension of number fields. The relative trace formula is the distribution on $X_{\mathbb{A}}$ given by

$$RTF(\Psi) = \int_{U \backslash U_{\mathbb{A}}} \left(\sum_{x \in X} \Psi({}^t\bar{u}xu) \right) \psi_U(u)\, du.$$

It can be expressed as a sum

$$\sum_{\{\xi\}} RTF_\xi(f_\xi) \tag{5.1}$$

over a set of representatives $\{\xi\}$ of the G-orbits in X where the functions $\{f_\xi\}$ in $C_c^\infty(G_{\mathbb{A}})$ are related to Ψ by

$$\Psi({}^t\bar{g}\xi g) = \int_{H_{\mathbb{A}}^\xi} f_\xi(hg)\ dh.$$

The distribution RTF_ξ of $G_{\mathbb{A}}$ is defined by

$$RTF_\xi(f) = \int_{H^\xi\backslash H_{\mathbb{A}}^\xi} \int_{U\backslash U_{\mathbb{A}}} K_f(h,u)\psi_U(u)\ du\ dh$$

where

$$K_f(x,y) = \sum_{\gamma\in G} f(x^{-1}\gamma y)$$

is the standard kernel function associated to the test function f acting on $L^2(G\backslash G_{\mathbb{A}}^1)$. For a fixed compact subset C of $X_{\mathbb{A}}$, there is a finite set Γ of representatives ξ of G-orbits, so that for any test function $\Psi \in C_c^\infty(X_{\mathbb{A}})$ with support contained in C and for any representative $\xi \notin \Gamma$, we have $RTF_\xi(f_\xi) = 0$ [**Jac95**, Lemma 1.1]. In particular, the sum (5.1) involves only finitely many non zero terms. From now on we focus on an individual term. Thus we fix $\xi \in X$, let $\mathbf{H} = \mathbf{H}^\xi$ and denote by $RTF = RTF_\xi$ the associated distribution on $G_{\mathbb{A}}$.

According to Langlands spectral decomposition of the L^2-space

$$L^2(G_F\backslash G_{\mathbb{A}}^1) = \bigoplus_{\chi\in\mathfrak{X}} L^2(G_F\backslash G_{\mathbb{A}}^1)_\chi$$

as a direct sum over cuspidal data (see [**Art78**, §3]), Arthur expanded in [**Art78**, §4] the kernel function as

$$K_f(x,y) = \sum_\chi K_\chi(x,y)$$

where

$$K_\chi(x,y) = \sum_M \frac{|W_M|}{|W|} \sum_\pi \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathrm{ob}(\mathcal{A}_P^\pi)} E(x, I(f,\lambda)\varphi,\lambda)\overline{E(y,\varphi,\lambda)}\ d\lambda. \tag{5.2}$$

The sum is over all standard parabolic subgroups $P = MU_P$ of G with standard Levi subgroup M and unipotent radical U_P, over the (finitely many) irreducible, discrete spectrum representations π in $L^2(M_F\backslash M_{\mathbb{A}}^1)_\chi$ and over an orthonormal basis of the space $\mathcal{A}_P^\pi$ of automorphic forms on $U_{P,\mathbb{A}}M_F\backslash G_{\mathbb{A}}^1$ in the parabolically induced representation $\mathrm{ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\pi)$. In [**Art82**], Arthur obtained the fine spectral expansion for the Arthur-Selberg trace formula, i.e., he expressed the trace formula explicitly as a sum of the contribution of each cuspidal data χ. The analogue for the distribution RTF was obtained by Lapid in [**Lap06**].

If we ignored convergence issues and naively interchanged integrals, we could have written

$$RTF(f) \text{ “} = \text{”} \sum_{\chi=(M,\pi)} \int_{i(\mathfrak{a}_M^G)^*} \mathfrak{B}_{\mathcal{A}_P^\pi, \mathcal{A}_P^\pi}^{\mathcal{P}^H(E(\cdot,\lambda)), \mathcal{W}^\psi(E(\cdot,\lambda))}(f)\ d\lambda$$

where

$$\mathcal{P}^H(\phi) = \int_{H \backslash H_{\mathbb{A}}} \phi(h)\ dh \tag{5.3}$$

is the H-period integral of an automorphic form ϕ on $G_{\mathbb{A}}$. For a cusp form ϕ of $G_{\mathbb{A}}$ the period integral (5.3) is convergent [**AGR93**]. Thus, if π is an irreducible, cuspidal representation of $G_{\mathbb{A}}$, its contribution to the RTF is indeed the relative Bessel distribution defined by

$$B_\pi(f) = \sum_{\phi \in \mathrm{ob}(\pi)} \mathcal{P}^H(\pi(f)\phi)\overline{\mathcal{W}}^\psi(\phi).$$

The H-period integral of a general automorphic form, however, is not convergent and in order to write the RTF as a sum of generalized Bessel distributions, it is necessary to first regularize the period integrals. This regularization of unitary periods and the analysis of regularized periods of Eisenstein series was obtained by Lapid and Rogawski in [**LR03**].

A pair $(\mathbf{Y}, \sigma)$ where $\mathbf{Y}$ is a reductive group and σ is an involution on $\mathbf{Y}$ both defined over F, is called quasi split if there exists a minimal F-parabolic subgroup which is σ-stable. The work of Lapid and Rogawski is, in fact, in the general context of a quasi split Galois pair, i.e., such that σ comes from a Galois action. For every $x \in X$ we denote by θ_x the involution $\theta_x(g) = x^{-1}\theta(g)x$, $g \in G$. The pair (G, θ_{w_0}) is a quasi split Galois pair. The regularized period integral

$$\mathcal{P}^H(\phi) = \int_{H_F \backslash H_{\mathbb{A}}}^{*} \phi(h)\ dh$$

is defined in [**LR03**, §8.4] for essentially all automorphic forms (with a non trivial closed condition on the exponents of ϕ). It is an $H_{\mathbb{A}_f}$-invariant linear functional ($\mathbb{A}_f$ denoting the finite adèles) that agrees with the period integral whenever it converges. For a cuspidal representation π of $M^1_{\mathbb{A}}$, we can now define (at least for generic λ) the relative Bessel distribution

$$B^G_{(M,\pi)}(f,\lambda) = B_{(M,\pi)}(f,\lambda) = \sum_{\varphi \in \mathrm{ob}(\mathcal{A}_P^\pi)} \mathcal{P}^H(E(I(f,\lambda)\varphi,\lambda))\overline{\mathcal{W}}^\psi(E(\varphi,-\bar{\lambda})).$$

In fact, the generalized Bessel distributions that occur in Lapid's expansion are more general. We need to consider the analogous distributions with respect to θ-stable Levi subgroups of G. Note that H is the fixed point group of the involution θ_ξ. One of the technical difficulties in [**LR03**] is that (G, θ_ξ) is not necessarily quasi split. This motivates Lapid and Rogawski to introduce in [**LR03**, §4.4] the *defect* of ξ (or of its G-orbit). Essentially, this is the standard Levi M° of a minimal θ_x-stable parabolic of G for some x in the G-orbit of ξ. It is convenient (and always possible) to choose $\xi \in M^\circ$ ([**LR03**, §4.5]). Once we do so, every θ_ξ-stable Levi subgroup L of G is also θ_{w_0}-stable, and $(L, \theta_{w_0|L})$ is a quasi split Galois pair. Let

$Q = LV$ be the standard parabolic subgroup with Levi L and unipotent radical V. Thus, for an automorphic form φ on $V_{\mathbb{A}}L\backslash G_{\mathbb{A}}$ that satisfies $\varphi(ag) = e^{\langle \rho_Q, H(a)\rangle}\varphi(g)$ for $a \in A_L$ we can define the regularized integral

$$\int^{*}_{Q_H\backslash H_{\mathbb{A}}} \varphi(h)\ dh = \int_{\mathbf{K_H}} \int^{*}_{L_H\backslash (L_H)_{\mathbb{A}}} \varphi(lk)\ dl\ dk$$

where $\mathbf{K_H} = \mathbf{K} \cap H_{\mathbb{A}}$ and for an algebraic subgroup $\mathbf{Y}$ of $\mathbf{G}$, we set $\mathbf{Y_H} = \mathbf{Y} \cap \mathbf{H}$. Accordingly, we define the relative Bessel distribution

$$B^L_{(M,\pi)}(f,\lambda) = \sum_{\varphi \in \mathrm{ob}(\mathcal{A}^{\pi}_P)} \left[\int^{*}_{Q_H\backslash H_{\mathbb{A}}} E^Q(h, I(f,\lambda)\varphi, \lambda)\ dh \right] \overline{\mathcal{W}}^{\psi}(E(\varphi, -\bar{\lambda})).$$

Remark 5.1. The regularized integrals are defined using the *mixed truncation* operators $\Lambda^{T,Q}_m$ for any θ_ξ stable parabolic subgroup Q. These are certain relative variants of Arthur's truncation operator, well adapted to the setting of the RTF. The operator $\Lambda^{T,Q}_m$ maps a function of moderate growth on $LV_{\mathbb{A}}\backslash G(\mathbb{A})^1$ to a function of rapid decay on $L_H(V_H)_{\mathbb{A}}\backslash H_{\mathbb{A}}$.

We now explain which triples (M,π,L) contribute to the spectral expansion. For standard Levi subgroups $M \subseteq L$ of G, we denote by w^L_M the longest w amongst the elements in the Weyl group W_L of L such that w is the shortest element in wW_M and wMw^{-1} is a standard Levi subgroup of L. In particular $w_0 = w^G_T$ is the longest Weyl element in W.

Definition 5.2. A Levi subgroup M is called θ_{w_0}-*elliptic* in G if $w^G_M M (w^G_M)^{-1} = \theta_{w_0}(M)$ and $w^G_M \theta_{w_0}$ acts as -1 on $\mathfrak{a}^G_M$. For a cuspidal representation π of $M^1_{\mathbb{A}}$, we say that (M,π) is θ_ξ-elliptic with respect to G if M is θ_{w_0}-elliptic in G and π is distinguished by $M \cap H^x$ for some $x \in \{{}^t\bar{g}\xi g : g \in G\} \cap (w^G_M w_0)^{-1}M$.

The contribution to the RTF comes only from triples (M,π,L) so that (M,π) is θ_ξ-elliptic with respect to L. The θ_ξ-stable Levi L is determined uniquely by (M,π). That other terms do not contribute can be seen from

Theorem 5.3. [**LR03**, Theorem 9.1.1] *Let π be a cuspidal representation of $M^1_{\mathbb{A}}$, then $\mathcal{P}^H(E(\varphi,\lambda)) = 0$ unless (M,π) is θ_ξ-elliptic in G. In this case*

$$\mathcal{P}^H(E(\varphi,\lambda)) = J(w^G_{\theta(M)}, \varphi, \lambda)$$

where the right hand side is the intertwining period defined by the sum over the set Ξ of M-orbits in $\{{}^t\bar{g}\xi g : g \in G\} \cap (w^G_M w_0)^{-1}M$ of

$$\int_{H\cap \eta P \eta^{-1}\backslash H_{\mathbb{A}}} e^{\left\langle \lambda, H(\eta^{-1}h)\right\rangle} \varphi(\eta^{-1}h)\ dh$$

where $\eta \in G$ is a representative such that ${}^t\bar{\eta}\xi\eta \in \Xi$ (this is independent of the choice of η).

The intertwining periods were first introduced in [**Jac95**] for GL_3 and studied further in [**JLR99**] for split Galois pairs. In [**LR03**] they were introduced in the general setting of a quasi split Galois pair. Lapid and Rogawski show that the (often infinite) sum of integrals defining the intertwining periods converges in some cone. Their meromorphic continuation is obtained in [**LR03**, Theorem 10.2.1] as a consequence of Theorem 5.3. More generally, we have that

$$\int^{*}_{Q_H\backslash H_{\mathbb{A}}} E^Q(h, I(f,\lambda)\varphi, \lambda)\ dh = 0$$

unless (M,π) is θ_ξ elliptic with respect to L in which case

$$\int^{*}_{Q_H\backslash H_{\mathbb{A}}} E^Q(h, I(f,\lambda)\varphi, \lambda)\ dh = J(w^L_{\theta(M)}, \varphi, \lambda).$$

Analyzing explicitly the results of Lapid-Rogawski for the case of unitary periods, Lapid obtained the spectral expansion [**Lap06**, Theorem 10.4] that we are now ready to explain. We choose the representative ξ of a G-orbit in X to be of the form

$$\xi = \begin{pmatrix} & & D \\ & x & \\ {}^t\bar{D} & & \end{pmatrix}$$

where D is anti diagonal in $GL_t(F)$ (t is the Witt index of the Hermitian form ξ) and x is an anisotropic hermitian form of size $d = n-2t$ (every orbit has a representative ξ of this form). For standard Levi subgroups $M \subseteq L \subseteq G$, the involution θ_{w_0} acts on $(\mathfrak{a}^L_M)^*$. We denote by $((\mathfrak{a}^L_M)^*)^+_{\theta_{w_0}}$ (resp. $((\mathfrak{a}^L_M)^*)^-_{\theta_{w_0}}$) the 1 (resp. -1) eigenspace. Furthermore, if L is θ_{w_0}-stable, let

$$(\mathfrak{a}^{L_H}_M)^* = (\mathfrak{a}^L_M)^* \oplus ((\mathfrak{a}^G_L)^*)^-_{\theta_{w_0}}.$$

THEOREM 5.4 (Fine spectral expansion of the RTF [**Lap06**]). *For every θ_ξ-elliptic pair (M,π) with respect to L, the relative Bessel distribution $B^L_{(M,\pi)}(f,\lambda)$ is holomorphic on $i\left(\mathfrak{a}^{L_H}_M\right)^*$. There are constants $\mathfrak{c}(M,\pi)$, so that*

$$RTF(f) = \sum_{(M,\pi)} \mathfrak{c}(M,\pi) \int_{i(\mathfrak{a}^{L_H}_M)^*} B^L_{(M,\pi)}(f,\lambda)\ d\lambda \tag{5.4}$$

where the sum is over all Levi subgroups M of G of type

$$(n_1, \dots, n_k, m_1, \dots, m_l, n_k, \dots, n_1)$$

and cuspidal representations π of $M^1_{\mathbb{A}}$ of the form

$$\pi = \sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \bar{\sigma}_k \otimes \cdots \otimes \bar{\sigma}_1 \tag{5.5}$$

where $\sigma_i \not\simeq \bar{\sigma}_i$, $i = 1, \dots, k$, and each τ_j is distinguished by some unitary group. For such a pair (M,π), the Levi subgroup L is then of type $(n_1, \dots, n_k, m_1 + \ldots + m_l, n_k, \dots, n_1)$ and $m_1 + \cdots + m_l \geq d$. Furthermore, the integral-sum (5.4) *is absolutely convergent.*

Very roughly speaking, the main technical difficulty is to interchange between two integrals where the inner integral is over the imaginary axis of a certain vector space and the outer integral is a unitary period. This is achieved using a shift of contour and coming back to the unitary access after interchanging the integrals. Lapid's approach, using complex analysis, is new. The formal manipulations are justified by a majorization of Eisenstein series, which is the technical heart of the paper [**Lap06**, Proposition 6.1]. The combinatorics of (G, M)-families, introduced by Arthur in [**Art81**], is applied to reduce the problem to lower bounds of Rankin-Selberg L-functions at the edge of the critical strip, which appear in the normalization of intertwining operators. Such lower bounds were obtained by Brumley [**Bru06**] for GL_n. For the absolute convergence, the uniform bound of Lou, Rudnick and Sarnak towards the Ramanujan conjecture is also applied [**LRS99**].

We will not get here into the deep analytic problems involved. However, in order to introduce the reader to the complexity of the problem, we wish to roughly explain the main 3 steps in the proof. Assume first that f is K_∞-finite, where $K_\infty = \prod_{v|\infty} K_v$. The first step was already obtained by Jacquet in [**Jac95**]. Jacquet obtained in [**Jac95**, Proposition 2.1] for any integer N and for y in a fixed compact set the bound

$$\sum_{\chi\in\mathfrak{X}} |K_\chi(x, y)| \le c\|x\|^{-N}.$$

Since $U\backslash U_{\mathbb{A}}$ is compact, using (5.2), this bound enables us to write

$$(5.6)\quad RTF(f) = \sum_{\chi\in\mathfrak{X}} \sum_{[(M,\pi)]} \frac{|W_M|}{|W|} \int_{H\backslash H_{\mathbb{A}}} \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathrm{ob}(\mathcal{A}_P^\pi)} E(h, I(f,\lambda)\varphi, \lambda)\overline{\mathcal{W}}^\psi(E(\varphi,\lambda))\, d\lambda\, dh$$

where the second sum is over equivalence classes of pairs such that π is in the discrete spectrum of $L^2(M\backslash M_{\mathbb{A}}^1)_\chi$. Based on the classification of the discrete spectrum of GL_n [**MW89**], representations in the residual spectrum are not generic. This way Jacquet showed that for a non cuspidal Eisenstein series $\mathcal{W}^\psi(E(\varphi, \lambda)) = 0$, i.e., that the only terms in (5.6) that contribute to the RTF are associated with pairs (M, π) where π is a cuspidal representation of $M_{\mathbb{A}}^1$ (see also [**Lap06**, Lemma 9.1]). Thus we obtain

$$(5.7)\quad RTF(f) = \sum_{[(M,\pi)]} \frac{|W_M|}{|W|} \int_{H\backslash H_{\mathbb{A}}} \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in\mathrm{ob}(\mathcal{A}_P^\pi)} E(h, I(f,\lambda)\varphi, \lambda)\overline{\mathcal{W}}^\psi(E(\varphi,\lambda))\, d\lambda\, dh$$

where the sum is now over all pairs (M, π) up to conjugation, where π is a cuspidal representation of $M(\mathbb{A})^1$. At this point, in order to expand $RTF(f)$ as a sum of relative Bessel distributions, we would formally want to change the order of integration. As we already explained, this naive approach cannot work and a shift of contour is first applied. To perform the shift of contour, Lapid applies an inversion formula for automorphic forms [**LR03**, Lemma 8.2.1] based on the mixed truncation. At the end of the day, every summand of (5.7) (associated to (M, π)) can be written as a sum of integrals of the form

$$(5.8)\quad \int_{\mathrm{Re}\,\lambda=\lambda_0} \frac{e^{\langle\lambda,T\rangle}F(\lambda)}{\prod_{\varpi^\vee\in\hat{\Delta}_{Q_H}^\vee}\langle-\lambda,\varpi^\vee\rangle}\, d\lambda$$

where Q ranges over certain parabolic subgroups, $\hat{\Delta}_{Q_H}^\vee$ is the basis of $\mathfrak{a}_{Q_H}^H$ dual to the set of non zero restrictions to L_H of the simple roots of H, λ_0 is a generic point sufficiently close to zero in the negative Weyl chamber of $\mathfrak{a}_L^*$ with respect to Q, and $F(\lambda)$ is holomorphic and rapidly decreasing in an appropriate domain.

