STABLE RELATIVE BESSEL DISTRIBUTIONS ON $GL(n)$ OVER A QUADRATIC EXTENSION

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Abstract. We study regularized periods of Eisenstein series over unitary groups. They play an important role in the study of unitary periods of cusp forms. A stabilization is used to express the periods in terms of L-functions. Thanks to recent developments of Jacquet and Lapid, we obtain certain Bessel identities that generalize identities obtained by Lapid-Rogawski. These identities are applied in "Compact unitary periods" by Lapid and Offen.

1. Introduction. Let $\mathbf{G}$ be a connected reductive algebraic group defined over a number field F and let θ be an involution on $\mathbf{G}$ defined over F. Let

$$\mathbf{Y} = \{y \in \mathbf{G} \mid y\theta(y) = e\}$$

where e denotes the identity. The group $\mathbf{G}$ acts on the symmetric space $\mathbf{Y}$ by $g \cdot y = gy\theta(g)^{-1}$. For every $\xi \in Y = \mathbf{Y}(F)$, let $\mathbf{H}^\xi$ be the stabilizer of ξ in $\mathbf{G}$. Let $\mathbb{A} = \mathbb{A}_F$ be the adèle ring of F and for an algebraic group $\mathbf{Q}$ defined over F let $Q_\mathbb{A} = \mathbf{Q}(\mathbb{A})$. The H^ξ-*period* of a cusp form ϕ on $G_\mathbb{A}$ is defined by the integral

$$\Pi^{H^\xi}(\phi) = \int_{H^\xi\backslash(H^\xi_\mathbb{A}\cap G^1_\mathbb{A})} \phi(h)\,dh,$$

known to converge by [AGR93]. A cuspidal automorphic representation π of $G_\mathbb{A}$ is called H^ξ-*distinguished* if there is an automorphic form ϕ in the space of π so that $\Pi^{H^\xi}(\phi) \neq 0$. For a general automorphic form the period integral may not converge. It is possible, however, to regularize it, as has been carried out in [JLR99] and [LR03] when θ is a Galois involution. The relative trace formula of Jacquet (RTF), is a tool to study distinguished representations. It is expected that to $(\mathbf{G}, \theta)$ there is attached a group $\mathbf{G}'$, so that the automorphic representations of $G_\mathbb{A}$ that are H^ξ-distinguished for some $\xi \in Y$ are precisely those in the image of a functorial transfer from $\mathbf{G}'$ to $\mathbf{G}$ [JLR93]. We now turn to the specific setting of this work. We refer to §2 for unexplained notation and for conventions regarding invariant measures. Let E/F be a quadratic extension of

Manuscript received September 13, 2005; revised January 8, 2006.
American Journal of Mathematics 129 (2007), 1183–1226.

number fields. Let $\mathbf{G}' = GL(n)$ be regarded as an algebraic group defined over F and let $\mathbf{G}$ be the restriction of scalars of GL_n from E to F. The associated groups of F-rational points are then $G' = GL(n,F)$ and $G = GL(n,E)$ respectively. Let θ be the Galois involution defined by

$$\theta(g) = w_n {}^t\bar{g}^{-1} w_n^{-1},$$

where $x \mapsto \bar{x}$ is the nontrivial Galois action on E/F and w_n is the permutation matrix in G with unit anti-diagonal. In this case the stabilizers $\mathbf{H}^\xi$ are the various unitary groups and the functorial transfer from $\mathbf{G}'$ to $\mathbf{G}$ is quadratic base change. It has recently been established by Jacquet, that every representation of $G_{\mathbb{A}}$ that is the base change of a cuspidal representation of $G'_{\mathbb{A}}$ is distinguished by some H^ξ [Jac05]. This is carried out by comparing the RTF for G with a Kuznetzov trace formula (KTF) for G'. The cuspidal contribution to the RTF appears as a sum of relative Bessel distributions attached to distinguished representations of G. It matches term by term with the corresponding sum in the KTF for Bessel distributions attached to cuspidal representations of G'. Thanks to the fine spectral expansion of the relative trace formula obtained by Lapid [Lap06], the contribution of the continuous spectrum to the RTF can also be expressed as an integral of relative Bessel distributions defined through regularized periods of Eisenstein series. The term wise comparison, however, cannot be carried out directly. It is shown in [LR00] for the case $n = 3$, that the comparison can be carried out using a stable version of the relative Bessel distributions. Our goal in this work is to generalize to $GL(n)$ the results of [LR00]. It has been made possible, thanks to the recent developments of Jacquet [Jac04], [Jac05] and Lapid [Lap06].

If we consider a cuspidal automorphic representation of $G_{\mathbb{A}}$ distinguished by some H^ξ then it is globally the base change of (essentially) only one cuspidal automorphic representation. Accordingly, the period integral is factorizable. If now we consider an Eisenstein automorphic representation then it is the base change of several automorphic representations. Accordingly, the regularized period integral is expected to be a finite sum of factorizable linear forms. The regularized period $\Pi^{H^\xi}(\phi)$ of an automorphic form ϕ of $G_{\mathbb{A}}$ is defined in [LR03]. When $\phi = E(\varphi,\lambda)$ is a cuspidal Eisenstein series, $\Pi^{H^\xi}(\phi)$ is computed in [LR03] in terms of the so called *intertwining periods* $J(\eta,\varphi,\lambda)$. We will be interested in Eisenstein series induced from the Borel. Our first result shows the above expectation holds in this case. We now explain the result more explicitly.