Thus, these integrals converge and, roughly speaking, the interchange of integrals with the unitary period at this stage has already been performed. Though the estimates obtained in [**Lap06**, §6] are used, arguments similar to those in [**Mül02**] suffice to get to this point. The full power of the majorization of Eisenstein series is used in the next step, getting back to an integral on the unitary axis. The main problem with directly getting back to the unitary axis is that the integrands may have singularities there. For this reason, Lapid introduces in [**Lap06**, §3] certain improper integrals for a family of meromorphic functions on a vector space. Let V be a real vector space and Λ a set of linearly independent linear functionals on V. For a *tame*, complex valued function F on $V_{\mathbb{C}} = V \otimes \mathbb{C}$ in the sense of [**Lap06**, §3] and a generic vector $v \in V$ (outside the kernel of each $\lambda \in \Lambda$), the improper integral

$$\int_{\nearrow v} \frac{F(u)}{\prod_{\lambda\in\Lambda} \lambda(u)}$$

is defined. If F is holomorphic and rapidly decreasing, then the improper integral equals

$$\int_{\mathrm{Re}\, u = v} \frac{F(u)}{\prod_{\lambda\in\Lambda} \lambda(u)}$$

(thus, the improper integral is a regularization of the latter for a wider family of functions). The expression (5.8) can thus be expressed as an improper integral in $v = \lambda_0$. The improper integrals do not quite depend on the vector v but rather on its connected component with respect to the hyperplanes defined by the kernels of $\lambda \in \Lambda$. Thus, in [**Lap06**, Lemma 3.3], the relation between the improper integrals for two generic vectors v and v' is given precisely. This is used in [**Lap06**, §9.3] to express (5.8) as a sum of improper integrals with respect to a fixed generic point in the positive Weyl chamber of $\mathfrak{a}_M^*$ with respect to P, sufficiently close to zero. Using the majorization of Eisenstein series, Lapid shows in [**Lap06**, Lemma 7.4] that $B^L_{(M,\pi)}(f, \lambda)$ is tame as a function of λ. Based on this and using [**Lap06**, Lemma 3.3] repeatedly, he finally expresses (5.8) as a sum of expressions of the form

$$\int_{i(\mathfrak{a}_{M_1}^{L_H})^*} B^L_{(M_1,\pi)}(f, \lambda)\, d\lambda.$$

Collecting together the terms associated to (M, π, L) whenever (M, π) is θ_ξ elliptic in L the fine spectral expansion is obtained for every K_∞-finite test function f. In fact, in order to compute the constants $\mathfrak{c}(M, \pi)$ that are of a combinatorial nature, one has to carefully follow the use of [**Lap06**, Lemma 3.3]. This seems to be rather complicated and is not carried out in the paper. However, once M is fixed the dependence is only on the type of π, i.e., for π in the form (5.5), $\mathfrak{c}(M, \pi)$ only depends on the integers k and l. Thus it only receives finitely many possible values and to prove the absolute convergence, it is enough to show that

$$\sum_{[(M,\pi)]} \int_{i(\mathfrak{a}_M^{L_H})^*} |B^L_{(M,\pi)}(f, \lambda)|\, d\lambda < \infty.$$

The absolute convergence then follows from bounds obtained in [**Mül02**, §6]. The fact that the expansion holds for any $f \in C_c^\infty(G_{\mathbb{A}})$ (dropping the K_∞-finiteness assumption) now follows from Lebesgue's dominant convergence theorem, using all the deep analytic bounds mentioned above.

REMARK 5.5. Of course, in order to understand the actual analytic difficulties that occur one will have to read [**Lap06**]. We do hope, however, that this somewhat vague description of the steps in the proof will make the paper of Lapid more approachable to the reader.

6. Spherical functions on Hermitian matrices

In this section F is a non-archimedean local field. The symmetric space

$$X = \{g \in G : {}^t\bar{g} = g\}$$

is the space of Hermitian matrices with respect to the quadratic extension E/F. The Hecke algebra $\mathcal{H}_G(K)$ acts on the space $C^\infty(X/K)$ of K-invariant functions on X by the convolution

$$f * \phi(x) = \int_G f(g^{-1})\phi({}^t\bar{g}xg)\ dg.$$

DEFINITION 6.1. A *spherical function* on X is a function $\Omega \in C^\infty(X/K)$ which is an $\mathcal{H}_G(K)$ eigenfunction.

Hironaka studied the spherical functions on X in a series of papers [**Hir88a, Hir88b, Hir89, Hir90, Hir99**]. When E/F is unramified she obtained explicit formulas for all spherical functions. For a ramified quadratic extension there are only partial results. In this section we recall the results of Hironaka. We begin with a few words about the status of the general theory of spherical functions on p-adic spaces.

The explicit computation of spherical functions on a reductive p-adic group was first obtained by Macdonald in [**Mac71**]. His formulas were reproved by Casselman in [**Cas80**] using the theory of unramified principal series representations. With this new approach, Casselman and Shalika obtained explicit formulas for Whittaker spherical functions [**CS80**] (generalizing Shintani's explicit formulas for GL_n). The method of Casselman-Shalika was then used to obtain explicit formulas for the spherical functions for various other cases of p-adic spaces, e.g., [**HS88, Off04, Hir05b, Sak06**]. In a recent work of Sakellaridis [**Saka**], much of the theory is developed in the general setting of a quasi affine p-adic spherical G-variety for a split reductive group G. The problem of computing the spherical functions explicitly will be addressed in this context in [**Sakb**]. Roughly speaking, once the Casselman-Shalika method is applied, there are still three main obstacles to obtaining explicit formulas for the spherical functions. The first obstacle is to obtain an analogue of the Cartan decomposition, i.e., a K-orbit decomposition on X. In the case of a p-adic symmetric space, Delorme and Sécherre provided recently a description of the K-orbits [**DS**]. The second obstacle is to explicitly describe certain functional equations satisfied by the spherical functions. In [**Hir05a, Hir06**], Hironaka suggests a strategy to reduce the computation of the functional equations to some low rank cases under some assumptions on X in the setting of a spherical G-variety (with G not necessarily split). The third obstacle is an explicit computation of certain integrals over a Iwahori subgroup. In many examples (but not in general) those are easy to compute.

Let $d_i(x)$ be the determinant of the $i \times i$ upper left block of $x \in X$. Thus d_i is a regular function on X which is B-equivariant with respect to the rational character $b \mapsto \mathrm{Nm}(b_1 b_2 \cdots b_i)$ on B, where $b = \mathrm{diag}(b_1, \dots, b_n)u$ and $u \in U$, i.e., $d_i({}^t\bar{b}xb) = \mathrm{Nm}(b_1 b_2 \cdots b_i) d_i(x)$. Note that the lattice spanned by these n rational characters is of finite index in $X^*(T)$ and that they provide a basis of $\mathfrak{a}^*_{0,\mathbb{C}}$. There is a unique open $\mathbf{B}$-orbit $\mathbf{X}^\circ$ in $\mathbf{X}$ given by

$$\mathbf{X}^\circ = \{x \in \mathbf{X} : d_i(x) \neq 0,\, i = 1, \dots, n\}.$$

The set of rational points X° consists of 2^n B-orbits parameterized by the abelian group

$$\Gamma = T' / \operatorname{Nm} T \simeq (F^\times / \operatorname{Nm} E^\times)^n$$

(this is not a coincidence, see [**Saka**, Corollary 3.3.2] when G is split and X is a quasi affine spherical G-variety). For $a = \mathrm{diag}(a_1, \dots, a_n) \in \Gamma$ we denote by X_a the associated B-orbit. It is given by

$$X_a = \{x \in X^\circ : d_i(x) \in a_1 a_2 \cdots a_i,\, i = 1, \dots, n\}.$$

Note that the entries of a are considered as cosets in $F^\times / \operatorname{Nm} E^\times$ and therefore it makes sense to write $y \in a_1 \cdots a_i$ for $y \in F^\times$. For $a \in \Gamma$ and $s = (s_1, \dots, s_n) \in \mathbb{C}^n$, let

$$\omega_a(x; s) = \int_K \mathbf{1}_{X_a}({}^t\bar{k}xk) \prod_{i=1}^n |d_i({}^t\bar{k}xk)|^{s_i}\, dk.$$

Hironaka's spherical functions $\{\omega_a(\cdot; s)\}_{a\in\Gamma}$ form a basis of the space of spherical functions on X with a fixed Hecke eigenvalue depending on s. Let $\lambda = \lambda(s) = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ be such that

$$f * \omega_a(\,\cdot\,; s) = \hat{f}(\lambda)\omega_a(\,\cdot\,; s) \tag{6.1}$$

and let

$$\omega_a^\lambda = \omega_a(\,\cdot\,; s).$$

Thus, for every Weyl element $w \in W$ the set $\{\omega_a^{w\lambda}\}_{a\in\Gamma}$ forms another basis for the same space of spherical functions and there are therefore matrices $B(w, \lambda) = (B_{a,a'}(w, \lambda))_{a,a'\in\Gamma} \in M_{2^n}(\mathbb{C}(q^\lambda))$ such that

$$(\omega_a^\lambda)_{a\in\Gamma} = B(w, \lambda)(\omega_a^{w\lambda})_{a\in\Gamma}. \tag{6.2}$$

Applying [**Hir99**, Theorem 1.9] to this setting Hironaka obtains that

$$(\omega_a^\lambda(x))_{a\in\Gamma} = \frac{1}{Q} \sum_{w\in W} c(w\lambda) B(w, \lambda)(I_a^{w\lambda}(x))_{a\in\Gamma}. \tag{6.3}$$

Here

$$c(\lambda) = \prod_{i<j} \frac{1 - q_E^{-(\lambda_i - \lambda_j + 1)}}{1 - q_E^{-(\lambda_i - \lambda_j)}},$$

$$Q = \sum_{w\in W} \frac{1}{[\mathcal{I}w\mathcal{I} : \mathcal{I}]} = \prod_{i=1}^n \frac{1 - q_E^{-i}}{1 - q_E^{-1}},$$

$$I_a^\lambda(x) = \int_{\mathcal{I}} \mathbf{1}_{X_a}({}^t\bar{k}xk) \prod_{i=1}^n |d_i({}^t\bar{k}xk)|^{s_i}\, dk,$$

and $\mathcal{I}$ denotes the Iwahori subgroup of K compatible with B.

Remark 6.2. In fact, an analogue of formula (6.3) is obtained in a more general context for certain spherical varieties by using the Casselman-Shalika method. It reduces the explicit computation of spherical functions to the three obstacles discussed above. It is enough to compute the spherical functions on a choice of representatives of the K-orbits in X. In many cases, an explicit choice of representatives is made so that the integrals over the Iwahori subgroup are easy to compute explicitly. Hironaka also suggests a general method to reduce the functional equation (6.2) for a simple reflection w to what is referred to as the case of small prehomogeneous spaces [**Hir05a**].

Remark 6.3. In the general context of spherical varieties, it is observed in [**Saka**, §3.3], (at least when G is split) that the B-orbits in X° are naturally parameterized by an abelian group Γ. Roughly speaking, if $\{\omega_a^\lambda\}_{a\in\Gamma}$ forms a basis of spherical functions of a given Hecke eigenvalue $f\mapsto\hat{f}(\lambda)$, with ω_a^λ supported on the B-orbit associated to a, then we can define the stable spherical functions $\omega_\eta^\lambda=\sum_{a\in\Gamma}\eta(a)\omega_a^\lambda$ for every character η on Γ (more precisely, in general the stabilization should be done with respect to a certain subgroup of Γ, see [**Saka**, §4.4.2]). It then follows from [**Saka**, Theorem 5.3.1] that ω_η^λ and $\omega_\eta^{w\lambda}$ lie in the same one dimensional space of spherical functions. Thus for the stable basis of spherical functions, the functional equations should be much simpler. As we shall soon see, Hironaka indeed considered this stabilization in order to obtain the functional equations in the case at hand when X is the space of Hermitian matrices.

For the space X of Hermitian matrices, the K-orbit decomposition was obtained in [**Jacob62**]. If E/F is unramified, then the functional equations (6.2) are given explicitly in [**Hir88b**, §2]. If E/F is ramified, then the functional equations are given explicitly only for a simple reflection w (see [**Hir88b**, §3] for odd residual characteristic and [**Hir90**] when $F=\mathbb{Q}_2$), but note that $B(w_1w_2,\lambda)=B(w_2,\lambda)B(w_1,w_2\lambda)$. In the unramified case, the integral $I_a^\lambda(x)$ is easy to compute for a special choice of a representative for each of the K-orbits in X. Thus all terms in (6.3) become explicit and this way Hironaka obtains the explicit formulas for the spherical functions. In the ramified case, explicit formulas are only available when $n=2$ [**Hir89, Hir90**] (see Remark 6.5 for the status of explicit formulas for general n).

In order to describe the functional equations it is more convenient, however, to introduce a different basis for the spherical functions. Let $\chi=(\chi_1,\dots,\chi_n)$ be a character of T' which is trivial on $\operatorname{Nm}T$. Hironaka introduced the spherical functions

$$L(x;\chi;s)=\int_{K^\circ}\prod_{i=1}^n|d_i({}^t\bar{k}xk)|^{s_i}\chi_i(d_i({}^t\bar{k}xk))\,dk$$

where $K^\circ=\{k\in K:{}^t\bar{k}xk\in X^\circ\}$. We can think of χ as an element of the dual $\hat{\Gamma}$ of Γ. Yet more convenient is to make a change in the variable χ. For χ as above we let $\nu=(\nu_1,\dots,\nu_n)\in\hat{\Gamma}$ be such that

$$\nu_i=\omega^i\prod_{j=n+1-i}^n\chi_j. \tag{6.4}$$

For $s\in\mathbb{C}^n$ the variable $\lambda=\lambda(s)=(\lambda_1,\dots,\lambda_n)$ that satisfies (6.1) can be defined by

$$\lambda_i=\frac{n+1}{2}-i-(s_{n+1-i}+\dots+s_n). \tag{6.5}$$

We set

$$\mathcal{L}(x;\nu;\lambda)=L(x;\chi;s)$$

where λ is related to s by (6.5) and ν is related to χ by (6.4). Note then that

$$\mathcal{L}(x;\nu;\lambda)=\sum_{a\in\Gamma}{}^{w_0}(\nu_0\nu)(a)\omega_a^\lambda(x)$$

where $\nu_0=(\omega,\omega^2,\dots,\omega^n)$ is the transfer factor γ viewed as an element of $\hat{\Gamma}$ and ${}^w\nu=(\nu_{w^{-1}(1)},\dots,\nu_{w^{-1}(n)})$ for every $w\in W$ and $\nu=(\nu_1,\dots,\nu_n)\in\hat{\Gamma}$. The basis $\{\mathcal{L}(\cdot;\nu;\lambda)\}_{\nu\in\hat{\Gamma}}$ consists of the stabilized spherical functions in the sense of Remark 6.3. Note that our change of variables from s to λ is slightly different then Hironaka's from s to z. In what follows we adjust her results accordingly.

6.1. The unramified Hermitian space. We assume here that E/F is an unramified quadratic extension of p-adic fields. Note then that $\omega=|\cdot|^{\epsilon_0}$ is an unramified character with $\epsilon_0=\frac{\pi i}{\log q}$ and that

$$L(\mu\omega,s)=L(\mu,s+\epsilon_0) \tag{6.6}$$

for any character μ of $F^\times$. Note further that the spherical functions $\omega_a^\lambda(x)$ (and hence also the functions $\mathcal{L}(x;\nu;\lambda)$) depend only on $\lambda \mod 2\epsilon_0\mathbb{Z}^n$ whereas their common Hecke eigenvalue $\hat{f}(\lambda)$ for $f\in\mathcal{H}_K(G)$ depends only on $\lambda \mod \epsilon_0\mathbb{Z}^n$.

The K-orbit decomposition on X is given by the disjoint union

$$X=\bigsqcup_{m\in\Lambda_n}\{{}^t\bar{k}\varpi^m k : k\in K\}$$

where

$$\Lambda_n=\{m=(m_1,\dots,m_n)\in\mathbb{Z}^n : m_1\ge\dots\ge m_n\},$$

and $\varpi^m=\operatorname{diag}(\varpi^{m_1},\dots,\varpi^{m_n})$ [**Jacob62**].

The expression

$$\mathbf{1}_{X_a}({}^t\bar{k}w_0\varpi^m w_0 k)\prod_{i=1}^n|d_i({}^t\bar{k}xk)|^{s_i}$$

is in fact constant for all $k\in\mathcal{I}$, and $I_a^\lambda(w_0\varpi^m w_0)$ is therefore easy to evaluate. We have

$$I_a^\lambda(w_0\varpi^m w_0)=\mathbf{1}_{X_a}(w_0\varpi^m w_0)q^{\langle\lambda-\rho,m\rangle} \tag{6.7}$$

for $m\in\Lambda_n$. This is [**Hir99**, Lemma 2.1]. Note that

$$L(x;\chi,s)=L(x;\mathbf{1}_\Gamma;s+\epsilon(\chi))$$

where $\epsilon(\chi)=(\epsilon(\chi_1),\dots,\epsilon(\chi_n))$ and

$$\epsilon(\mu)=\begin{cases}0 & \mu=1\\ \epsilon_0 & \mu=\omega\end{cases}$$

and that $\lambda(s+\epsilon(\chi))\equiv\lambda(s)-\epsilon(\nu\nu_0) \mod 2\epsilon_0\mathbb{Z}^n$ where ν is related to χ by (6.4). Thus

$$\mathcal{L}(x;\nu;\lambda)=\mathcal{L}(x;\nu_0;\lambda-\epsilon(\nu\nu_0)) \tag{6.8}$$

and it is therefore enough to compute $\mathcal{L}(x;\nu_0;\lambda)=\sum_{a\in\Gamma}\omega_a^\lambda(x)$. Let

$$\tau(\nu;\lambda)=\prod_{i<j}\frac{L(\nu_i\nu_j^{-1},\lambda_i-\lambda_j+1)}{L(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j)}.$$

With our notation, Hironaka obtains in [**Hir88b**, §2] the functional equation

$$\tau(\nu_0;\lambda)\mathcal{L}(x;\nu_0,\lambda)=\tau({}^w\nu_0;w\lambda)\mathcal{L}(x;{}^w\nu_0,w\lambda)$$

for every $w\in W$. Applying (6.6) and (6.8) this functional equation generalizes. For any $\nu\in\hat{\Gamma}$, $\lambda\in\mathbb{C}^n$ and $w\in W$, we have

$$\tau(\nu;\lambda)\mathcal{L}(x;\nu,\lambda)=\tau({}^w\nu;w\lambda)\mathcal{L}(x;{}^w\nu,w\lambda). \tag{6.9}$$

This functional equation is the one obtained in [**Hir99**, p. 570] for $L(x;\chi;s)$. Our normalization of variables is more natural and simplifies the functional equation.

Remark 6.4. The functional equations are obtained by Hironaka by relating the spherical functions to the classical local densities associated with Hermitian forms. The formula obtained in [**Hir88a**, §2], expresses the spherical functions as generating functions for local densities.

For every $w\in W$, let

$$\Delta^w=({}^{w_0}(\nu_0{}^w\nu)(a))_{\nu\in\hat{\Gamma}\,;\,a\in\Gamma}\text{ and }T^w(\lambda)=(\delta_{\nu,\nu'}\ \tau({}^w\nu,w\lambda))_{\nu,\,\nu'\in\hat{\Gamma}}.$$

When w is the identity element, we sometimes omit the superscript w. From the formula (6.2) that defines $B(w,\lambda)$ and from the functional equation (6.9), we get that

$$\Delta B(w,\lambda)=T(\lambda)^{-1}T^w(\lambda)\Delta^w.$$

It follows from (6.7) that for $m\in\Lambda_n$, we have

$$\Delta^w(I_a^{w\lambda}(w_0\varpi^m w_0))_{a\in\Gamma}=((\nu_0{}^w\nu)(\varpi^m)q^{\langle m,w\lambda-\rho\rangle})_{\nu\in\hat{\Gamma}}.$$

It is now convenient to denote by $Z=Z(\nu;\lambda)=(Z_1,\dots,Z_n)$ the variable given by

$$Z_i=\nu_i(\varpi)q^{\lambda_i}. \tag{6.10}$$

Applying Δ to both sides of (6.3) after some cancellation, we obtain

$$\mathcal{L}(\varpi^m;\nu;\lambda)=\frac{1}{Q}\nu_0(\varpi^m)q^{-\langle m,\rho\rangle}\prod_{i<j}\frac{Z_i-q^{-1}Z_j}{Z_i+Z_j}\sum_{w\in W}w\Big(Z^m\prod_{i<j}\frac{Z_i+q^{-1}Z_j}{Z_i-Z_j}\Big)$$

where $Z^m=\prod_{i=1}^n Z_i^{m_i}$. This can be expressed in terms of the mth Hall-Littlewood polynomial:

$$P_m(Z_1,\dots,Z_n;t)=\frac{(1-t)^n}{V_m(t)}\sum_{w\in W}w\Big(Z^m\prod_{i<j}\frac{Z_i-tZ_j}{Z_i-Z_j}\Big). \tag{6.11}$$

It is well-known that this is a symmetric Laurent polynomial in Z. Here the combinatorial factor $V_m(t)$ is determined by the requirement that $P_m(Z;t)$ is monic (i.e., that the leading monomial symmetric polynomial, that associated to m, has coefficient one). Explicitly, let $v_n(t)=\prod_{i=1}^n(1-t^i)$; thus,

$$V_m(t)=\prod_j v_{N_i(\lambda)}(t)$$

where $N_i(\lambda)=\#\{j:1\le j\le n,\ m_j=i\}$. In fact $\{P_m(Z;t):m\in\Lambda_n\}$ forms a basis of $\mathbb{Z}[t][Z_1^{\pm1},\dots,Z_n^{\pm1}]^W$. We then have

$$\begin{aligned}(6.12)\quad &\mathcal{L}(\varpi^m;\nu;\lambda)\\ &=\nu_0(\varpi^m)q^{-\langle m,\rho\rangle}(1-q^{-1})^n\frac{V_m(-q^{-1})}{v_n(q^{-2})}\Big[\prod_{i<j}\frac{Z_i-q^{-1}Z_j}{Z_i+Z_j}\Big]P_m(Z_1,\dots,Z_n;-q^{-1}).\end{aligned}$$

This gives an explicit formula for all spherical functions on X.

REMARK 6.5. If E/F is ramified, Hironaka also obtained the functional equations for the spherical functions. Thus her results can give explicitly formulas for the spherical functions in terms of the integrals $I_a^\lambda(x)$. If there is a diagonal matrix in the K-orbit of x, then choosing a convenient diagonal representative the integrals over the Iwahori subgroup can easily be obtained as in (6.7) and $\mathcal{L}(x;\nu;\lambda)$ can be given explicitly. However, in the ramified case not all Hermitian forms are integrally equivalent to diagonal ones. For such x, explicit formulas for $\mathcal{L}(x;\nu;\lambda)$ are not yet available.

Next, Hironaka defines the spherical Fourier transform. Again, we define the transform slightly differently so that it will be more convenient to apply with our notation. We consider the normalized spherical function

$$\Omega(x;\nu;\lambda) = \frac{\mathcal{L}(x;\nu;\lambda)}{\mathcal{L}(e;\nu;\lambda)}.$$

Thus for $m \in \Lambda_n$ we have

$$\Omega(\varpi^m;\nu;\lambda) = \nu_0(\varpi^m) q^{-\langle m,\rho\rangle} \frac{V_m(-q^{-1})}{v_n(-q^{-1})} P_m(Z;-q^{-1}).$$

Recall that $\mathcal{H}_X(K) = C_c^\infty(X/K)$ is an $\mathcal{H}_G(K)$-module. The spherical Fourier transform is defined for $\Psi \in \mathcal{H}_X(K)$ by

$$\hat{\Psi}(\nu;\lambda) = \mathrm{vol}(X\cap K)^{-1} \int_X \Psi(x)\Omega(x^{-1};\nu;\lambda)\, dx. \tag{6.13}$$

THEOREM 6.6. [**Hir99**, Theorem 2] *The spherical Fourier transform* (6.13) *defines an isomorphism of $\mathcal{H}_K(G)$-modules*

$$\mathcal{H}_K(X) \simeq \mathbb{C}[Z_1^{\pm 1},\dots,Z_n^{\pm 1}]^W.$$

We now obtain an identity between spherical Fourier transforms that is applied in §7 in order to solve the transfer factor dichotomy. Let $c_m = \mathbf{1}_{\{{}^t\bar{k}\varpi^m k : k\in K\}}$ be the characteristic function of the K-orbit of ϖ^m. Since E/F is unramified c_0 is the characteristic function of $X\cap K$. Note that $\{c_m : m \in \Lambda_n\}$ is a basis of $\mathcal{H}_X(K)$ and

$$\hat{c}_m(\lambda) = \frac{\int_X c_m(x)\, dx}{\int_X c_0(x)\, dx}\ \Omega_\lambda(\varpi^{\tilde{m}})$$

where we recall that as in §4 we have $\tilde{m} = (-m_n,\dots,-m_1)$. Hironaka computed the volume of every K-orbit using explicit formulas for certain local densities and obtained that

$$\frac{\int_X c_m(x)\, dx}{\int_X c_0(x)\, dx} = q^{2\langle m,\rho\rangle}\frac{v_n(-q^{-1})}{V_m(-q^{-1})},\ m\in\Lambda_n.$$

Since $P_{\tilde{m}}(Z;t) = P_m(Z^{-1};t)$ where $Z^{-1} = (Z_1^{-1},\dots,Z_n^{-1})$ we see that

$$\hat{c}_m(\lambda) = ({}^{w_0}\nu_0)(\varpi^m) q^{\langle m,\rho\rangle} P_m(Z^{-1};-q^{-1}). \tag{6.14}$$

For $m\in\Lambda_n$ we denote by τ_m the symmetric Laurent polynomial

$$\tau_m(Z_1,\dots,Z_n) = \sum_{w\in W} Z^{wm}.$$

Recall that the functions $\Phi^{(k)} \in \mathcal{H}_{G'}(K')$ and $\Psi^{(k)} \in \mathcal{H}_X(K')$ were introduced in §4. For $m \in \mathbb{Z}^n$, let $|m| = m_1 + \cdots + m_n$ and set

$$\Lambda_n^+(k) = \{m \in \Lambda_n : m_n \geq 0 \text{ and } |m| = k\}.$$

In fact, $\Psi^{(k)} = \sum_{m \in \Lambda_n^+(k)} c_m$ and similarly $\Phi^{(k)} = \sum_{m \in \Lambda_n^+(k)} \mathbf{1}_{K'\varpi^m K'}$.

LEMMA 6.7. *For every integer $k \geq 0$ we have*

$$(-1)^{nk}\widehat{\Psi^{(k)}}(\nu;\lambda) = \widehat{\Phi^{(k)}}(\nu;\lambda) = q^{k\frac{n-1}{2}} \sum_{m \in \Lambda_n^+(k)} \tau_m(Z^{-1}).$$

PROOF. We first note that it is easy to compute $\hat{\Phi}^{(k)}$ explicitly. Using the Iwasawa decomposition $G = UTK$, we get that

$$\widehat{\Phi^{(k)}}(\nu;\lambda) = \sum_{m \in \mathbb{Z}^n} q^{\langle \rho, m \rangle} Z^{-m} \int_U \Phi^{(k)}(u\varpi^m)\, du.$$

Let

$$M^+(k) = \{m \in \mathbb{Z}^n : m_1, \dots, m_n \geq 0 \text{ and } |m| = k\}.$$

Note that

$$\int_U \Phi^{(k)}(u\varpi^m) = \begin{cases} q^{\sum_{i=1}^n (i-1)m_i} & m \in M^+(k) \\ 0 & \text{else} \end{cases}$$

and that

$$q^{\langle \rho, m\rangle + \sum_{i=1}^n (i-1)m_i} = q^{|m|\frac{n-1}{2}}.$$

We therefore indeed see that

$$\widehat{\Phi^{(k)}}(\nu;\lambda) = q^{k\frac{n-1}{2}} \sum_{m \in \Lambda_n^+(k)} \tau_m(Z^{-1}).$$

On the other hand, from Macdonald's computation of the spherical functions (e.g., [**Mac95**, p. 299]), we have

$$\hat{\mathbf{1}}_{K\varpi^m K}(\nu;\lambda) = q^{\langle \rho, m\rangle} P_m(Z^{-1}; q^{-1}) = q^{|m|\frac{n-1}{2}} (q^{-1})^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1}; q^{-1}).$$

We therefore get the identity

$$\sum_{m \in \Lambda_n^+(k)} \tau_m(Z^{-1}) = \sum_{m \in \Lambda_n^+(k)} (q^{-1})^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1}; q^{-1}).$$

Since it holds for infinitely many values of q we obtain the algebraic identity

$$\sum_{m \in \Lambda_n^+(k)} \tau_m(Z^{-1}) = \sum_{m \in \Lambda_n^+(k)} t^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1}; t). \tag{6.15}$$

Applying (6.14), we see that

$$\widehat{\Psi^{(k)}}(\nu;\lambda) = q^{k\frac{n-1}{2}} \sum_{m \in \Lambda_n^+(k)} ({}^{w_0}\nu_0)(\varpi^m)(q^{-1})^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1}; -q^{-1}).$$

Since $\omega(\varpi) = -1$, we see that

$$({}^{w_0}\nu_0)(\varpi^m) = (-1)^{n|m|}(-1)^{\sum_{i=1}^n (i-1)m_i}.$$

It follows from (6.15), now applied to $t = -q^{-1}$, that

$$\widehat{\Psi^{(k)}}(\nu;\lambda) = (-1)^{nk} q^{k\frac{n-1}{2}} \sum_{m \in \Lambda_n^+(k)} \tau_m(Z^{-1}).$$

□

7. Bessel identities for principal series representations

In this section we recall and refine the results of [**Off07**]. The main result is a local identity between a distribution on X and a distribution on G'. The notation in [**Off07**] was slightly different. The space

$$Y = \{g \in G : gw_0{}^t\bar{g}^{-1}w_0^{-1} = e\}$$

was considered as a left G-space with the action $g \cdot y = gyw_0{}^t\bar{g}w_0^{-1}$. The map $y \mapsto x = (yw_0)^{-1}$ is used to translate results from the left G-space Y to the right G-space X. Note that $g^{-1} \cdot y = {}^t\bar{g}xg$ if $y \mapsto x$ and that $H^x = \{g \in G : g \cdot y = y\}$. We present the results of [**Off07**] in the notation of this work without further remarks regarding the different G-actions.

We start with the main local result. Assume then that E/F is a quadratic extension of local fields of characteristic zero. For a character $\nu = (\nu_1, \dots, \nu_n)$ of T' and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, we let

$$\gamma(\nu, \lambda, \psi') = \prod_{i<j} \gamma(\nu_i\nu_j^{-1}\omega, \lambda_i - \lambda_j, \psi')$$

where for a character μ of $F^\times$ and $s \in \mathbb{C}$, we let

$$\gamma(\mu, s, \psi') = \frac{L(\mu, s)}{\epsilon(\mu, s, \psi')L(\mu^{-1}, 1-s)}$$

be the Tate γ-factor. Let χ be a unitary character of T which is a base change from T', i.e., it factors through the norm map. Thus the set $\mathcal{B}(\chi)$ of characters ν of T' such that $\chi = \nu \circ \mathrm{Nm}$ is not empty. Recall that $\Gamma = T'/\mathrm{Nm}(T)$. For $x \in X$ and $\nu \in \mathcal{B}(\chi)$, we define a family of H^x-invariant linear functionals $J^{st,x}(\nu, \varphi, \lambda)$ on $I(\chi, \lambda)$ as follows. First, for $a \in \Gamma$, if the G-orbit of x does not contain a, we set $J^x_\nu(a, \varphi, \lambda) = 0$; otherwise, let $\eta \in G$ and $t \in a$ be such that ${}^t\bar{\eta}x\eta = t$ and let $H^x_\eta = H^x \cap \eta B\eta^{-1}$. The linear functional $J^x_\nu(a, \varphi, \lambda)$ is defined for $\varphi \in I(\chi)$ and $\lambda \in \mathbb{C}^n$ so that $\mathrm{Re}\,\lambda$ lies in a certain positive cone, by the convergent integral

$$J^x_\nu(a, \varphi, \lambda) = (\nu_0\nu)(t)e^{\frac{1}{2}\langle\lambda+\rho, H(t)\rangle} \int_{H^x_\eta\backslash H^x} e^{\langle\lambda, H(\eta^{-1}h)\rangle}\varphi(\eta^{-1}h)\, dh$$

and is independent of the choices of η and t. We define the stable linear functional

$$J^{st,x}(\nu, \varphi, \lambda) = \sum_{a \in \Gamma} J^x_\nu(a, \varphi, \lambda).$$

Using Bernstein's principle of analytic continuation, the proof of [**LR00**, Proposition 2] shows that if E/F is a quadratic extension of p-adic fields, then $J^{st,x}(\nu, \varphi, \lambda)$ admits a meromorphic continuation to a rational function in q^λ. In the archimedean case, the meromorphic continuation then follows from an analogous global statement that we shall soon come to.