Let $\mathbf{T}$ (resp. $\mathbf{T}'$) be the maximal torus in $\mathbf{G}$ (resp. $\mathbf{G}'$) so that the group T (resp. T'), of F-rational points, is the group of diagonal matrices in G (resp. G') and let $\mathbf{B} = \mathbf{TU}$ (resp. $\mathbf{B}' = \mathbf{T}'\mathbf{U}'$) be the Borel subgroup of $\mathbf{G}$ (resp. $\mathbf{G}'$) containing $\mathbf{T}$ (resp. $\mathbf{T}'$) where U (resp. U') consists of the upper triangular unipotent matrices. Let χ be a character of $T\backslash T_{\mathbb{A}}$, let λ be in the complex vector space $\mathfrak{a}^*_{0,\mathbb{C}} \simeq \mathbb{C}^n$ where the roots of G with respect to T live, and let $\varphi\colon G_{\mathbb{A}} \to \mathbb{C}$ be a smooth function such that

$$\varphi(bg) = \delta_B^{\frac{1}{2}} \chi(b)\varphi(g),\ b \in B_{\mathbb{A}},\ g \in G_{\mathbb{A}}. \tag{1}$$

The Eisenstein series

$$E(g, \varphi, \lambda) = \sum_{\gamma \in B \backslash G} \varphi(\gamma g) e^{\langle \lambda, H(\gamma g) \rangle}$$

converges absolutely for $\operatorname{Re} \lambda$ sufficiently positive and admits a meromorphic continuation to $\mathbb{C}^n$. According to a result of Springer [Spr85], every B-orbit in Y has a representative $x \in Y$ that lies in the normalizer $N_G(T)$ of T. Let $[x]$ denote the class of such an x in the Weyl group W of G. We obtain a natural map

$$\iota\colon\ B \backslash Y \to W$$

sending $B \cdot x$ to $[x]$. Fix $\xi \in Y$ and let $\eta \in G$ be such that $\eta \cdot \xi = x \in N_G(T)$. For such η let

$$\mathbf{H}_\eta^\xi = \mathbf{H}^\xi \cap \eta^{-1} \mathbf{B} \eta.$$

The intertwining period attached to η is the integral

$$J^\xi(\eta, \varphi, \lambda) = \int_{(H_\eta^\xi)_{\mathbb{A}} \backslash H_{\mathbb{A}}^\xi} e^{\langle \lambda, H(\eta h) \rangle} \varphi(\eta h)\, d\dot{h}.$$

The result of [LR03] applied to this case gives us that the integral defining $J^\xi(\eta, \varphi, \lambda)$ is convergent for suitable χ and λ and that

$$\Pi^{H^\xi}(E(\varphi, \lambda)) = \sum_{\iota(\eta) = w_n} \operatorname{vol}(H_\eta^\xi \backslash (H_\eta^\xi)_{\mathbb{A}}) J^\xi(\eta, \varphi, \lambda). \tag{2}$$

Denote by $\omega = \omega_{E/F}$ the idèle class character attached to E/F by class field theory. Let Nm: $T \to T'$ be induced from the norm map from E to F. Let χ be a unitary character on $T \backslash T_{\mathbb{A}}$ which is a base change with respect to Nm of a unitary character on $T' \backslash T'_{\mathbb{A}}$. Denote by $\mathcal{B}(\chi)$ the set of 2^n characters ν on $T' \backslash T'_{\mathbb{A}}$ such that $\chi = \nu \circ \mathrm{Nm}$. The stable intertwining period $J^{st,\xi}(\nu, \varphi, \lambda)$ is defined in §4 as the product over all places of F of its local analogue $J^{st,\xi_v}(\nu_v, \varphi_v, \lambda)$ defined in §3. The period $J^{st,\xi}(\nu, \varphi, \lambda)$ is invariant under $H^\xi_{\mathbb{A}_f}$ where $\mathbb{A}_f$ is the ring of finite adèles of F. We can now state our first result. It is the generalization of Theorem 1 in [LR00].

THEOREM 1.

$$\Pi^{H^\xi}(E(\varphi,\lambda)) = 2^{-n}\,\mathrm{vol}\,(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}})\sum_{\nu\in\mathcal{B}(\chi)} J^{st,\xi}(\nu,\varphi,\lambda).$$

In particular, this expresses the left-hand side as a sum of factorizable linear functionals.

Proposition 3 and (13) show that the local factors of the summands at the unramified places are given as ratios of L-functions. The local unramified computation was carried out by Y. Hironaka in [Hir99]. We interpret her computation to suit our setting and hence we obtain a description of $\Pi^{H^\xi}(E(\varphi,\lambda))$ in terms of L-functions. Let $\varphi=\otimes\varphi_v$ be a factorizable section that satisfies (1). Let S be a finite set of places containing the archimedean places and the places where E/F is ramified, so that for $v\notin S$, $\xi_v\in K_v$, χ_v is an unramified character of T_v and φ_v is K_v-invariant and normalized such that $\varphi_v(e)=1$.