The Whittaker functional $\mathcal{W}^\psi(\varphi, \lambda)$ and the Whittaker function $W^\psi(g, \varphi, \lambda)$ were defined in §3.3. The local stable relative Bessel distribution is defined for $\Psi \in C_c^\infty(X)$ by

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = \sum_{\varphi \in \mathrm{ob}(I(\chi))} \int_X \Psi(x) J^{st,x}(\nu, \varphi, \lambda)\, dx\, \overline{\mathcal{W}}^\psi(\varphi, -\bar{\lambda}).$$

The local Bessel distribution is defined for $\Phi \in C_c^\infty(G')$ by

$$B'(\Phi,\nu,\lambda) = \sum_{\varphi' \in \mathrm{ob}(I'(\nu))} W^{\psi'}(w_0, I'(\Phi,\lambda)\varphi',\lambda)\overline{\mathcal{W}}^{\psi'}(\varphi',-\bar{\lambda}).$$

THEOREM 7.1. *There exists a root of unity $\kappa_{E/F} = \kappa_{E/F}(\psi', n)$ so that for any unitary character ν of T' and matching functions $\Phi \overset{\psi'}{\longleftrightarrow} \Psi$ we have*

$$\tilde{B}^{st}(\Psi,\nu,\lambda) = \kappa_{E/F}\ \gamma(\nu,\lambda,\psi')\ B'(\Phi,\nu,\lambda).$$

If $\psi'_a = \psi'(a\ \cdot)$ for some $a \in F^\times$, then

$$\kappa_{E/F}(\psi'_a, n) = \omega(a)^{\frac{(n-1)n}{2}}\kappa_{E/F}(\psi', n).$$

Furthermore, if E/F is unramified of odd residual characteristic and ψ' has conductor $\mathcal{O}_F$, then $\kappa_{E/F}(\psi', n) = 1$.

The motivation for this local identity of distributions was already explained in the introduction. It allows us to express the unitary period in the left hand side of (1.1) explicitly in terms of Hironaka's spherical functions. For the case $n = 3$ the Bessel identity was first obtained in [**LR00**]. The results of Jacquet, Lapid and Hironaka explained in §4-§6 enable us to apply the method of Lapid and Rogawski and to prove Theorem 7.1 for general n.

The proof uses global methods and we also prove an analogous global identity. Before explaining the method of proof, let us first recall the global analogue. Assume now that E/F is a quadratic extension of number fields. Let χ be a unitary character of $T\backslash T_{\mathbb{A}}$ which is in the image of base change, i.e., so that the set $\mathcal{B}(\chi)$ of characters ν of $T'\backslash T'_{\mathbb{A}}$ such that $\chi = \nu \circ \mathrm{Nm}$ is not empty. For $x \in X_{\mathbb{A}}$, $\nu \in \mathcal{B}(\chi)$ and $\varphi = \otimes_v \varphi_v \in I(\chi)$ we define the stable intertwining period $J^{x,st}(\varphi,\nu,\lambda)$ by the meromorphic continuation of

$$J^{x,st}(\varphi,\nu,\lambda) = \prod_v J^{x_v,st}(\varphi_v,\nu_v,\lambda).$$

We shall soon see that the right hand side is an Euler product, convergent in some positive cone, that admits a meromorphic continuation in λ. The global stable relative Bessel distribution is defined for $\Psi \in C_c^\infty(X_{\mathbb{A}})$ by

$$\tilde{B}^{st}(\Psi,\nu,\lambda) = \sum_{\varphi \in \mathrm{ob}(I(\chi))} \int_{X_{\mathbb{A}}} \Psi(x) J^{x,st}(\varphi,\nu,\lambda)\ dx\ \overline{\mathcal{W}}^{\psi}(\varphi,-\bar{\lambda}).$$

The global Bessel distribution is defined for $\Phi \in C_c^\infty(G'_{\mathbb{A}})$ by

$$B'(\Phi,\nu,\lambda) = \sum_{\varphi' \in \mathrm{ob}(I'(\nu))} W^{\psi'}(w_0, I'(\Phi,\lambda)\varphi',\lambda)\overline{\mathcal{W}}^{\psi'}(\varphi,-\bar{\lambda}).$$

THEOREM 7.2. *Let ν be a unitary character of $T'\backslash T'_{\mathbb{A}}$. For $\Phi \in C_c^\infty(G'_{\mathbb{A}})$ and $\Psi \in C_c^\infty(X_{\mathbb{A}})$ such that $\Phi \overset{\psi'}{\longleftrightarrow} \Psi$, we have*

$$\tilde{B}^{st}(\Psi,\nu,\lambda) = B'(\Phi,\nu,\lambda).$$

The stable relative Bessel distribution $\tilde{B}^{st}(\Psi,\nu,\lambda)$ contributes to the most continuous part of Lapid's spectral expansion for the RTF, i.e., to the summands in (5.4) where the integration is over the $(n-1)$-dimensional space $i(\mathfrak{a}_0^G)^*$. We now explain in what sense. Let $\{\xi\}$ be a set of representatives for the G-orbits in X.

Recall that for a function $\Psi \in C_c^\infty(X_{\mathbb{A}})$ we associate a family $\{f^\xi\}$ of functions in $C_c^\infty(G_{\mathbb{A}})$ so that

$$\Psi({}^t\bar{g}\xi g) = \int_{H^\xi_{\mathbb{A}}} f^\xi(hg)\ dh. \tag{7.1}$$

As already mentioned in §5, the distribution RTF on $X_{\mathbb{A}}$ can be expressed as a sum of the distributions RTF_ξ on $G_{\mathbb{A}}$, the spectral expansion of which is described in §5. The most continuous part of the spectrum is the distribution

$$\sum_\chi \int_{i(\mathfrak{a}_0^G)^*} \tilde{B}(\Psi, \chi, \lambda)\ d\lambda$$

where the sum is over all unitary Hecke characters χ of $T_{\mathbb{A}}$ that lie in the image of base change from $T'_{\mathbb{A}}$. The relative Bessel distribution $\tilde{B}(\Psi, \chi, \lambda)$ is defined by

$$\tilde{B}(\Psi, \chi, \lambda) = \sum_\xi \sum_{\varphi \in \mathrm{ob}(I(\chi))} \mathcal{P}^{H^\xi}(E(I(f^\xi, \lambda)\varphi, \lambda)))\ \overline{\mathcal{W}}^\psi(\varphi, -\bar{\lambda}).$$

We also recall the definition of certain global intertwining periods from [**LR03**]. These are special cases, for principal series representations, of the intertwining periods in Theorem 5.3. When in the global setting we also denote by Γ the group $T'/\operatorname{Nm} T$. For every place v of F we denote by Γ_v the analogous local group $T'_v/\operatorname{Nm} T_v$ and let $\Gamma_{\mathbb{A}} = \bigoplus_v \Gamma_v$. The group Γ naturally embeds in $\Gamma_{\mathbb{A}}$ in the diagonal and it follows from class field theory that $\#(\Gamma_{\mathbb{A}}/\Gamma) = 2^n$. Let $x \in X$ and let $\Gamma(x) = \{a \in \Gamma : \exists g \in G,\ {}^t\bar{g}xg \in a\}$. For $a \in \Gamma(x)$ and λ such that $\operatorname{Re}\lambda$ is sufficiently large, we define

$$J^x(a, \varphi, \lambda) = \int_{(H^x_\eta)_{\mathbb{A}} \backslash H^x_{\mathbb{A}}} e^{\langle \lambda, H(\eta^{-1}h)\rangle} \varphi(\eta^{-1}h)\ dh$$

where $\eta \in G$ is such that ${}^t\bar{\eta}x\eta \in a$. This integral converges and is independent of the choice of η. As explained in [**LR00**, Remark 3], however, it is not expected to have a meromorphic continuation in λ. As a special case of Theorem 5.3, the result of Lapid and Rogawski gives

$$\mathcal{P}^{H^x}(E(\varphi, \lambda)) = 2^n \sum_{a \in \Gamma(x)} J^x(a, \varphi, \lambda). \tag{7.2}$$

Thus, the (infinite) sum on the right hand side does admit a meromorphic continuation in λ. A simple argument in Fourier analysis of finite groups gives that for any factorizable, absolutely summable function $g(a) = \prod_v g_v(a_v)$ on $\Gamma_{\mathbb{A}}$ we have

$$2^n \sum_{a \in \Gamma} g(a) = \sum_{\kappa \in (\Gamma_{\mathbb{A}}/\Gamma)^*} \prod_v \hat{g}_v(\kappa_v)$$

where the sum on the right hand side is over the 2^n characters κ of $\Gamma_{\mathbb{A}}/\Gamma$, κ_v is the restriction of κ to Γ_v and $\hat{g}_v(\kappa_v) = \sum_{a \in \Gamma_v} \kappa_v(a_v) g_v(a_v)$. Applying this to the function

$$g(a) = \begin{cases} J^x(a, \varphi, \lambda) & a \in \Gamma(x) \\ 0 & \text{else,} \end{cases}$$

the equality (7.2) implies that

$$\mathcal{P}^{H^x}(E(\varphi,\lambda)) = \sum_{\nu\in\mathcal{B}(\chi)} J^{st,x}(\nu,\varphi,\lambda). \tag{7.3}$$

Thus we express the regularized period of an Eisenstein series associated to a principal series representation as a finite sum of factorizable linear functionals. Using the fact that the local period $J^{st,x_v}(\nu_v,\varphi_v,\lambda)$ in a p-adic place v is meromorphic, it is not difficult to deduce from (7.3) that the stable intertwining period $J^{st,x}(\nu,\varphi,\lambda)$ (and hence also each archimedean factor) admits a meromorphic continuation to $\lambda\in\mathbb{C}^n$ (see [**Off07**, Lemma 6]). For $x\in X_{\mathbb{A}}$ and $f\in C_c^\infty(G_{\mathbb{A}})$, we also define the stable relative Bessel distribution

$$B^{st,x}(f,\nu,\lambda) = \sum_{\varphi\in\mathrm{ob}(I(\chi))} J^{st,x}(I(f,\lambda)\varphi,\nu,\lambda)\ \overline{\mathcal{W}}(\varphi,-\bar\lambda).$$

Using formal manipulation of integrals and following the definitions, we get that

$$\tilde B^{st}(\Psi,\nu,\lambda) = \sum_{[\xi]\in X_{\mathbb{A}}/G_{\mathbb{A}}} B^{st,\xi}(f^\xi,\nu,\lambda)$$

where the sum is over all $G_{\mathbb{A}}$-orbits in $X_{\mathbb{A}}$ (with a choice $\{\xi\}$ of representatives) and Ψ and $\{f^\xi\}$ are related by (7.1). For $\nu=(\nu_1,\dots,\nu_n)\in\mathcal{B}(\chi)$ we let $\omega\nu=(\omega\nu_1,\dots,\omega\nu_n)\in\mathcal{B}(\chi)$. It is also easy to see that for $x\in X_{\mathbb{A}}$, we have

$$J^{st,x}(\varphi,\omega\nu,\lambda) = \omega(\det x)J^{st,x}(\varphi,\nu,\lambda).$$

It follows from the local to global principle for Hermitian forms that $\omega(\det x)=1$ if and only if the $G_{\mathbb{A}}$-orbit of x contains a rational point. Therefore, when summing over $\nu\in\mathcal{B}(\chi)$, the summands associated to $G_{\mathbb{A}}$-orbits with no point in X cancel out and we obtain

$$\sum_{\nu\in\mathcal{B}(\chi)} \tilde B^{st}(\Psi,\nu,\lambda) = \sum_{\nu\in\mathcal{B}(\chi)}\sum_{[\xi]\in X/G} B^{st,\xi}(f^\xi,\nu,\lambda)$$

where the inner sum on the right hand side is now only over the G-orbits in X. Combined with (7.3) we now get that

$$\tilde B(\Psi,\chi,\lambda) = \sum_{\nu\in\mathcal{B}(\chi)} \tilde B^{st}(\Psi,\nu,\lambda).$$

Thus, for a principal series representation $I(\chi)$ of $G_{\mathbb{A}}$ where ξ is in the image of base change from $T'_{\mathbb{A}}$, its contribution to the RTF is expressed as a finite sum of factorizable distributions. The Bessel distribution $B'(\Phi,\nu,\lambda)$ is the contribution of the principal series representation $I'(\nu)$ of $G'_{\mathbb{A}}$ to the KTF. We then see that the most continuous part of the spectral expansion of both the RTF and the KTF is a sum of distributions parameterized by characters ν of $T'\backslash T'_{\mathbb{A}}$. The content of Theorem 7.2 is that these distributions can be compared termwise for matching functions.

Roughly speaking, the idea behind the proof of Theorem 7.1 is as follows. The local Bessel identities are first proved directly in the split case and in the unramified inert case when all the data is unramified. In order to prove the identity in the general local case, the problem is embedded into a favorable global setting, where the local identities are known at all places except the relevant local extension. We then use the RTF in order to obtain the global identity for this case and deduce the relevant local identity. The root of unity $\kappa_{E/F}$ is not determined in general, since we can only choose a suitable global setting so that the relevant extension may occur at more then one place. Once the general local identity is obtained, Theorem 7.2 follows immediately. We begin our description of the proof by recalling the local identities that are obtained directly.

Consider first the split case, i.e., when $E = F \oplus F$. In this case $G = G' \times G'$ and $X = \{({}^t g, g) : g \in G'\}$ is a unique G-orbit. A unitary character of $T = T' \times T'$ is in the image of base change if and only if it is of the form $\chi = (\nu, \nu)$ for a character ν of T' and in this case $\mathcal{B}(\chi) = \{\nu\}$. For $\Psi \in C_c^\infty(X)$ let $\Phi \in C_c^\infty(G')$ be defined by

$$\Phi(g) = \Psi({}^t g, g) \tag{7.4}$$

then $\Phi \xleftrightarrow{\psi'} \Psi$.

LEMMA 7.3 ([**Off07**], Proposition 4). *For every Φ and Ψ as in* (7.4)*, we have*

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = \gamma(\nu, \lambda, \psi') B'(\Phi, \nu, \lambda).$$

The proof is based on [**LR00**, Proposition 4] and relies on a functional equation of Shahidi for Whittaker functions given in [**Sha81**].

We now turn to the case where E/F is an unramified quadratic extension of p-adic fields. Recall that the spherical Fourier transform of Hironaka on $\mathcal{H}_X(K)$ was defined in §6. The following is a generalization of [**Off07**, Proposition 5].

LEMMA 7.4. *Assume that ψ' has conductor $\mathcal{O}_F$ and let $\Phi \in \mathcal{H}_{G'}(K')$ and $\Psi \in \mathcal{H}_X(K)$ be such that $\hat{\Phi} = \hat{\Psi}$. Then, for every unramified unitary character ν of T' and $\lambda \in \mathbb{C}^n$ we have*

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = \gamma(\nu, \lambda, \psi') B'(\Phi, \nu, \lambda).$$

PROOF. Let $\chi = \nu \circ \mathrm{Nm}$ and let φ_χ denote the K-invariant section in $I(\chi)$ normalized so that $\varphi_\chi(e) = 1$. It follows from the definitions that for $\varphi \in I(\chi)$ we have

$$J^{st, {}^t g x g}(\varphi, \nu, \lambda) = J^{st,x}(I(g,\lambda)\varphi, \nu, \lambda) \tag{7.5}$$

and since Ψ is K-invariant, we then have

$$\int_X \Psi(x) J^{st,x}(\varphi, \nu, \lambda)\, dx = 0$$

whenever $(\varphi, \varphi_\chi) = 0$. It follows that

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = \int_X \Psi(x) J^{st,x}(\varphi_\chi, \nu, \lambda)\, dx\ \overline{\mathcal{W}}(\varphi_\chi, -\bar{\lambda}). \tag{7.6}$$

We show in [**Off07**, §3.2] that

$$J^{st,x}(\varphi_\chi, \nu, \lambda) = \frac{\mathrm{vol}(H^e \cap K)}{\mathrm{vol}(H^e_e)} (1 - q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(x^{-1}; \nu; \lambda). \tag{7.7}$$

The terms on the right hand side were defined in §6. This gives an interpretation of the stable period at the unramified section in terms of Hironaka's spherical functions. It is a straightforward computation to obtain this identity up to a constant depending only on the G-orbit of x. That the constant is independent of the G-orbits then follows from the algebraic nature of the consistent measures on the different unitary groups. The constant is then computed using an asymptotic formula for the intertwining periods [**Off07**, Lemma 4]. Note that under our assumptions, $\mathrm{vol}(H^e \cap K) = \mathrm{vol}(X \cap K)^{-1}$ and $\mathrm{vol}(H_e^e) = 1$. It follows that

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = \hat{\Psi}(\nu, \lambda)\ (1 - q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(e; \nu; \lambda)\ \overline{\mathcal{W}}^{\psi}(\varphi_\chi, -\bar{\lambda}).$$

On the other hand,

$$B'(\Phi, \nu, \lambda) = \hat{\Phi}(\nu, \lambda)\ \mathcal{W}^{\psi'}(\varphi_\nu, \lambda)\ \overline{\mathcal{W}}^{\psi'}(\varphi_\nu, -\bar{\lambda}).$$

Thus the lemma would follow from the identity

$$(1-q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(e; \nu; \lambda)\ \overline{\mathcal{W}}^{\psi}(\varphi_\chi, -\bar{\lambda}) = \gamma(\nu, \lambda, \psi')\ \mathcal{W}^{\psi'}(\varphi_\nu, \lambda)\ \overline{\mathcal{W}}^{\psi'}(\varphi_\nu, -\bar{\lambda}).$$

Using (6.12) and the Shintani, Casselman-Shalika formula for the spherical Whittaker functions, we have

$$\begin{aligned} &(1 - q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(e; \nu; \lambda)\ \overline{\mathcal{W}}(\varphi_\chi, -\bar{\lambda}) \\ &\qquad = \prod_{i<j} \frac{L(\nu_i \nu_j^{-1} \omega, \lambda_i - \lambda_j)}{L(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j + 1)} L(\chi_j \chi_i^{-1}, \lambda_j - \lambda_i + 1)^{-1} \end{aligned}$$

whereas

$$\begin{aligned} &\gamma(\nu, \lambda, \psi')\ \mathcal{W}(\varphi_\nu, \lambda)\ \overline{\mathcal{W}}(\varphi_\nu, -\bar{\lambda}) \\ &= \prod_{i<j} \frac{L(\nu_i \nu_j^{-1} \omega, \lambda_i - \lambda_j)}{L(\nu_j \nu_i^{-1} \omega, \lambda_j - \lambda_i + 1)} L(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j + 1)^{-1} L(\nu_j \nu_i^{-1}, \lambda_j - \lambda_i + 1)^{-1}. \end{aligned}$$

Since $L(\mu, s) L(\mu\omega, s) = L(\mu \circ \mathrm{Nm}, s)$ for every character μ of $F^\times$ the identity follows. $\square$

Lemma 7.4 allows us to reduce Theorems 7.1 and 7.2 to [**Off07**, Theorem 3] as follows. In [**Off07**, Theorem 3] we show the weaker statement, that there exists $\delta = \delta(n) \in \{0, 1\}$ depending only on n, so that the Bessel identities of Theorem 7.1 hold for functions Φ and Ψ matching with respect to the transfer factor $\omega(\det a)^\delta \gamma(a)$ (we shall then write $\Phi \overset{\delta}{\longleftrightarrow} \Psi$). Thus our refinement here is the claim that $\delta = 0$. Recall from (4.14) that Jacquet showed, in particular, that $\Phi^{(1)} \overset{0}{\longleftrightarrow} (-1)^n \Psi^{(1)}$. Since $\Phi^{(1)}$ is supported on $(\omega \circ \det)^{-1}(-1)$, we then have

$$\Phi^{(1)} \overset{\delta}{\longleftrightarrow} (-1)^{\delta+n} \Psi^{(1)}. \tag{7.8}$$

It also follows from Lemma 6.7 that $\widehat{\Phi}^{(1)} = (-1)^n \widehat{\Psi}^{(1)}$. Hence, from Lemma 7.4 we have

$$\tilde{B}^{st}((-1)^n \Psi^{(1)}, \nu, \lambda) = \gamma(\nu, \lambda, \psi') B'(\Phi^{(1)}, \nu, \lambda) \tag{7.9}$$

and this expression is not identically zero on (ν, λ). But also, [**Off07**, Theorem 3] with $\delta = \delta(n)$ together with (7.8) imply that

$$(-1)^\delta \tilde{B}^{st}((-1)^n \Psi^{(1)}, \nu, \lambda) = \gamma(\nu, \lambda, \psi') B'(\Phi^{(1)}, \nu, \lambda).$$

Since (7.9) is generically non zero, we must have $\delta = 0$.

REMARK 7.5. Note that the fundamental lemma of Jacquet in the form of Theorem 4.4 is valid even if we replace the transfer factor $\gamma(a)$ by the transfer factor $\omega(\det a)\gamma(a)$ in the definition of matching, since the functions involved are supported in the kernel of $\omega \circ \det$. This is what we refer to as the transfer factor dichotomy. The matching (4.14) for odd k, however, only holds with the transfer factor $\gamma(a)$. As we just explained, this fact, together with Lemma 7.4 and Lemma 6.7 solves the dichotomy and determines uniquely the transfer factor $\gamma(a)$ for which Theorems 7.1 and 7.2 hold. Furthermore, Lemma 7.4 can be viewed as a spectral version of the fundamental lemma. It inspired me to predict Theorem 10.1.

From now on, we explain the proof of [**Off07**, Theorem 3]. Let us go back to a global setting. Lapid and Rogawski generalized in [**LR00**, Lemma 4] the argument of Langlands in [**Lan80**, §11], of linear independence of characters, to suit the setting of the RTF. From the fundamental Lemma of Jacquet and the fine spectral expansion of Lapid for the RTF it can then be shown that

$$\sum_{\nu \in \mathcal{B}(\chi)} \tilde{B}^{st}(\Psi, \nu, \lambda) = \sum_{\nu \in \mathcal{B}(\chi)} B'(\Phi, \nu, \lambda) \tag{7.10}$$

whenever $\Phi \overset{\psi'}{\longleftrightarrow} \Psi$ and χ is a character of $T\backslash T_{\mathbb{A}}$ that lies in the image of base change. Indeed, as in the proof of [**LR00**, Proposition 6] the identity (7.10) can be proved up to a combinatorial constant $\mathfrak{c}$ that depends only on n (the constant $\mathfrak{c}(T, \chi)$ in (5.4) does not depend on χ and not even on the quadratic field extension E/F). In [**Off07**] we used the fact that $\mathfrak{c} = 1$ without justification. To complete the argument, we explain this here. Let

$$\alpha(\chi, \lambda) = \prod_v \alpha_v(\chi_v, \lambda) = \prod_{i<j} L(\chi_j \chi_i^{-1}, \lambda_j - \lambda_i + 1).$$

From the unramified local computations, it follows that at least when $\operatorname{Re} \lambda$ is sufficiently large each summand of (7.10) is a factorizable distribution, indeed we can write for $\operatorname{Re} \lambda$ sufficiently large

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = \alpha(\chi, \lambda)^{-1} \prod_v \alpha_v(\chi_v, \lambda) \tilde{B}^{st}(\Psi_v, \nu_v, \lambda)$$

and

$$B'(\Phi, \nu, \lambda) = \alpha(\chi, \lambda)^{-1} \prod_v \gamma(\nu_v, \lambda, \psi_v) \alpha_v(\chi_v, \lambda) \tilde{B}'(\Phi_v, \nu_v, \lambda)$$

and our local unramified computations imply that the Euler products converge. Choosing an extension E/F which is split at all archimedean places and unramified at all finite places it follows from Lemma 7.3 and Lemma 7.4 that $\mathfrak{c} = 1$. From the global identity (7.10) we can deduce the following identity on the summands.

LEMMA 7.6 (Corollary 3 [**Off07**]). *Let E/F be a quadratic extension of number fields which is split at all real places of F, let $\delta \in \{0, 1\}$ and let χ be a unitary Hecke character of $T_{\mathbb{A}}$ that is a base change from $T'_{\mathbb{A}}$. There exists a permutation $\tau = \tau_{\delta,\chi}$ on $\mathcal{B}(\chi)$ so that*

$$\tilde{B}^{st}(\Psi, \nu, \lambda) = B'(\Phi, \tau(\nu), \lambda)$$

whenever $\Phi \overset{\delta}{\longleftrightarrow} \Psi$.

To deduce Lemma 7.6 from (7.10), we proceed as in [**LR00**]. Using the localization principle of Gelfand-Kazdhan [**GK75**], Jacquet observed that, for an inert p-adic place v of F, the local Bessel distribution on G'_v depends only on the orbital integrals of the test functions and may therefore be viewed as a distribution on X_v via matching. Since this is also the case at the split places, under the assumption of Lemma 7.6 on E/F, (7.10) may be viewed as an identity between distributions on $X_{\mathbb{A}}$. The key for deducing the termwise identity from (7.10) is a lemma in linear algebra [**LR00**, Lemma 5]. We now recall its content. Let V_1, V_2 and V_3 be vector spaces. Let $\{x_i^j\}_{i=1}^m$ and $\{y_i^j\}_{i=1}^m$ be two sets of m vectors in V_j so that the vectors $\{x_i^j\}_{i=1}^m$ are linearly independent for $j=1,2,3$ and such that

$$\sum_{i=1}^m x_i^1 \otimes x_i^2 \otimes x_i^3 = \sum_{i=1}^m y_i^1 \otimes y_i^2 \otimes y_i^3.$$

Then there exists a permutation τ of $\{1,2,\dots,m\}$ such that

$$y_i^1 \otimes y_i^2 \otimes y_i^3 = x_{\tau(i)}^1 \otimes x_{\tau(i)}^2 \otimes x_{\tau(i)}^3$$

for all $i=1,\dots,m$. In [**Off07**, Lemma 8], we show that for an inert p-adic place v of F the local relative Bessel distributions $\{\tilde{B}^{st}(\cdot,\nu_v,\lambda)\}_{\nu_v\in\mathcal{B}(\chi_v)}$ on X_v are linearly independent. This allows us to apply the above statement in linear algebra, using two auxiliary p-adic inert places of F and to obtain Lemma 7.6 from (7.10).

Using this global identity we first prove the local Bessel identity in the p-adic case. The idea is as follows. Fix a quadratic extension E^0/F^0 of p-adic fields. We embed the local p-adic setting in the following global setting. There exists a quadratic extension of number fields E/F so that the set S^0 of all places v of F so that $E_v/F_v \simeq E^0/F^0$ is not empty and so that for all $v \notin S^0$ the extension E_v/F_v is either split or an unramified quadratic extension of p-adic fields for p odd. For $\delta \in \{0,1\}$ and a unitary Hecke character χ of $T_{\mathbb{A}}$ which is a base change from $T'_{\mathbb{A}}$, let $\tau_{\delta,\chi}$ be the permutation on $\mathcal{B}(\chi)$ given by Lemma 7.6. Since $B'(\omega\circ\det\ \Phi,\nu,\lambda) = B'(\Phi,\omega\nu,\lambda)$ and $\Phi \xleftrightarrow{\delta} \Psi$ if and only if $(\omega\circ\det)\Phi \xleftrightarrow{1-\delta} \Psi$, it follows from the linear independence of the local p-adic relative Bessel distributions that

$$\tau_{1-\delta,\chi}(\nu) = \omega\tau_{\delta,\chi}(\nu) \tag{7.11}$$

for $\nu \in \mathcal{B}(\chi)$. We now pick an odd prime p relatively prime to the residual characteristic of F^0 and so that denoting by S^p the set of all places of F with residual characteristic p, the character ψ_v has conductor $\mathcal{O}_v$ for all $v \in S^p$. For a finite set of finite places S of F we denote by $\mathcal{U}_S$ the set of unramified unitary characters of $T'_S = \prod_{v\in S} T'_v$.

Lemma 7.7. [**Off07**, Lemma 11] *There is a non empty open set $U^p \subset \mathcal{U}_{S^p}$ so that if ν is a unitary Hecke character of $T'_{\mathbb{A}}$ such that $\nu_{S^p} \in U^p$ then $\tau_{\delta,\chi}(\nu) \in \{\nu,\omega\nu\}$.*

The lemma is proved by writing explicitly the identity of Lemma 7.6 for a choice of decomposable matching functions $\Phi \xleftrightarrow{\delta} \Psi$ so that $\Phi_v = \mathbf{1}_{K'_v}$ and $\Psi_v = \mathbf{1}_{X_v\cap K_v}$ for almost all v. Using the fundamental Lemma of Jacquet and the explicit Bessel identities in the split and unramified cases, we obtain an identity between two Dirichlet series. Comparing the p part of the Dirichlet series, we show that $\tau_{\delta,\chi}(\nu) \notin \{\nu,\omega\nu\}$ provides a non trivial closed condition on ν_{S^p}.

Let U^p be the set of characters given by Lemma 7.7. It follows from (7.11) that for every ν such that $\nu_{S^p} \in U^p$ there exists $\delta = \delta(\nu) \in \{0,1\}$ such that $\tau_{\delta,\chi}(\nu) = \nu$. In order to apply Lemma 7.7, we use a Lemma of Lapid-Rogawski on the distribution of Hecke characters.

LEMMA 7.8. [**LR00**, Corollary 2] *Let S_f be a finite set of finite places of F and let $S = S_\infty \cup S_f$. Given a place $w \notin S$, a unitary character $\eta = (\eta_v)_{v \in S_f}$ of T'_{S_f} and an open set $U \subset \mathcal{U}_{S_f}$, there exists a Hecke character ϱ of $T'_{\mathbb{A}}$ which is unramified outside $S \cup \{w\}$ such that $\varrho_{S_f}^{-1}\eta \in U$.*

We apply Lemma 7.8 with $S_f = S^0 \cup S^p$, $\eta = (\eta_v)_{v \in S_f}$ with $\eta_v = \mathbf{1}_{T'_v}$ for $v \in S^p$ and $\eta_v = \mu$ for a fixed character μ of $(F^0)^\times$ for all $v \in S^0$, $U = U^0 \times U^p$ where U^p is given by Lemma 7.7; U^0 will soon be specified and w is a place of F which is split in E. Thus, there exists a unitary Hecke character ν of $T'_{\mathbb{A}}$ such that $\nu_{S_f} = \eta$. In particular, we have $\tau_{\delta(\nu),\chi}(\nu) = \nu$. Since $\tau_{\delta(\nu),\chi}$ is given by Lemma 7.6, there exists constants $\kappa_v(\nu_v, \lambda)$ such that

$$\tilde{B}^{st}(\Psi_v, \nu_v, \lambda) = \kappa_v(\nu_v, \lambda)\gamma(\nu_v, \lambda_v, \psi)B'(\Phi_v, \nu_v, \lambda)$$

whenever $\Phi_v \overset{\delta(\nu)}{\longleftrightarrow} \Psi_v$. From the split and unramified explicit identities, we get that $\kappa_v(\nu_v, \lambda) = 1$ for $v \notin S^0$ and therefore the global identity implies that

$$\prod_{v \in S^0} \kappa_v(\nu_v, \lambda) = 1. \tag{7.12}$$

Note that for $v \in S^0$ there exists $\lambda_v \in i\mathbb{R}^n$ such that $\nu_v = \mu|\cdot|^{\lambda_v}$ and the function $\kappa_v(\nu_v, \lambda) = \kappa_v(\mu, \lambda + \lambda_v)$ is a rational function in $q_{F^0}^{\lambda}$. Assume now that the function $\kappa_v(\mu, \lambda)$ is not constant in λ. Then (7.12) is a closed condition on ν_{S^0} and we may choose $U^0 \subset \mathcal{U}_{S^0}$ for which (7.12) does not hold. This will contradict Lemma 7.8. Thus we see that $\kappa_v(\mu, \lambda) = \kappa_v(\mu)$ is a $|S^0|$-root of unity independent of λ. Repeating the argument, but changing η_v to be a different character μ_1 of $(F^0)^\times$ only at a single place $v \in S^0$ we shall get similarly that $\kappa(\mu)^{|S^0|-1}\kappa(\mu_1) = 1$ and therefore that $\kappa(\mu)$ is independent of μ. We therefore obtain the desired p-adic Bessel identity for $\delta(\mu)$-matching functions. To see that $\delta(\mu)$ is independent of μ, given any two quadratic extensions of p-adic fields E_i/F_i and characters ν_i for $i = 1, 2$, there exists a global setting of the following form: a quadratic extension of number fields E/F that splits at all archimedean places, places v_i of F and a Hecke character ν of $T'\backslash T'_{\mathbb{A}}$ such that $E_{v_i}/F_{v_i} \simeq E_i/F_i$ and $\nu_{v_i} = \nu_i$. This shows that $\delta(\nu_1) = \delta(\nu) = \delta(\nu_2)$. This proves the local Bessel identity [**Off07**, Theorem 3] in the p-adic case. The archimedean case now follows from (7.10) applied to $E/F = \mathbb{Q}[\sqrt{-1}]/\mathbb{Q}$, using the linear independence of the local Bessel distributions at the inert p-adic places. As we explained, now that [**Off07**, Theorem 3] holds we also know that $\delta = 0$ and we get Theorem 7.1. Theorem 7.2 follows from Theorem 7.1 applying again (7.10) and the linear independence of the local distributions.

7.1. Stable intertwining periods and Hironaka's spherical functions. In order to express the unitary periods of cusp forms in terms of Hironaka's spherical functions, it is necessary in the p-adic case to relate between the local functionals $J^{st,x}(\varphi_\chi, \nu, \lambda)$ (when χ is unramified and $\nu \in \mathcal{B}(\chi)$) and Hironaka's spherical functions $\mathcal{L}(x; \nu; \lambda)$. This is achieved in [**Off07**, §3.2]. Since we only consider unramified principal series on G, we may as well let $\chi = \mathbf{1}_T$ be the trivial character.

Assume first that E/F is unramified. Then the relation is given by (7.7). For every $\nu \in \mathcal{B}(\mathbf{1}_T)$ and $m \in \Lambda_n^+$, we then have

$$\begin{aligned}(7.13)\quad &\frac{\mathrm{vol}(H_e^e \cap K)}{\mathrm{vol}(H^e \cap K)} J^{st,\varpi^m}(\varphi_{\mathbf{1}_T}, \nu, \lambda) \\ &= \nu_0(\varpi^m) q^{-\langle 2\rho, m\rangle} \frac{V_m(\omega(\varpi)q^{-1})}{v_n(\omega(\varpi)q^{-1})} \prod_{i<j} \frac{L(\lambda_i - \lambda_j, \nu_i \nu_j^{-1}\omega)}{L(\lambda_i - \lambda_j + 1, \nu_i \nu_j^{-1})} P_m(Z; \omega(\varpi)q^{-1})\end{aligned}$$

where $Z = Z(\nu, \lambda)$ is given by (6.10). In fact, applying [**Off07**, Proposition 2] and Macdonald's formula for the spherical functions on G', we observed in [**LO07**] that the identity (7.13) also holds when $E = F \oplus F$.