COROLLARY 1.

$$\begin{aligned}\Pi^{H^\xi}(E(\varphi,\lambda)) \;=\; & \frac{\mathrm{vol}\,(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}})\,\mathrm{vol}\,((H^{w_n})^S\cap K^S)}{2^n\,\mathrm{vol}\,((H_e^{w_n})^S\cap K^S)}\\ & \times\sum_{\nu\in\mathcal{B}(\chi)}\left(\prod_{v\in S}J^{st,\xi_v}(\nu_v,\varphi_v,\lambda)\right)\\ & \times\prod_{1\le i<j\le n}\frac{L^S(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j)}{L^S(\nu_i\nu_j^{-1},\lambda_i-\lambda_j+1)},\end{aligned}\tag{3}$$

where L^S stands for the associated partial L-function.

When $n=2$ the formula (3) was obtained in some special cases in ([EGM87], (8.13)) (see [CO, LO07] for the interpretation of an anisotropic unitary period as a finite weighted sum of point evaluations over the genus of the hermitian form). The formula of Grunewald-Mennicke-Elstrodt for the special case determines all local terms explicitly and is applied to obtain new proofs of old identities and new identities for representation numbers associated with binary Hermitian and with ternary quadratic forms. In [CO] we apply (3) in the higher rank setting and obtain information on familiar representation numbers as well as on a new type of representation numbers associated with Hermitian forms.

We now describe the identities of Bessel distributions. In general, if (π,V) is a unitary, admissible representation of $G_{\mathbb{A}}$ and L_1, L_2 are continuous linear functionals on V, we may define a distribution on the space of compactly supported $\mathbf{K}$-finite functions f on $G_{\mathbb{A}}$ by the formula

$$\mathcal{B}_{L_1,L_2}(f)=\sum_{\{\phi\}}L_1(\pi(f)\phi)\overline{L_2(\phi)}$$

where $\{\phi\}$ is an orthonormal basis of V consisting of $\mathbf{K}$-finite vectors. The sum is then finite and $\mathcal{B}_{L_1,L_2}(f)$ is independent of the choice of basis. The distribution $\mathcal{B}_{L_1,L_2}$, can then be extended to compactly supported smooth functions on $G_{\mathbb{A}}$ (cf. [JLR04, §4.1]). The distributions occurring in the KTF and in the RTF are all of this type. They are referred to as Bessel distributions and relative Bessel distributions, respectively. We will consider factorizable functions $\Phi = \otimes_v \Phi_v$ (resp. $f' = \otimes_v f'_v$) on $Y_{\mathbb{A}}$ (resp. $G'_{\mathbb{A}}$) smooth and of compact support so that Φ_v is the characteristic function of $K_v \cdot w_n$ (resp. f'_v is the characteristic function of K'_v) for almost all v. The Bessel identities that we wish to formulate are for matching functions Φ and f'. The concept of local matching depends on a transfer factor. At this stage we can only say that the transfer factor is one of two possibilities. Locally, if ψ is an additive character of F and $\delta \in \{0, 1\}$, we say that the functions Φ and f' δ-match for ψ and write $\Phi \overset{\psi,\delta}{\leftrightarrow} f'$ if

$$\int_{U'\times U'} f'(u_1 w_n a u_2)\psi_{U'}(u_1u_2)\, d_{U'}u_1\, d_{U'}u_2 \\ = \omega^{\delta}\nu_{\omega}(a)\int_U \Phi(u^{-1}\cdot(a^{-1}w_n))\psi_U(u)\, d_U u$$

for all $a \in T'$. Here ω is the quadratic character of F attached to E/F by class field theory and

$$\nu_{\omega} = (\omega, \omega^2, \dots, \omega^n). \tag{4}$$

Globally, there is an analogue notion of δ-matching and decomposable functions Φ and f' have δ-matching orbital integrals for an additive character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}$ whenever Φ_v and f'_v δ-match for ψ_v at every place v of F. We often suppress ψ from the notation. From the definition it follows that both in the local case and in the global case we have

$$\Phi \overset{\delta}{\leftrightarrow} f' \text{ if and only if } \Phi \overset{1-\delta}{\leftrightarrow} f'_{\omega} \tag{5}$$

where $f'_{\omega}(g) = \omega(\det g) f'(g)$. Jacquet established the trace formula identity

$$RTF(\Phi) = KTF(f'),$$

whenever $\Phi \overset{\delta}{\leftrightarrow} f'$. As already mentioned, the cuspidal contribution to the identity of trace formulas can be compared term wise. Our goal is to formulate and prove an analogue for the distributions coming from the most continuous part of the spectrum. Let χ be a unitary character of $T\backslash T_{\mathbb{A}}$ and let

$$I(\chi, \lambda) = \mathrm{Ind}_{B_{\mathbb{A}}}^{G_{\mathbb{A}}}(\chi e^{\langle\lambda, H(\cdot)\rangle})$$

be the associated representation on $G_{\mathbb{A}}$ induced from the Borel. Fix a set of representatives $\{\xi\}$ for the set of G-orbits in Y. For a function Φ, we associate a family of smooth functions of compact support $\{f^\xi\}$ on $G_{\mathbb{A}}$ so that