Assume now that E/F is ramified. In [**Off07**, Lemma 5], we show that there is a constant $c \neq 0$ such that

$$\frac{\mathrm{vol}(H_e^e \cap K)}{\mathrm{vol}(H^e \cap K)} J^{st,x}(\varphi_{\mathbf{1}_T}, \nu, \lambda) = c\ \mathcal{L}(x^{-1}; \nu; \lambda).$$

We also obtain in [**Off07**, Corollary 2] the asymptotic formula

$$\frac{\mathrm{vol}(H_e^e \cap K)}{\mathrm{vol}(H^e \cap K)} \lim_{\lambda \to \infty} J^{st,e}(\varphi_{\mathbf{1}_T}, \nu, \lambda) = 2^{n-1} \mathbf{1}_{\{\nu_0, \omega\nu_0\}}(\nu)$$

where the limit as $\lambda \to \infty$ means that $\lambda_i - \lambda_{i+1} \to \infty$ for $i = 1, \dots, n-1$. Combining the two formulas we see, in particular, that $\lim_{\lambda\to\infty} \mathcal{L}(e; \nu_0; \lambda) \neq 0$ and that

$$c = \frac{2^{n-1}}{\lim_{\lambda\to\infty} \mathcal{L}(e; \nu_0; \lambda)}.$$

We therefore obtain in the ramified p-adic case that

$$(7.14)\qquad \frac{\mathrm{vol}(H_e^e \cap K)}{\mathrm{vol}(H^e \cap K)} J^{st,x}(\varphi_{\mathbf{1}_T}, \nu, \lambda) = 2^{n-1} \frac{\mathcal{L}(x^{-1}; \nu; \lambda)}{\lim_{\mu\to\infty} \mathcal{L}(e; \nu_0; \mu)}.$$

Finally, we also need the archimedean case. Assume that $E/F = \mathbb{C}/\mathbb{R}$. Then $H^e = K$ and it is easy to see, as observed in [**LO07**, (7)] that

$$(7.15)\qquad \frac{\mathrm{vol}(H_e^e \cap K)}{\mathrm{vol}(H^e \cap K)} J^{st,\pm e}(\varphi_{\mathbf{1}_T}, \nu, \lambda) = \nu\nu_0(\pm e).$$

8. On the proof of Theorem 1.1

Here we illustrate the proof for Jacquet's characterization of the image of quadratic base change in terms of non vanishing of unitary periods. It is based on the results of §4 and §5.

If π is an irreducible cuspidal representation of $G_{\mathbb{A}}$ which is distinguished by some unitary group, then an argument, essentially due to Harder, Langlands and Rapoport [**HLR86**], shows that it is Galois invariant. Indeed, if v is a split place then $G_v \simeq G'_v \times G'_v$ and accordingly we write $\pi_v = \pi'_{1,v} \otimes \pi'_{2,v}$. Any local unitary subgroup is then of the form $\{(g, \xi^t g^{-1} \xi^{-1} : g \in G'_v\}$ for some $\xi \in G'_v$. Thus if π_v admits a non zero linear functional invariant under a unitary group, then $\pi'_{1,v} \simeq \pi'_{2,v}$, i.e., $\pi_v = \bar{\pi}_v$. If v is now an inert place so that π_v is unramified, then it is also Galois invariant. It follows that π and $\bar{\pi}$ are irreducible, cuspidal automorphic representations of $G_{\mathbb{A}}$ with equivalent local factors at almost all places. By strong multiplicity one $\pi = \bar{\pi}$. Arthur and Clozel then showed that π is a base change from $G'_{\mathbb{A}}$ [**AC89**].

To prove the other direction, Jacquet uses his RTF. If $\pi = \mathrm{bc}(\pi')$ is irreducible and cuspidal, then we also have $\pi = \mathrm{bc}(\pi' \otimes \omega)$ and $\pi' \not\simeq \pi' \otimes \omega$. The contribution of π to the relative trace formula is given by the distribution

$$\tilde{B}_\pi(\Psi) = \sum_{\phi \in \mathrm{ob}(\pi)} \int_{G\backslash G_{\mathbb{A}}} \left[\sum_{x \in X} \Psi({}^t\bar{g}xg)\right] \phi(g)\ dg \overline{\mathcal{W}}(\phi). \tag{8.1}$$

It can also be expressed as

$$\tilde{B}_\pi(\Psi) = \sum_\xi B^\xi_\pi(f^\xi) \tag{8.2}$$

where the sum is over a set $\{\xi\}$ of representatives for the G-orbits in X, Ψ is related to $\{f^\xi\}$ by (7.1) and

$$B^\xi_\pi = \mathfrak{B}_{\pi,\pi}^{\mathcal{P}^{H^\xi}, \mathcal{W}^\psi, (\ ,\ )_{G\backslash G^1_{\mathbb{A}}}}$$

is the generalized Bessel distribution already mentioned in §5. It is therefore clear that if $B_\pi(\Psi) \neq 0$ for some Ψ then π is distinguished by some unitary group. The contribution of π' to the KTF is

$$B_{\pi'} = \mathfrak{B}_{\pi',\pi'}^{\mathcal{W}^{\psi'} \circ \pi'(w_0), \mathcal{W}^{\psi'}, (\ ,\ )_{G'\backslash (G'_{\mathbb{A}})^1}}.$$

Theorem 4.4 together with Theorem 5.4 allow us to apply the linear independence of characters of Lapid-Rogawski [**LR00**, Lemma 4]. For matching functions $\Phi \xleftrightarrow{\psi'} \Psi$, we get

$$\tilde{B}_\pi(\Psi) = B_{\pi'}(\Phi) + B_{\pi' \otimes \omega}(\Phi). \tag{8.3}$$

The Bessel distribution $B_{\pi'}(\Phi)$ is factorizable. For decomposable test functions, it can be expressed as a product $B_{\pi'}(\Phi) = \prod_v B_{\pi_v}(\Phi_v)$ of local Bessel distributions (see §9.1 for more details). Jacquet showed that he can choose $\Phi = \otimes_v \Phi_v$, such that $\Phi_v = \mathbf{1}_{K'_v}$ for almost all v and such that at all places $B_{\pi'_v}(\Phi_v) \neq 0$. Furthermore, since there is a place v_0 inert in E such that $\pi'_{v_0} \not\simeq \pi'_{v_0} \otimes \omega_{v_0}$ the function Φ_{v_0} can be chosen so that $B_{\pi'_{v_0} \otimes \omega_{v_0}}(\Phi_{v_0}) = 0$. For the real places, the existence of such Φ_v, supported on the big Bruhat cell, is guaranteed by [**Jac05**, Lemma 23] and it is easy to see then that there exist a matching Ψ_v. Smooth matching together with the fundamental lemma then shows that there is a matching $\Psi = \otimes \Psi_v$. We then get that $\tilde{B}_\pi(\Psi) = B_{\pi'}(\Phi) \neq 0$ and therefore π is distinguished by some unitary group.

9. Anisotropic unitary periods – On the proof of Theorem 1.2

Here we shall see how the results of §4-§7 are applied in order to obtain our formula (1.1). We first recall the setting.

Fix a CM-extension E/F, an everywhere unramified cuspidal automorphic representation π of $G_{\mathbb{A}}$ which is a base change from $G'_{\mathbb{A}}$ and a cuspidal automorphic representation π' such that $\pi = \mathrm{bc}(\pi')$. Then $\pi' \not\simeq \pi' \otimes \omega$ and we also have $\pi = \mathrm{bc}(\pi' \otimes \omega)$. We denote by ϕ_0 the everywhere unramified, L^2-normalized cusp form in the space of π. Let $\alpha \in X$ be such that $\alpha_v = \pm {}^t\bar{\theta}_v \theta_v$ for some $\theta_v \in G_v$, is either positive or negative definite for every real place v of F and let $\mathbf{H} = \mathbf{H}^\alpha$ be the associated unitary group. For a finite place v of F let $\theta_v = e$ and let $\theta = (\theta_v) \in G_{\mathbb{A}}$. Our goal is to compute $|\mathcal{P}^H(\pi(\theta^{-1})\phi_0)|^2$.

The RTF of Jacquet is taking all unitary periods into consideration. Since we focus on a single unitary group $H_{\mathbb{A}}$, we may simplify the RTF and consider it as a distribution on $G_{\mathbb{A}}$ as follows. We choose a set of representatives $\{\xi\}$ containing α for the G-orbits in X. For every $\xi \neq \alpha$ we set $f^{\xi} = 0$ and we let $f = f^{\alpha} \in C_c^{\infty}(G_{\mathbb{A}})$. If Ψ is now associated to $\{f^{\xi}\}$ by (7.1) and we express the RTF as a sum over the G-orbits as in (5.1), then we obtain

$$RTF(\Psi) = RTF_{\alpha}(f).$$

For a function $f' \in C_c^{\infty}(G'_{\mathbb{A}})$, we shall also say that f and f' have matching orbital integrals for ψ' and write

$$f' \xleftrightarrow{\psi'} f \quad \text{whenever} \quad f' \xleftrightarrow{\psi'} \Psi.$$

Note that this matching depends on the Haar measure dh on $H_{\mathbb{A}}$, since the dependence of f on Ψ is inverse proportional to dh. As in §8 we now have

$$B_{\pi}^{\alpha}(f) = B_{\pi'}(f') + B_{\pi'\otimes\omega}(f')$$

whenever $f' \xleftrightarrow{\psi'} f$. This identity is independent of dh since the distribution B_{π}^{α} is proportional to dh and therefore cancels out the dependence of f on dh. If in addition we assume that the support of f' is contained in $\ker(\omega \circ \det)$, then $B_{\pi'}(f') = B_{\pi'\otimes\omega}(f')$ and we obtain the identity of Bessel distributions

$$B_{\pi}^{\alpha}(f) = 2B_{\pi'}(f'). \tag{9.1}$$

Fix $g \in G_{\mathbb{A}}$ such that $W^{\psi}(\phi_0, g) \neq 0$. Let S be a finite set of places of F containing all real places, all even places and all inert ramified places and such that for all $v \notin S$ the character ψ_v has conductor $\mathcal{O}_v$ and $g_v, \alpha_v \in K_v$. Consider a function $f \in C_c^{\infty}(G_{\mathbb{A}})$ of the form

$$f = \prod_{v\in S} f_v \prod_{v\notin S} \mathrm{vol}(H_v \cap K_v)^{-1}\mathbf{1}_{K_v}$$

where $f_v \in \mathcal{H}_{G_v}(K_v)$ for all $v \in S$. Let $f_{\theta}^{g}(x) = f(\theta x g)$, $x \in G_{\mathbb{A}}$. Based on Jacquet's smooth matching at the finite places and the fundamental lemma for the Hecke unit element it is not hard to choose a function $f' \in C_c^{\infty}(G'_{\mathbb{A}})$ of the form

$$f' = \prod_{v\in S} f'_v \prod_{v\notin S} \mathbf{1}_{K'_v}$$

with support contained in $\ker(\omega \circ \det)$ and such that

$$f' \xleftrightarrow{\psi'} f_{\theta}^{g} \tag{9.2}$$

(see [**LO07**, §4] for the choice of f'_v when v is real). Choosing a basis $\mathrm{ob}(\pi)$ containing $\pi(g)\phi_0$ and taking the K-invariance of f into consideration, we get that

$$B_{\pi}^{\alpha}(f_{\theta}^{g}) = \mathrm{vol}(H^S \cap K^S)^{-1}\hat{f}_S(\pi_S)\mathcal{P}^H(\pi(\theta^{-1})\phi_0)\overline{\mathcal{W}}^{\psi}(\pi(g)\phi_0) \tag{9.3}$$

where

$$\hat{f}_S(\pi_S) = \prod_{v\in S} \hat{f}_v(\pi_v)$$

is the spherical Fourier transform of f. Applying (9.1) to the pair of matching functions (9.2), we obtain from (9.3) that

$$|\mathcal{P}^H(\pi(\theta^{-1})\phi_0)|^2 = 4\,\mathrm{vol}(H^S \cap K^S)^2|\hat{f}_S(\pi_S)W^{\psi}(\phi_0, g)|^{-2}|B_{\pi'}(f')|^2. \tag{9.4}$$

9.1. Application of a formula of Jacquet for the inner product of cusp forms. Jacquet showed in [**Jac01**] that the Bessel distribution $B_{\pi'}(f')$ is decomposable. This observation is based on a formula he obtained for the inner product of cusp forms on $G'_{\mathbb{A}}$ that, in turn, is based on a Rankin-Selberg integral that expresses the inner product in terms of Whittaker functions [**JS81**]. We recall the formula here and refer to [**LO07**, §2.2] for details.

In the local setting, for an irreducible, generic, unitary representation π' of G' let $\mathcal{W}^{\psi'}(\pi')$ be the Whittaker model of π'. An inner product on $\mathcal{W}^{\psi'}(\pi')$ was given by Bernstein in the non-archimedean case [**Ber84**] and by Baruch in the archimedean case [**Bar03**] by the formula

$$[W_1, W_2] = \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^\times}) \int\limits_{U'_{n-1}\backslash G'_{n-1}} W_1(\mathrm{diag}(g,1))\overline{W_2(\mathrm{diag}(g,1))}\, dg. \tag{9.5}$$

The normalization factor outside the integral appears merely for our convenience. We define the local Bessel distribution

$$B_{\pi'}^{\psi'}(f') = \mathfrak{B}^{\delta_{w_0},\delta_e,[\cdot,\cdot]}_{\mathcal{W}^{\psi'}(\pi'),\mathcal{W}^{\psi'}(\pi')}(f')$$

where $\delta_g(W) = W(g)$.

Globally, for a cusp form ϕ in the space of $\pi' = \otimes_v \pi'_v$ which is a pure tensor, we may write $W^{\psi'}(\phi, g) = \prod_v W_v(g_v)$ with $W_v \in \mathcal{W}^{\psi'_v}(\pi_v)$ and $W_v(e) = 1$ almost everywhere. Let S be a finite set of places containing the archimedean places, so that for $v \notin S$, π_v is unramified, ψ'_v has conductor $\mathcal{O}_v$, W_v is spherical and $W_v(e) = 1$. Then

$$(\phi,\phi)_{G'\backslash (G'_{\mathbb{A}})^1} = \mathrm{Res}_{s=1} L^S(s, \pi' \times \tilde{\pi}') \prod_{v\in S} [W_v, W_v].$$

We get that

$$|W^{\psi'}(g,\phi)|^2 = \frac{(\phi,\phi)_{G'\backslash (G'_{\mathbb{A}})^1}}{\mathrm{Res}_{s=1} L^S(s,\pi'\times\tilde{\pi}')} \prod_{v\in S} |W^{\psi'}_{1,v}(g_v)|^2 \tag{9.6}$$

where $W^{\psi'}_{1,v}$ is a spherical Whittaker coefficient of π' normalized so that $[W^{\psi'}_{1,v}, W^{\psi'}_{1,v}] = 1$. The inner product formula also gives rise to the decomposition of the Bessel distribution. Let $f' = \otimes_{v\in S} f'_v \otimes_{v\notin S} \mathbf{1}_{K'_v}$, then

$$B^{\psi'}_{\pi'}(f') = \frac{1}{\mathrm{Res}_{s=1} L^S(s,\pi'\times\tilde{\pi}')} \prod_{v\in S} B^{\psi'_v}_{\pi'_v}(f'_v). \tag{9.7}$$

In order to evaluate $B^{\psi'}_{\pi'}(f')$ in the setting of (9.4), we shall apply the local Bessel identities. Note that locally, if $\pi' = I'(\nu, \lambda)$ is a principal series representation then the distributions $B^{\psi'}_{\pi'}$ and $B^{\psi'}(\cdot, \nu, \lambda)$ are normalized differently. The normalizing factor was computed in [**LO07**, Proposition 1] and the paragraph that follows the proof. If $\lambda \in \mathbb{C}^n$ is such that $|\mathrm{Re}\,\lambda_i| < \frac{1}{2}$ (which is the case, in particular, when π' is unitary), then the integral in (9.5) converges and defines a G'-invariant pairing between $\mathcal{W}^{\psi'}(\pi')$ and $\mathcal{W}^{\psi'}((\pi')^*)$ where $(\pi')^*$ denotes the conjugate contragredient of π' and can be identified with $I(\nu, -\bar{\lambda})$. We showed that for such λ and for φ'_1, $\varphi'_2 \in I'(\nu, \lambda)$, we have

$$(\varphi'_1, \varphi'_2) = \frac{[W^{\psi}(\varphi'_1, \lambda), W^{\psi}(\varphi'_2, -\bar{\lambda})]}{L(1, \mathbf{1}_{F^\times})^n}. \tag{9.8}$$

This implies that

$$B^{\psi'}(f',\nu,\lambda) = L(1,\mathbf{1}_{F^\times})^n \cdot \mathfrak{B}^{\delta_{w_0},\delta_e,[\cdot,\cdot]}_{\mathcal{W}^{\psi'}(\pi'),\mathcal{W}^{\psi'}((\pi')^*)}(f').$$

In particular, if $I'(\nu,\lambda)$ is unitary, then

$$B^{\psi'}(f',\nu,\lambda) = L(1,\mathbf{1}_{F^\times})^n B^{\psi}_{I'(\nu,\lambda)}(f'). \tag{9.9}$$

We also note that for the normalized unramified section φ'_ν, (9.8) gives

$$|W^{\psi'}(\varphi'_\nu,-\bar{\lambda},g)W^{\psi'}(\varphi'_\nu,\lambda,g)| = L(1,\mathbf{1}_{F^\times})^n |W^{\psi'}_1(g)|^2. \tag{9.10}$$