$$\Phi(g^{-1}\cdot\xi) = \int_{H^\xi_{\mathbb{A}}} f^\xi(hg)\,dh.$$

The relative Bessel distribution is defined in terms of the regularized periods as follows

$$\tilde{B}(\Phi,\chi,\lambda) = \sum_\xi\sum_\varphi \Pi^{H^\xi}(E(I(f^\xi,\chi,\lambda)\varphi,\lambda))\overline{\mathcal{W}}(\varphi,-\bar{\lambda})$$

where φ runs through an orthonormal basis of $I(\chi,\lambda)$. For $\nu\in\mathcal{B}(\chi)$, the Bessel distribution is defined by

$$B'(f',\nu,\lambda) = \sum_{\varphi'} \mathcal{W}'(I'(f',\nu,\lambda)\varphi',\lambda)\overline{\mathcal{W}}'(\varphi',-\bar{\lambda}).$$

The Whittaker functionals $\mathcal{W}(\varphi,\lambda)$ and $\mathcal{W}'(\varphi,\lambda)$ are defined in §2. We denote by $\overline{\mathcal{W}}(\varphi,\lambda)$ and $\overline{\mathcal{W}}'(\varphi,\lambda)$ their complex conjugates respectively. Since there is more then one representation whose base change is $I(\chi,\lambda)$ some stabilization of the relative Bessel distributions is required in order to obtain a comparison. For $\nu\in\mathcal{B}(\chi)$ we define

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = \sum_\varphi\left[\int_{Y_{\mathbb{A}}}\Phi(y)J^{st,y}(\nu,\varphi,\lambda)\,dy\right]\overline{\mathcal{W}}(\varphi,-\bar{\lambda}).$$

It is shown in §6 that

$$\tilde{B}(\Phi,\chi,\lambda) = 2^{-n}\operatorname{vol}(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}})\sum_{\nu\in\mathcal{B}(\chi)}\tilde{B}^{st}(\Phi,\nu,\lambda).$$

The next result generalizes to $GL(n)$ Theorem 2 in [LR00]. Furthermore, we do not make any assumptions on the archimedean places.

THEOREM 2. *There exists $\delta=\delta(n)\in\{0,1\}$, depending only on n, such that whenever χ is a unitary character on $T\backslash T_{\mathbb{A}}$, $\nu\in\mathcal{B}(\chi)$ and $\Phi\overset{\psi,\delta}{\leftrightarrow}f'$ we have,*

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = \frac{2^n\operatorname{vol}(B\backslash B^1_{\mathbb{A}})}{\operatorname{vol}(B'\backslash B'^1_{\mathbb{A}})\operatorname{vol}(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}))}B'(f',\nu,\lambda).$$

Next we state the local analogue of this theorem. The distributions in Theorem 2 are factorizable. Their local counterparts are defined in §5. Locally, for a character μ of $F^\times$ we introduce the Tate γ-factor

$$\gamma(\mu, s, \psi) = \frac{L(\mu, s)}{\epsilon(\mu, s, \psi)L(\mu^{-1}, 1-s)}.$$

For a character $\nu = (\nu_1, \dots, \nu_n)$ on T' and $\lambda \in \mathbb{C}^n$ set,

$$\gamma(\nu, \lambda, \psi) = \prod_{1 \le i < j \le n} \gamma(\nu_i \nu_j^{-1} \omega, \lambda_i - \lambda_j, \psi). \tag{6}$$

For an additive character ψ of F, let d_F^ψ be the self dual Haar measure on F with respect to ψ. If ψ' is another character of F we denote by $(d_F^{\psi'} : d_F^\psi)$ the positive number for which $d_F^{\psi'} = (d_F^{\psi'} : d_F^\psi) d_F^\psi$. We set $e(\psi) = (d_F^\psi : d_F^{\psi_0})$ where ψ_0 is a character of conductor $\mathcal{O}_F$ in the p-adic case and $\psi_0(x) = e^{2\pi i \operatorname{Tr}_{F/\mathbb{R}}(x)}$ in the archimedean case. If $\psi' = \psi(a\cdot)$ for some $a \in F^\times$ is another character then, $e(\psi') = |a|^{\frac{1}{2}} e(\psi)$. For a quadratic extension E/F of local fields, we define in §2.3 a certain quantity $[d_B\colon\, d_{B'} \times d_{H_e^{w_n}}]$ which depends proportionally on the Haar measure on B and inverse proportionally on the Haar measures on B' and on $H_e^{w_n}$.

THEOREM 3. *For a quadratic extension E/F of local fields of characteristic zero, there exists a root of unity $\kappa_{E/F} = \kappa_{E/F}(\psi)$, depending only on n, on ψ and on the extension E/F, so that for any unitary character ν of T' and any $\Phi \overset{\psi,\delta(n)}{\leftrightarrow} f'$ we have*

$$\tilde{B}^{st}(\Phi, \nu, \lambda) = \kappa_{E/F}(\psi)\, e(\psi)^{-\dim U'} [d_B\colon\, d_{B'} \times d_{H_e^{w_n}}] \gamma(\nu, \lambda, \psi) B'(f', \nu, \lambda).$$

If $\psi' = \psi(a\cdot)$ for some $a \in F^\times$ is another character then, $\kappa_{E/F}(\psi') = \omega(a)^{\dim U'} \kappa_{E/F}(\psi)$. Moreover, if E/F is unramified and of odd residual characteristic and if ψ has conductor $\mathcal{O}_F$ then $\kappa_{E/F}(\psi) = 1$.

The element $\delta(n) \in \{0, 1\}$ is determined in Theorem 2. In the unramified places, we determine $\kappa_{E/F}$ by taking Φ to be the characteristic function of $Y \cap K$ and f' to be the unit element of the Hecke algebra $\mathcal{H}_{G'}$. We then directly compare both sides of the equation. As already mentioned, for this we use Hironaka's computation. The identity we obtain is also valid in the case of even residual characteristic, but in this case we do not know the fundamental lemma, and therefore cannot determine $\kappa_{E/F}$. We do not attempt to determine $\kappa_{E/F}$ in general. We remark however that in a global situation, as in [LR00], we have $\prod_v \kappa_{E_v/F_v} = 1$.

Theorem 3 is our main motivation for this work. In a joint work with E. Lapid we apply the local Bessel identities in order to obtain a new expression for the compact unitary period of certain cusp forms in terms of special values of L-functions [LO07]. This in turn, has an application towards a recent conjecture of P. Sarnak about the L^∞-norms of automorphic forms [Sar04].

Although not necessary for the application we have in mind, it would be interesting to determine $\delta(n)$. In [Off] we conjecture a generalization of the fundamental lemma of Jacquet that would, in particular, determine $\delta(n)$. We prove the conjecture for the case $n = 2$ and obtain $\delta(2) = 1$.

Acknowledgments. During the preparation of this work, my research at the Weizmann Institute of Science is being supported by an Edith and Edward F. Anixter Postdoctoral Fellowship. This project was suggested to me by E. Lapid. I would like to thank him for many fruitful conversations and constant encouragement.

2. Notation and preliminaries. Throughout, E/F denotes a quadratic extension of number fields or local fields of characteristic zero. In the global setting we set $E_v = F_v \otimes_F E$ for every place v of F. Thus E_v/F_v is a quadratic extension of local fields whenever v is inert and $E_v \simeq F_v \oplus F_v$ whenever v is split. In the local setting we therefore also consider the split case $E = F \oplus F$. Denote by $\mathrm{Nm}\,(x) = x\bar{x}$ the norm map from $E^\times$ to $F^\times$ and by ω the quadratic character attached to E/F by class field theory. In the global setting it is an idèle class character and in the local setting it is a character on $F^\times$. If $E = F \oplus F$ then ω is trivial and Nm is the map $(x, y) \mapsto xy$. Fix an algebraic closure $\overline{F}$ of F. We will use bold letters such as $\mathbf{X}$ to denote an algebraic set defined over F, which we identify with the set $\mathbf{X}(\overline{F})$. We will use plain letters such as X to denote the set of F-rational points of an algebraic set, i.e. $X = \mathbf{X}(F)$. If F is global we also denote $X_v = \mathbf{X}(F_v)$ for every place v of F and $X_{\mathbb{A}} = \mathbf{X}(\mathbb{A})$ where $\mathbb{A}$ is the adèle ring of F. We keep the notation introduced in §1. In particular $\mathbf{G}' = GL(n)$, $\mathbf{G} = \mathrm{Res}_{E/F}(GL_n)$ and $\mathbf{Y} = \{y \in \mathbf{G} \mid y\theta(y) = e\}$ are all defined over F. Note that $\mathbf{Y}w_n$ is the space of Hermitian matrices in $\mathbf{G}$ and for each $\xi \in Y$, $\mathbf{H}^\xi$ is the unitary group with respect to E/F and the Hermitian form $(\xi w_n)^{-1}$. We shall fix some further notation and conventions for G; similar notation and conventions will apply for G' with a prime appended. In the global setting, the standard maximal compact of $G_{\mathbb{A}}$ is denoted by $\mathbf{K}$. In the local setting, the standard maximal compact of G is denoted by K. Globally, we have $\mathbf{K} = \prod_v K_v$, the product being over all places v of F. We denote by W the Weyl group of G. Let $\mathfrak{a}_0^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X^*(T)$ is the lattice of rational characters of T and denote the dual space by $\mathfrak{a}_0$. We identify $\mathfrak{a}_0^*$ and its dual space with $\mathbb{R}^n$. The W-invariant pairing $\langle \cdot, \cdot \rangle\colon\ \mathfrak{a}_0^* \times \mathfrak{a}_0 \to \mathbb{R}$ is then the standard inner product on $\mathbb{R}^n$. The height map $H\colon\ G_{\mathbb{A}} \to \mathfrak{a}_0$ is characterized by the condition $e^{\langle \alpha, H(utk) \rangle} = |\alpha(t)|$ for all $\alpha \in X^*(T)$, $u \in U_{\mathbb{A}}$, $t \in T_{\mathbb{A}}$ and $k \in \mathbf{K}$. For an algebraic group $\mathbf{Q}$ defined over F, we denote by δ_Q the modulus function of $Q_{\mathbb{A}}$ in the global setting and of Q in the local setting. Denote by $\rho \in \mathfrak{a}_0^*$ half the sum of the positive roots in $X^*(T)$ with respect to B, thus