9.2. Application of the local Bessel identities for principal series representations. Here E/F is a quadratic extension of local fields. Let $\pi = I(\lambda)$ be a unitary, unramified principal series representation of G and let $\nu \in \mathcal{B}(\mathbf{1}_T)$. Thus, $\pi' = I'(\nu,\lambda)$ is such that $\pi = \mathrm{bc}(\pi')$. Once again, we focus on a single unitary group and we therefore consider the relative Bessel distribution on G defined by

$$B^{st,\alpha}(f,\nu,\lambda) = \mathfrak{B}^{J^{st,\alpha}(\cdot,\nu,\lambda),\mathcal{W}(\cdot,-\bar{\lambda}),(\cdot,\cdot)}_{\pi,\pi^*}.$$

Note that if Ψ is supported on the G-orbit of α and $\Psi({}^t g\alpha g) = \int_H f(hg)\,dh$, then

$$\tilde{B}^{st}(\Psi,\nu,\lambda) = B^{st,\alpha}(f,\nu,\lambda).$$

For θ, $g \in G$ and a Hecke function $f \in \mathcal{H}_K(G)$ let $f^g_\theta(y) = f(\theta y g)$, $y \in G$ and let $f' \in C^\infty_c(G')$ be such that $f' \overset{\psi'}{\longleftrightarrow} f^g_\theta$. Combining (9.9) with Theorem 7.1, we get that

$$B_{I'(\nu,\lambda)}(f') = (L(1,\mathbf{1}_{F^\times})^n \kappa_{E/F}\gamma(\nu,\lambda,\psi'))^{-1} B^{st,\alpha}(f^g_\theta,\nu,\lambda).$$

Since f is bi-K-invariant, choosing an orthonormal basis containing $\pi(g)\varphi_{\mathbf{1}_T}$, we see that

$$B^{st,\alpha}(f^g_\theta,\nu,\lambda) = \hat{f}(\lambda) J^{st,\alpha}(I(\theta^{-1},\lambda)\varphi_{\mathbf{1}_T},\nu,\lambda)\overline{W}^{\psi}(g,\varphi_{\mathbf{1}_T},-\bar{\lambda})$$

and therefore that

$$B_{I'(\nu,\lambda)}(f')$$
$$= (\kappa_{E/F} L(1,\mathbf{1}_{F^\times})^n \gamma(\nu,\lambda,\psi'))^{-1} \hat{f}(\lambda) J^{st,\alpha}(I(\theta^{-1},\lambda)\varphi_{\mathbf{1}_T},\nu,\lambda)\overline{W}^{\psi}(g,\varphi_{\mathbf{1}_T},-\bar{\lambda}).$$

Since $I'(\nu,\lambda)$ is unitary it is isomorphic to $I'(\nu,-\bar{\lambda})$. Therefore $\hat{f}(\lambda) = \hat{f}(-\bar{\lambda})$ and also $B_{I'(\nu,\lambda)}(f')$ is invariant under $\lambda \mapsto -\bar{\lambda}$. We therefore also have

$$\overline{B_{I'(\nu,\lambda)}(f')}$$
$$= \kappa_{E/F}(L(1,\mathbf{1}_{F^\times})^n \gamma(\nu,-\lambda,\bar{\psi}'))^{-1} \overline{\hat{f}(\lambda)} J^{st,\alpha}(I(\theta^{-1},\lambda)\varphi_{\mathbf{1}_T},\nu,-\lambda) W^{\psi}(g,\varphi_{\mathbf{1}_T},\lambda).$$

Observe that since

$$\varepsilon(s,\omega,\psi')\varepsilon(-s,\omega,\overline{\psi'}) = \left(\frac{\mathfrak{d}_F}{\mathfrak{d}_E}\right)^2,$$

we also have

$$\gamma(\nu,\lambda,\psi')\gamma(\nu,-\lambda,\bar{\psi}') = \left(\frac{\mathfrak{d}_F}{\mathfrak{d}_E}\right)^{(n-1)n} \prod_{i\neq j} \frac{L(\lambda_i-\lambda_j,\omega\nu_i\nu_j)}{L(\lambda_i-\lambda_j+1,\omega\nu_i\nu_j)}$$

and therefore

$$\frac{L(1,\mathbf{1}_{E^\times})^n}{L(1,\mathbf{1}_{F^\times})^{2n}} \frac{1}{\gamma(\nu,\lambda,\psi')\gamma(\nu,-\lambda,\bar{\psi}')} = \frac{L(0,\omega)^n}{L(1,\mathbf{1}_{F^\times})^n} \frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{L(0,\pi'\times\tilde{\pi}'\otimes\omega)}.$$

Applying (9.10) to the group G, we get that

$$|B_{I'(\nu,\lambda)}(f')|^2 = \frac{L(0,\omega)^n}{L(1,\mathbf{1}_{F^\times})^n} \frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{L(0,\pi'\times\tilde{\pi}'\otimes\omega)} \tag{9.11}$$

$$|\hat{f}(\lambda)|^2 \left(\frac{\mathfrak{d}_F}{\mathfrak{d}_E}\right)^{(n-1)n} J^{st,\alpha}(I(\theta^{-1},\lambda)\varphi_{\mathbf{1}_T},\nu,\lambda) J^{st,\alpha}(I(\theta^{-1},\lambda)\varphi_{\mathbf{1}_T},\nu,-\lambda)|W_1^\psi(g)|^2.$$

9.3. Proof of Theorem 1.2. Since π is everywhere unramified, each component of $\pi' = \otimes_v \pi'_v$ is a unitary, principal series representation $\pi'_v = I'(\nu_v,\lambda_v)$ with $\nu_v \in \mathcal{B}(\mathbf{1}_{T_v})$ and $\lambda_v \in \mathbb{C}^n$. In order to obtain the formula for $|\mathcal{P}^H(\pi(\theta^{-1})\phi_0)|^2$ we now only need to collect together the results we have already recalled. We apply (9.6) to the group G, thus

$$|W^\psi(g,\phi_0)|^2 = \frac{1}{\mathrm{Res}_{s=1} L^S(s,\pi\times\tilde{\pi})} \prod_{v\in S} |W^\psi_{1,v}(g_v)|^2.$$

We now apply the factorization (9.7) of the Bessel distribution and (9.11) in order to obtain an expression for $|B_{\pi'}(f')|^2$. Plugging all this into (9.4) and taking into consideration the fact that

$$\mathrm{Res}_{s=1} L(s,\pi\times\tilde{\pi}) = \mathrm{Res}_{s=1} L(s,\pi'\times\tilde{\pi}') L(1,\pi'\times\tilde{\pi}'\otimes\omega),$$

we get that

$$|\mathcal{P}^H(\pi(\theta^{-1})\phi_0)|^2 = 4\,\mathrm{vol}(H^e_{\mathbb{A}}\cap K)^2 \left|\frac{\Delta_E}{\Delta_F}\right|^{\frac{n(n+1)}{2}} \frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{\mathrm{Res}_{s=1} L(s,\pi'\times\tilde{\pi}')} \prod_{v\in S} \mathcal{P}^*_{\alpha_v}(\pi'_v)$$

where

$$\mathcal{P}^*_{\alpha_v}(\pi'_v) = \mathrm{vol}(H^e_v\cap K_v)^{-2} \frac{L(1,\pi'_v\times\tilde{\pi}'_v)}{L(1,\mathbf{1}_{F_v^\times})} \frac{L(0,\omega_v)}{L(0,\pi'_v\times\tilde{\pi}'_v\otimes\omega_v)} \left(\frac{\mathfrak{d}_{E_v}}{\mathfrak{d}_{F_v}}\right)^{2n}$$
$$\times J^{st,\alpha_v}(I(\theta_v^{-1},\lambda_v)\varphi_{\mathbf{1}_{T_v}},\nu_v,\lambda_v) J^{st,\alpha_v}(I(\theta_v^{-1},\lambda_v)\varphi_{\mathbf{1}_{T_v}},\nu_v,-\lambda_v).$$

We now apply the formulas of §7.1 and express the left hand side explicitly in terms of Hironaka's spherical functions of §6.

Let S_∞ be the set of real places of F, S_r the set of inert places of F ramified in E and $S_u = S\setminus(S_r\cup S_\infty)$. Note first that if v is real, then it follows from (7.5) that

$$J^{st,\alpha}(I(\theta_v^{-1},\lambda)\varphi_{\mathbf{1}_{T_v}},\nu_v,\lambda_v) = J^{st,\pm e}(\varphi_{\mathbf{1}_{T_v}},\nu_v,\lambda_v)$$

where the sign in front of e is positive (resp. negative) if α_v is positive (resp. negative) definite. Recall that if v is a finite place then $\theta_v = e$. If v is unramified (split or inert), let $m_v = m_v(\alpha) \in \Lambda_n$ be such that α is in the K_v-orbit of ϖ^{m_v}. We also have

$$\mathrm{vol}((H^e_e)_v\cap K_v) = \begin{cases} \left(\frac{\mathfrak{d}_{E_v}}{\mathfrak{d}_{F_v}}\right)^n & v \text{ is unramified (split or inert)} \\ \left(2\frac{\mathfrak{d}_{E_v}}{\mathfrak{d}_{F_v}}\right)^n & v \text{ is real or ramified in } E. \end{cases}$$

Let us now define

$$\mathcal{P}_{\alpha_v}(\pi'_v) = \begin{cases} \mathcal{P}^*_{\alpha_v}(\pi'_v) & v\in S_u \\ 4\mathcal{P}^*_{\alpha_v}(\pi'_v) & v\in S_r \\ 4^n\mathcal{P}^*_{\alpha_v}(\pi'_v) & v\in S_\infty. \end{cases}$$

If $v \in S_u$, we now obtain from (7.13) that

$$\mathcal{P}_{\alpha_v}(\pi'_v) = (q_v^{-\langle 2\rho, m_v(\alpha)\rangle} \frac{V_{m_v(\alpha)}(\omega_v(\varpi_v)q_v^{-1})}{v_n(\omega_v(\varpi_v)q_v^{-1})})^2 \times P_{m_v(\alpha)}(Z_v; \omega_v(\varpi_v)q_v^{-1}) P_{\widetilde{m}_v(\alpha)}(Z_v; \omega_v(\varpi_v)q_v^{-1}) \tag{9.12}$$

where $Z_v = Z(\nu_v; \lambda_v)$ is given by (6.10), and $P_m(Z; t)$ is the Hall-Littlewood polynomial defined in (6.11). If $v \in S_r$, we obtain from (7.14) that

$$\mathcal{P}_{\alpha_v}(\pi'_v) = \frac{L(1, \pi'_v \times \tilde{\pi}'_v)}{L(1, \mathbf{1}_{F_v^\times})} \frac{L(0, \omega_v)}{L(0, \pi'_v \times \tilde{\pi}'_v \otimes \omega_v)} \times \Big(\lim_{\mu \to \infty} \mathcal{L}(e; \nu_0; \mu)\Big)^{-2} \mathcal{L}(\alpha^{-1}; \nu; \lambda) \mathcal{L}(\alpha^{-1}; \nu; -\lambda). \tag{9.13}$$

Finally, if $v \in S_\infty$ we obtain from (7.15) that

$$\mathcal{P}_{\alpha_v}(\pi'_v) = \frac{L(1, \pi'_v \times \tilde{\pi}'_v)}{L(1, \mathbf{1}_{F_v^\times})} \frac{L(0, \omega_v)}{L(0, \pi'_v \times \tilde{\pi}'_v \otimes \omega_v)}. \tag{9.14}$$

Note further that in the real case if $\nu_v = \mathbf{1}_{T'_v}$, then

$$\frac{L(1, \pi'_v \times \tilde{\pi}'_v)}{L(1, \mathbf{1}_{F_v^\times})} \frac{L(0, \omega_v)}{L(0, \pi'_v \times \tilde{\pi}'_v \otimes \omega_v)} = 1$$

and therefore if $v \in S_\infty$ and π'_v is unramified, we have $\mathcal{P}_{\alpha_v}(\pi'_v) = 1$. This completes the proof of Theorem 1.2.

10. A generalization of Jacquet's explicit Kloosterman identities

The main result of this section is in the context of an unramified quadratic extension of p-adic fields. We obtain explicit matching for a more general space of functions then in Theorem 4.4 and in (4.14). The proof relies on the fundamental lemma of Jacquet, the spherical Fourier transform of Hironaka and the local Bessel identities of §7 and the method is global using a simple relative trace formula.

Recall that the root of unity $\kappa_{E/F}(\psi', n)$ is defined by Theorem 7.1 and that if F has odd residual characteristic, E/F is unramified and ψ' has conductor $\mathcal{O}_F$, then $\kappa_{E/F}(\psi', n) = 1$.

Theorem 10.1. *Let E/F be an unramified quadratic extension of p-adic fields and assume that ψ' has conductor $\mathcal{O}_F$. For $\Phi \in \mathcal{H}_{G'}(K')$ and $\Psi \in \mathcal{H}_X(K)$ such that $\hat{\Phi} = \hat{\Psi}$, we also have*

$$\Phi \xleftrightarrow{\psi'} \kappa_{E/F}(\psi', n)\Psi.$$

In particular, if F has odd residual characteristic, then $\Phi \xleftrightarrow{\psi'} \Psi$.

Remark 10.2. Note that there is no restriction on the characteristic of the residual field of F. In particular, in the case of even residual characteristic, this proves the fundamental lemma up to the undetermined root of unity $\kappa_{E/F}$.

Remark 10.3. Hironaka's spherical functions are parameterized by the same parameter (ν, λ) as the unramified principle series representations of G'. As explained in §3.2, in the unramified local case, the Hecke algebra $\mathcal{H}_{G'}(K')$ is realized as a free $\mathcal{H}_G(K)$-module of rank 2^n via the spherical Fourier transform. Similarly, the spherical Fourier transform of Hironaka realizes $\mathcal{H}_X(K)$ as a free $\mathcal{H}_G(K)$-module of rank 2^n. The respective spherical Fourier transforms naturally identify the two $\mathcal{H}_G(K)$-modules. Theorem 10.1 asserts that functions that correspond under this identification have matching orbital integrals. Note that Theorem 4.4 is the special case where Φ (resp. Ψ) lies in the rank one submodule $\mathcal{H}_G(K) * \Phi^{(0)}$ (resp. $\mathcal{H}_G(K) * \Psi^{(0)}$). The more general matching (4.14) proves Theorem 10.1 for certain other $\mathcal{H}_G(K)$-orbits, but not for all orbits (even when $n = 2$). In [**Off06**], we verified Theorem 10.1 for $n = 2$ by straightforward computation and conjectured that it is true for all n.

Proof. In order to prove the local matching, we embed the setting into a global setting and apply the RTF identity. It will therefore be convenient to denote by E^0/F^0 our fixed, unramified quadratic extension of p-adic fields and let E/F be a quadratic extension of number fields which is split at all infinite places and such that there is a place $v_{\rm rel}$ of F such that $E_{v_{\rm rel}}/F_{v_{\rm rel}} \simeq E^0/F^0$. This is the relevant place where we wish to prove the explicit matching. We fix a character ψ' of $F\backslash\mathbb{A}$ such that $\psi'_{v_{\rm rel}}$ has conductor $\mathcal{O}_{v_{\rm rel}}$. Globally, we shall only consider decomposable test functions $\Phi = \otimes_v \Phi_v \in C_c^\infty(G'_{\mathbb{A}})$ and $\Psi = \otimes_v \Psi_v \in C_c^\infty(X_{\mathbb{A}})$. Recall that the geometric expansion of the RTF is the sum of orbital integrals

$$RTF(\Psi) = \sum_{M'} \sum_{a \in T'_{M'}} \Omega[\Psi, \psi, E/F : w_{M'}a].$$

Similarly, the geometric expansion of the Kuznetzov trace formula is the sum of orbital integrals

$$KTF(\Phi) = \sum_{M'} \sum_{a \in T'_{M'}} \Omega[\Phi, \psi : w_{M'}a].$$

The global orbital integrals are given as the product over all places of F of their local counterparts.

Let $\Phi^\circ_{v_{\rm rel}} \in \mathcal{H}_{G'_{v_{\rm rel}}}(K'_{v_{\rm rel}})$ and $\Psi^\circ_{v_{\rm rel}} \in \mathcal{H}_{X_{v_{\rm rel}}}(K_{v_{\rm rel}})$ be such that $\hat{\Phi}^\circ_{v_{\rm rel}} = \hat{\Psi}^\circ_{v_{\rm rel}}$ and let $\alpha \in T'_{v_{\rm rel}}$. Our goal is to prove the identity

$$\Omega[\Phi^\circ_{v_{\rm rel}}, \psi_{v_{\rm rel}} : \alpha] = \kappa_{E_{v_{\rm rel}}/F_{v_{\rm rel}}}\ \gamma(\alpha)\ \Omega[\Psi^\circ_{v_{\rm rel}}, \psi_{v_{\rm rel}}, E_{v_{\rm rel}}/F_{v_{\rm rel}} : \alpha]. \tag{10.1}$$

For every place v of F, let $\mathfrak{D}_v \subset G'_v$ be the kernel of the character $\omega_v \circ \det$ on G'_v. By linearity, it is enough to prove (10.1) for $\Phi^\circ_{v_{\rm rel}}$ with support contained either in $\mathfrak{D}_{v_{\rm rel}}$ or in $G'_{v_{\rm rel}} \setminus \mathfrak{D}_{v_{\rm rel}}$. Let $D \in \{\mathfrak{D}_{v_{\rm rel}}, G'_{v_{\rm rel}} \setminus \mathfrak{D}_{v_{\rm rel}}\}$ and assume from now on that the support of $\Phi^\circ_{v_{\rm rel}}$ is contained in D. Let ϵ be such that $\omega_{v_{\rm rel}}(\det(D)) = \{\epsilon\}$. We shall use a relative version of the simple trace formula of Deligne-Kazhdan. For this purpose, we fix a finite place $v_{\rm cusp}$ of F which is split in E where the test function we choose is cuspidal. We will also fix an auxiliary place $v_{\rm aux}$ of F where the support of the test functions we shall choose will be restricted to suit our application, but let us defer this choice to later on in the proof.