$$\delta_B = e^{\langle 2\rho, H(\cdot)\rangle}.$$

Let χ be a unitary character of $T_{\mathbb{A}}$. For all $\lambda \in \mathfrak{a}^*_{0,\mathbb{C}}$ we identify the spaces of the induced representations $I(\chi) = I(\chi, \lambda)$ with the pre-Hilbert space of smooth functions $\varphi\colon\ G_{\mathbb{A}} \to \mathbb{C}$ such that

$$\varphi(utg) = \delta_B^{\frac{1}{2}}(t)\chi(t)\varphi(g)$$

for $u \in U_{\mathbb{A}}$, $t \in T_{\mathbb{A}}$ and $g \in G_{\mathbb{A}}$. The scalar product on $I(\chi)$ is given by

$$(\varphi_1, \varphi_2) = \int_{B_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi_1(\dot{g})\overline{\varphi_2(\dot{g})}\, d\dot{g}.$$

The representation $I(\chi, \lambda)$ is defined by

$$I(g, \chi, \lambda)\varphi(g') = e^{\langle \lambda, H(g'g) - H(g')\rangle}\varphi(g'g).$$

It is unitary if $\lambda \in i\mathfrak{a}^*_0$. Let $w \in W$ and let ${}^w\chi(t) = \chi(wtw^{-1})$. The (un-normalized) intertwining operator

$$M(w, \lambda)\colon\ I(\chi, \lambda) \to I({}^w\chi, w\lambda)$$

is defined by

$$(M(w, \lambda)\varphi)(g) = e^{-\langle w\lambda, H(g)\rangle} \int_{(U_{\mathbb{A}} \cap w^{-1}U_{\mathbb{A}}w)\backslash U_{\mathbb{A}}} e^{\langle \lambda, H(wug)\rangle}\varphi(wug)\, du.$$

It is absolutely convergent in a suitable cone and admits a meromorphic continuation in λ. We will use similar notation for induced representations and intertwining operators in the local setting. In the nonarchimedean setting, let $\mathcal{H}_G$ be the Hecke algebra of compactly supported, bi-K-invariant functions on G. When χ is unramified, let $\hat{f}(\chi, \lambda)$ denote the spherical Fourier transform of $f \in \mathcal{H}_G$ evaluated at $I(\chi, \lambda)$. Denote by bc the base change homomorphism

$$\mathrm{bc}\colon\ \mathcal{H}_G \to \mathcal{H}_{G'}.$$

It satisfies $\hat{f}(\chi, \lambda) = \hat{f}'(\nu, \lambda)$ whenever $f' = \mathrm{bc}(f)$ and $\chi = \nu \circ \mathrm{Nm}$. We fix a nontrivial character ψ of $\mathbb{A}/F$. For $u \in U_{\mathbb{A}}$ we set

$$\psi_U(u) = \psi\left(\mathrm{Tr}_{E/F}(u_{1,2}) + \cdots + \mathrm{Tr}_{E/F}(u_{n-1,n})\right).$$

For $\varphi \in I(\chi, \lambda)$ we let $E(\varphi, \lambda)$ be the Eisenstein series on $G_{\mathbb{A}}$ induced from φ. The ψ-th Fourier coefficient $\mathcal{W}^{\psi} = \mathcal{W}$ of $E(\varphi, \lambda)$ is defined by

$$\mathcal{W}(\varphi, \lambda) = \int_{U \backslash U_{\mathbb{A}}} E(u, \varphi, \lambda) \overline{\psi_U(u)} \, du.$$

We define similarly the character $\psi_{U'}$ of $U'_{\mathbb{A}}$ by

$$\psi_{U'}(u') = \psi \left(u'_{1,2} + \cdots + u'_{n-1,n} \right).$$

For a character ν of $T' \backslash T'_{\mathbb{A}}$ and $\varphi' \in I'(\nu, \lambda)$, the ψ-th Fourier coefficient $\mathcal{W}'(\varphi', \lambda)$ of the Eisenstein series $E(\nu, \lambda)$ on $G'_{\mathbb{A}}$ is defined in a similar way with respect to $\psi_{U'}$.

2.1. Orbits in Y. Denote by $G \backslash Y$ the set of G-orbits in Y. Given a set of representatives $\{\xi\}$ in Y for $G \backslash Y$ the map $g \mapsto g^{-1} \cdot \xi$ defines a bijection

$$\bigsqcup_{\{\xi\}} H^{\xi} \backslash G \to Y. \tag{7}$$

Next, we consider B-orbits in Y. According to a result of T. Springer [Spr85], every B-orbit in Y intersects the normalizer $N_G(T)$. In fact the map $\mathcal{O} \mapsto \mathcal{O} \cap N_G(T)$ is a bijection between B-orbits in Y and T-orbits in $Y \cap N_G(T)$. We define a map

$$\iota\colon\ B \backslash Y \to W$$

by $\mathcal{O} \mapsto T(\mathcal{O} \cap N_G(T))$. We wish to analyze the B-orbits in $\iota^{-1}(w_n)$. It is observed in [LR00] that for $t \in T$ we have $tw_n \in Y$ if and only if $t \in T'$, and that the T-orbit of tw_n is $\mathrm{Nm}\,(T) tw_n$. Set

$$A = T' / \mathrm{Nm}\,(T).$$

As in [LR00] we have,

LEMMA 1. *There is a bijection*

$$\iota^{-1}(w_n) \simeq A$$

defined by $\mathcal{O} \leftrightarrow (\mathcal{O} \cap T' w_n) w_n^{-1}$.