Locally, a function $\Phi \in C_c^\infty(G')$ is called cuspidal if for any g_1, $g_2 \in G'$ and V the unipotent radical of a parabolic subgroup of G', we have

$$\int_V \Phi(g_1 v g_2)\ dv = 0.$$

Cuspidal functions form a (two-sided) ideal in $C_c^\infty(G')$. Let $\pi_{v_{\rm cusp}}$ be a supercuspidal representation of $G'_{v_{\rm cusp}}$. Then the operator $\pi_{v_{\rm cusp}}(\Phi)$ for $\Phi \in C_c^\infty(G'_{v_{\rm cusp}})$ depends only on the projection to the ideal of cuspidal test functions. Since a supercuspidal representation has a Whittaker model, the Bessel distribution $\mathfrak{B}^{\mathcal{W},\mathcal{W}}_{\pi_{v_{\rm cusp}}}$

defined with respect to the Whittaker functional $\mathcal{W}$ on $\pi_{v_{\rm cusp}}$, is not identically zero. It follows that there exists a cuspidal function $\Phi_{v_{\rm cusp}} \in C_c^\infty(G'_{v_{\rm cusp}})$ such that $\mathfrak{B}^{\mathcal{W},\mathcal{W}}_{\pi_{v_{\rm cusp}}}(\Phi_{v_{\rm cusp}}) \neq 0$. This, in turn, implies that there exists $\beta \in T'_{v_{\rm cusp}}$ such that $\Omega[\Phi_{v_{\rm cusp}}, \psi' : \beta] \neq 0$. Indeed, if $\Omega[\Phi_{v_{\rm cusp}}, \psi' : a] = 0$ for all $a \in T'_{v_{\rm cusp}}$, then it follows from Theorem 4.1 that $\Omega[\Phi_{v_{\rm cusp}}, \psi' : g] = 0$ for every relevant $g \in G'_{v_{\rm cusp}}$ and therefore by the localization principal of Gelfand-Kazhdan [**GK75**] that $\mathfrak{B}^{\mathcal{W},\mathcal{W}}_{\pi_{v_{\rm cusp}}}(\Phi_{v_{\rm cusp}})$ is identically zero.

Let $\Psi_{v_{\rm cusp}}$ be a cuspidal function such that $\Phi_{v_{\rm cusp}} \longleftrightarrow \Psi_{v_{\rm cusp}}$. We then have

$$\Omega[\Psi_{v_{\rm cusp}}, \psi_{v_{\rm cusp}}, E_{v_{\rm cusp}}/F_{v_{\rm cusp}} : \beta] = \Omega[\Phi_{v_{\rm cusp}}, \psi'_{v_{\rm cusp}} : \beta] \neq 0.$$

Since the local orbital integrals are locally constant in $a \in T'_v$ and since the diagonal embedding of T' into $T'_{v_{\rm rel}} \times T'_{v_{\rm cusp}}$ is dense, there exists $a \in T'$ close enough to (α, β) so that

$$\Omega[\Phi^\circ_{v_{\rm rel}}, \psi'_{v_{\rm rel}}; \alpha] = \Omega[\Phi^\circ_{v_{\rm rel}}, \psi'_{v_{\rm rel}} : a],$$
$$\Omega[\Psi^\circ_{v_{\rm rel}}, \psi_{v_{\rm rel}}, E_{v_{\rm rel}}/F_{v_{\rm rel}} : \alpha] = \Omega[\Psi^\circ_{v_{\rm rel}}, \psi_{v_{\rm rel}}, E_{v_{\rm rel}}/F_{v_{\rm rel}} : a]$$

and

$$\Omega[\Phi_{v_{\rm cusp}}, \psi'_{v_{\rm cusp}} : \beta] = \Omega[\Phi_{v_{\rm cusp}}, \psi'_{v_{\rm cusp}}, E_{v_{\rm cusp}}/F_{v_{\rm cusp}} : a].$$

In particular, it is enough to prove (10.1) with α replaced by a.

Let S be a finite set of places of F containing all infinite places all even places and the places $v_{\rm rel}$, $v_{\rm cusp}$ so that $a \in K_v$ and ψ'_v has conductor $\mathcal{O}_v$ for all $v \notin S$. For every $v \in S \setminus \{v_{\rm cusp}\}$, fix $\Phi_v \longleftrightarrow \Psi_v$ such that $\Omega[\Phi_v, \psi'_v; a] \neq 0$. For every $v \in S$, let C_v be a compact subset of G'_v such that the support of Φ_v is contained in C_v and such that the support of $\Phi^\circ_{v_{\rm rel}}$ is also contained in $C_{v_{\rm rel}}$ and let $C = \prod_{v\in S} C_v \times \prod_{v\notin S} K_v$. We now choose an auxiliary place $v_{\rm aux} \notin S$. For $v \notin S \cup \{v_{\rm aux}\}$ let $\Phi_v = \mathbf{1}_{K'_v}$ and $\Psi_v = \mathbf{1}_{K_v \cap X_v}$. Thus $\Phi_v \longleftrightarrow \Psi_v$ and $\Omega[\Phi_v, \psi'_v : a] \neq 0$. We also let $\Phi_{v_{\rm aux}} = \mathbf{1}_{G'_{v_{\rm aux}}[\ell]} \cdot \mathbf{1}_{K'_{v_{\rm aux}}}$ and $\Psi_v = \mathbf{1}_{X_{v_{\rm aux}}[\ell]} \cdot \mathbf{1}_{K_{v_{\rm aux}} \cap X_{v_{\rm aux}}}$ where

$$G'_v[\ell] = \{g \in G'_v : d_i(g) \in d_i(a) + \mathfrak{p}_v^\ell,\ i = 1 \dots, n\}$$

and

$$X'_v[\ell] = \{x \in X_v : d_i(x) \in d_i(a) + \mathfrak{p}_v^\ell,\ i = 1 \dots, n\}.$$

Since d_i is invariant under the actions of U_v and of $U'_v \times U'_v$ on T'_v we also have, independent of the positive integer ℓ, that $\Phi_{v_{\rm aux}} \longleftrightarrow \Psi_{v_{\rm aux}}$ and $\Omega[\Phi_{v_{\rm aux}}, \psi'_{v_{\rm aux}} : a] \neq 0$. Let $\Phi = \otimes_v \Phi_v$. We claim that we may choose ℓ large enough so that for every relevant representative $w_{M'}b$, $b \in T'_{M'}$ such that $w_{M'}b \neq a$, we have

$$\Omega[\Phi, \psi' : w_{M'}b] = 0. \tag{10.2}$$

Indeed, first depending only on the valuations at $v_{\rm aux}$ of the diagonal entries of a, we may take ℓ large enough so that for all $i = 1, \dots, n$, we have $0 \notin d_i(a) + \mathfrak{p}^\ell_{v_{\rm aux}}$. Thus if $M' \neq T'$, i.e., if $w_{M'}b$ is not diagonal, then (10.2) is satisfied. Note that the set $\{d_i(g) : g \in C\}$ is contained in a set of the form $\prod_{v\in S} c_v \prod_{v\notin S} \mathcal{O}_v \subset \mathbb{A}$ where $c_v \subseteq F_v$ is compact and therefore its intersection with $F^\times$ is finite. Since a diagonal matrix $b \in T'$ is determined by $d_i(b)$, $i = 1, \dots, n$ it follows that $T' \cap C$ is finite. We now choose ℓ so large, that if b is in $T' \cap C$ and also in $G'_{v_{\rm aux}}[\ell]$, then $b = a$. We then have $\Phi \longleftrightarrow \Psi = \otimes_v \Psi_v$ and

$$RTF(\Psi) = \Omega[\Psi, \psi, E/F : a] = \Omega[\Phi, \psi' : a] = KTF(\Phi) \neq 0.$$

Furthermore, let $\Psi^\circ = \Psi^\circ_{v_{\mathrm{rel}}} \otimes (\otimes_{v \neq v_{\mathrm{rel}}} \Psi_v)$ and $\Phi^\circ = \Phi^\circ_{v_{\mathrm{rel}}} \otimes (\otimes_{v \neq v_{\mathrm{rel}}} \Phi_v)$, then

$$\Omega[\Psi^\circ, \psi, E/F : w_{M'} b] = \Omega[\Phi^\circ, \psi' : w_{M'} b] = 0$$

for every relevant representative $w_{M'} b \neq a$. We therefore also have

$$RTF(\Psi^\circ) = \Omega[\Psi^\circ, \psi, E/F : a] \text{ and } KTF(\Phi^\circ) = \Omega[\Phi^\circ, \psi' : a]. \tag{10.3}$$

Thus (10.1) and therefore the theorem will follow once we show that

$$\kappa_{E_{v_{\mathrm{rel}}}/F_{v_{\mathrm{rel}}}} \; RTF(\Psi^\circ) = KTF(\Phi^\circ).$$

To show this equality, we turn to the respective spectral expansions. Recall that both $\Phi_{v_{\mathrm{cusp}}}$ and $\Psi_{v_{\mathrm{cusp}}}$ are cuspidal. We then have

$$RTF(\Psi^\circ) = \sum_\pi \tilde{B}_\pi(\Psi^\circ) \text{ and } KTF(\Phi^\circ) = \sum_{\pi'} B_{\pi'}(\Phi^\circ)$$

where each sum is only over cuspidal representations. If $\pi' \simeq \pi' \otimes \omega$, then $\mathrm{bc}(\pi')$ is not cuspidal and therefore if $\Psi' = \otimes_v \Psi'_v$ is such that $\Phi^\circ \longleftrightarrow \Psi'$ and $\Psi'_{v_{\mathrm{cusp}}} = \Psi_{v_{\mathrm{cusp}}}$, then the contribution of $\mathrm{bc}(\pi')$ to $RTF(\Psi')$ is 0. The linear independence of characters implies then that $B_{\pi'}(\Phi^\circ) = 0$. Thus we may assume that $\pi' \not\simeq \pi' \otimes \omega$ and set $\pi = \mathrm{bc}(\pi')$. To complete the proof, we need to show that

$$\kappa_{E_{v_{\mathrm{rel}}}/F_{v_{\mathrm{rel}}}} \; \tilde{B}_\pi(\Psi^\circ) = B_{\pi'}(\Phi^\circ) + B_{\pi' \otimes \omega}(\Phi^\circ). \tag{10.4}$$

Recall that if $\Phi' \overset{\psi'}{\longleftrightarrow} \Psi'$, then as in (8.3) we do have

$$\tilde{B}_\pi(\Psi') = B_{\pi'}(\Phi') + B_{\pi' \otimes \omega}(\Phi'). \tag{10.5}$$

If $\pi'_{v_{\mathrm{rel}}}$ is ramified (i.e., does not have a $K'_{v_{\mathrm{rel}}}$-invariant vector), then it is clear that the right hand side of (10.4) is zero. From (8.1) it is also easy to see that the left hand side vanishes. We may therefore assume that $\pi'_{v_{\mathrm{rel}}} = I'(\lambda_{v_{\mathrm{rel}}})$ and $\pi_{v_{\mathrm{rel}}} = I(\lambda_{v_{\mathrm{rel}}})$ are unramified principal series representations. Recall that factorizing as in (9.7), we may write

$$B_{\pi'}(\Phi) = B_{\pi'_{v_{\mathrm{rel}}}}(\Phi_{v_{\mathrm{rel}}}) B_{(\pi')^{v_{\mathrm{rel}}}}(\Phi^{v_{\mathrm{rel}}}).$$

Essentially, this can be used to factor out the place v_{rel} also for $\tilde{B}_\pi$. Let $\Psi' = \Psi'_{v_{\mathrm{rel}}} \otimes (\Psi')^{v_{\mathrm{rel}}}$ be such that the support of $\Psi'_{v_{\mathrm{rel}}}$ is contained in $\{x \in X : \omega(\det x) = \epsilon\}$. There exists $\Phi'_{v_{\mathrm{rel}}}$ with support contained in D such that $\Phi'_{v_{\mathrm{rel}}} \longleftrightarrow \Psi'_{v_{\mathrm{rel}}}$. Then for $\Phi' = \Phi'_{v_{\mathrm{rel}}} \otimes (\Phi')^{v_{\mathrm{rel}}} \longleftrightarrow \Psi'$ we have

$$\tilde{B}_\pi(\Psi') = B_{\pi'_{v_{\mathrm{rel}}}}(\Phi'_{v_{\mathrm{rel}}})(B_{(\pi')^{v_{\mathrm{rel}}}} + \epsilon B_{(\pi' \otimes \omega)^{v_{\mathrm{rel}}}})((\Phi')^{v_{\mathrm{rel}}}).$$

It now follows from Theorem 7.1 and (9.9) that

$$B_{\pi'_{v_{\mathrm{rel}}}}(\Phi'_{v_{\mathrm{rel}}}) = \tilde{B}_{\pi_{v_{\mathrm{rel}}}}(\Psi'_{v_{\mathrm{rel}}})$$

where we set

$$\tilde{B}_{\pi_{v_{\mathrm{rel}}}}(\Psi'_{v_{\mathrm{rel}}}) := (L(1, \mathbf{1}_{F^\times})^n \kappa_{E_{v_{\mathrm{rel}}}/F_{v_{\mathrm{rel}}}} \gamma(\mathbf{1}_{T'_{v_{\mathrm{rel}}}}, \lambda_{v_{\mathrm{rel}}}, \psi'_{v_{\mathrm{rel}}}))^{-1} \tilde{B}^{st}(\Psi'_{v_{\mathrm{rel}}}, \mathbf{1}_{T'_{v_{\mathrm{rel}}}}, \lambda_{v_{\mathrm{rel}}})$$

and therefore the identity

$$\tilde{B}_\pi(\Psi') = \tilde{B}_{\pi_{v_{\mathrm{rel}}}}(\Psi'_{v_{\mathrm{rel}}})(B_{(\pi')^{v_{\mathrm{rel}}}} + \epsilon B_{(\pi' \otimes \omega)^{v_{\mathrm{rel}}}})((\Phi')^{v_{\mathrm{rel}}})$$

now holds, whenever $(\Phi')^{v_{\rm rel}} \longleftrightarrow (\Psi')^{v_{\rm rel}}$ and in particular, if we replace Ψ' with $\kappa_{E_{v_{\rm rel}}/F_{v_{\rm rel}}}\ \Psi^\circ$ and Φ' with Φ°. Applying Lemma 7.4, we therefore obtain

$$\kappa_{E_{v_{\rm rel}}/F_{v_{\rm rel}}}\ \tilde{B}_\pi(\Psi^\circ) = \kappa_{E_{v_{\rm rel}}/F_{v_{\rm rel}}}\ \tilde{B}_{\pi_{v_{\rm rel}}}(\Psi^\circ_{v_{\rm rel}})(B_{(\pi')^{v_{\rm rel}}} + \epsilon B_{(\pi'\otimes\omega)^{v_{\rm rel}}})((\Phi)^{v_{\rm rel}})$$
$$= B_{\pi'_{v_{\rm rel}}}(\Phi^\circ_{v_{\rm rel}})(B_{(\pi')^{v_{\rm rel}}} + \epsilon B_{(\pi'\otimes\omega)^{v_{\rm rel}}})((\Phi)^{v_{\rm rel}}) = B_{\pi'}(\Phi^\circ) + B_{\pi'\otimes\omega}(\Phi^\circ).$$

Theorem 10.1 now follows. □

REMARK 10.4. If E/F is either ramified p-adic or $\mathbb{C}/\mathbb{R}$, then we do NOT expect a similar explicit matching for the spherical test functions. Let us explain the heuristics behind it. If $\Phi \in \mathcal{H}_{G'}(K')$ and $\Psi \in \mathcal{H}_X(K)$ are such that $\Phi \longleftrightarrow \Psi$, it follows from Theorem 7.1 that

$$\tilde{B}^{st}(\Psi,\nu,\lambda) = \kappa_{E/F}\gamma(\nu,\lambda,\psi')B'(\Phi,\nu,\lambda)$$

for every $\lambda \in \mathbb{C}^n$ and $\nu \in \mathcal{B}(\mathbf{1}_T)$. If $\nu \neq \mathbf{1}_{T'}$, then $I(\nu)$ is ramified and therefore $B'(\Phi,\nu,\lambda) = 0$. On the other hand, as in (7.6)

$$B^{st}(\Psi,\nu,\lambda) = \int_X \Psi(x)J^{st,x}(\varphi_{\mathbf{1}_T},\nu,\lambda)\ dx\ \overline{\mathcal{W}}(\varphi_{\mathbf{1}_T},-\bar{\lambda})$$

where we recall that $\varphi_{\mathbf{1}_T}$ is the normalized spherical section in $I(\mathbf{1}_T)$. We expect that in general $\int_X \Psi(x)J^{st,x}(\varphi_{\mathbf{1}_T},\nu,\lambda)\ dx$ is generically not zero as a function of λ for some $\nu \neq \mathbf{1}_T$.

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