We then have a decomposition

$$A = \bigsqcup_{\mathcal{C} \in G \backslash Y} A_{\mathcal{C}}$$

where

$$A_{\mathcal{C}} = \{a \in A\colon\ aw_n \subset \mathcal{C}\}.$$

Set

$$X = F^{\times} / \operatorname{Nm}(E^{\times}).$$

Thus, $A \simeq X^n$. In the global setting X_v is then $F_v^{\times} / \operatorname{Nm}(E_v^{\times})$. Let

$$X_{\mathbb{A}} = \oplus_v X_v$$

and define $A_{\mathbb{A}}$ similarly. The group X imbeds diagonally in $X_{\mathbb{A}}$. By class field theory, $[X_{\mathbb{A}} : X] = 2$ and therefore $[A_{\mathbb{A}} : A] = 2^n$. Let

$$d\colon\ A_{\mathbb{A}} \to X_{\mathbb{A}}$$

be the map defined by the determinant. We also denote by $G_{\mathbb{A}} \backslash Y_{\mathbb{A}}$ the set of $G_{\mathbb{A}}$-orbits in $Y_{\mathbb{A}}$. It is identified with elements $\mathcal{C} = (\mathcal{C}_v)_v$ where for each place v of F, $\mathcal{C}_v$ is a G_v-orbit in Y_v and for almost all v, $\mathcal{C}_v = G_v \cdot w_n$. By abuse of notation we will also denote by d the map

$$d\colon\ G_{\mathbb{A}} \backslash Y_{\mathbb{A}} \to X_{\mathbb{A}}$$

defined by $d(\mathcal{C}) = \det(\mathcal{C} w_n)$. For $\mathcal{C} \in G_{\mathbb{A}} \backslash Y_{\mathbb{A}}$ we denote

$$A_{\mathcal{C},\mathbb{A}} = \{a \in A_{\mathbb{A}}\colon\ \forall v,\ a_v w_n \subset \mathcal{C}_v\}.$$

We have $d(a) = d(\mathcal{C})$ for all $a \in A_{\mathcal{C},\mathbb{A}}$. There is a natural map

$$i\colon\ G \backslash Y \to G_{\mathbb{A}} \backslash Y_{\mathbb{A}}$$

sending a G-orbit to the $G_{\mathbb{A}}$-orbit that contains it. The local to global principle for Hermitian forms says that this map is injective. We thus have the following commuting diagram

$$\begin{array}{ccc} A & \hookrightarrow & A_{\mathbb{A}} \\ \downarrow & & \downarrow \\ G\backslash Y & \overset{i}{\hookrightarrow} & G_{\mathbb{A}} \backslash Y_{\mathbb{A}} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X_{\mathbb{A}}, \end{array}$$

where in both sides the upper vertical map sends a to the orbit containing aw_n and the other vertical map is d. An orbit $\mathcal{C} \in G_{\mathbb{A}} \backslash Y_{\mathbb{A}}$ lies in $i(G \backslash Y)$ if and only if $d(\mathcal{C}) \in X$. Another way to describe the local to global principle is to say that for $\mathcal{C} \in G\backslash Y$ and for $\xi \in Y$ we have

$$\mathrm{ch}_{\mathcal{C}}(\xi) = \prod_v \mathrm{ch}_{\mathcal{C}_v}(\xi_v), \tag{8}$$

where ch_Γ denotes the characteristic function of a set Γ.

2.2. Fourier inversion and stabilization. We recall here some Fourier analysis on $A_{\mathbb{A}}$ from [LR00]. If g is an absolutely summable function on $A_{\mathbb{A}}$, we define its Fourier transform

$$\hat{g}(\kappa) = \sum_{a \in A_{\mathbb{A}}} \kappa(a) g(a)$$

for any character κ of $A_{\mathbb{A}}$. We have the following inversion formula

$$2^n \sum_{a \in A} g(a) = \sum_{\kappa \in (A_{\mathbb{A}}/A)^*} \hat{g}(\kappa).$$

If in addition g is of the form

$$g(a) = \prod_v g_v(a_v)$$

where g_v is a function on A_v for all v and the infinite product absolutely converges then by Lemma 2 of [LR00] we have

$$\hat{g}(\kappa) = \prod_v \hat{g}_v(\kappa_v),$$

where κ_v is the restriction of κ to A_v and

$$\hat{g}_v(\kappa_v) = \sum_{a_v \in A_v} \kappa_v(a_v) g_v(a_v).$$

2.3. Measures. Since we want to emphasize the dependence of the Bessel distributions on the various invariant measures, we do not, at this stage, make explicit choices of measures.

We denote by d_Q a right Haar measure on a locally compact group Q. If R is a closed subgroup of Q, we denote by $d_{R\backslash Q}$ the equivariant measure on $R\backslash Q$ determined by d_R and d_Q. To be precise, it is the functional on the space—smooth functions f on Q of compact support modulo R, such that $f(rq) = \delta_R \delta_Q^{-1}(r) f(q)$

for all $r \in R$ and $q \in Q$—such that

$$\int_Q \phi(q)\, d_Q q = \int_{R\backslash Q} \int_R \phi(r\dot{q}) \delta_Q \delta_R^{-1}(r)\, d_R r\, d_{R\backslash Q}\dot{q}$$

for every smooth function of compact support ϕ on Q. When this relation holds we denote symbolically

$$d_Q = d_R \times d_{R\backslash Q}.$$

If ϕ: $Q_1 \to Q_2$ is an isomorphism of locally compact groups, and if R_i is a closed subgroup of Q_i, $i = 1, 2$ such that $R_2 = \phi(R_1)$, then ϕ and $d_{R_2\backslash Q_2}$ determine an equivariant measure $d^{\phi}_{R_1\backslash Q_1}$ on $R_1\backslash Q_1$. We denote by $(\frac{d_{R_1\backslash Q_1}}{d_{R_2\backslash Q_2}})_\phi = (d_{R_1\backslash Q_1}\colon\ d_{R_2\backslash Q_2})_\phi$ the positive number such that

$$d_{R_1\backslash Q_1} = (d_{R_1\backslash Q_1}\colon\ d_{R_2\backslash Q_2})_\phi d^{\phi}_{R_1\backslash Q_1}.$$

Clearly

$$(d_{R_1\backslash Q_1}\colon\ d_{R_2\backslash Q_2})_\phi^{-1} = (d_{R_2\backslash Q_2}\colon\ d_{R_1\backslash Q_1})_{\phi^{-1}}.$$

When clear from the context, we will often supress the index ϕ from our notation.

The Haar measures will be bound by the following constrains. Discrete groups will be endowed with the counting measure. If $\mathbf{Q}$ is an algebraic group defined over a global field F, we fix a decomposition $d_{Q_{\mathbb{A}}} q = \otimes_v d_{Q_v} q_v$. The groups $\mathbf{H}^\xi$ are all inner forms of one another. We assume that the Haar measures

$$d_{H^\xi_{\mathbb{A}}} = \otimes_v d_{H^\xi_v},\ \xi \in Y$$

are chosen compatibly in the following sense. If $\mathbf{Q}$ and $\mathbf{Q}'$ are two reductive algebraic groups defined over a local field F which are inner forms of one another, it is explained in §15 of [JL70] how a Haar measure $d_Q q$ on Q determines a Haar measure $d_{Q'} q'$ on Q' by pulling back the associated invariant differential form via an inner twist. Globally, we assume that for ξ, $\xi' \in Y$ the measures $d_{H^\xi_{\mathbb{A}}}$ and $d_{H^{\xi'}_{\mathbb{A}}}$ are such that for every place v of F the measure $d_{H^{\xi'}_v}$ is the pull back of $d_{H^\xi_v}$ in the above sense.

Fix a set of representatives $\{\xi\}$ for $G_{\mathbb{A}}\backslash Y_{\mathbb{A}}$. For a representative ξ, we let $H^\xi_{\mathbb{A}} = \prod'_v H^{\xi_v}_v$ be the restricted product with respect to $H^{\xi_v}_v \cap K_v$ for almost all v. The measure $d_{H^\xi_{\mathbb{A}}} = \otimes_v d_{H^{\xi_v}_v}$ is a Haar measure on $H^\xi_{\mathbb{A}}$. The $G_{\mathbb{A}}$-invariant measure $d_{Y_{\mathbb{A}}}$ on $Y_{\mathbb{A}}$ is determined by the isomorphism $g \mapsto g^{-1} \cdot \xi$ from the disjoint union over $\{\xi\}$ of $H^\xi_{\mathbb{A}}\backslash G_{\mathbb{A}}$ to $Y_{\mathbb{A}}$ and by the invariant measures $d_{H^\xi_{\mathbb{A}}\backslash G_{\mathbb{A}}}$. By our choice of compatible measures on the unitary groups, the measure $d_{Y_{\mathbb{A}}}$ is independent of a choice of representatives for the $G_{\mathbb{A}}$-orbits and decomposes as

$d_{Y_{\mathbb{A}}}(y) = \otimes_v d_{Y_v}(y_v)$ where d_{Y_v} is the G_v-invariant measure on Y_v determined by the isomorphism (7) and the measures $d_{H_v^\xi \backslash G_v}$.

Let $\xi \in Y$, $a \in A$ and $\eta \in G$ be such that $\eta \cdot \xi \in aw_n$. Denote

$$\mathbf{H}_\eta^\xi = \mathbf{H}^\xi \cap \eta^{-1}\mathbf{B}\eta.$$

It is observed in [LR00] that the groups $\eta H_\eta^\xi \eta^{-1}$ are independent of ξ, a and η as above and consist of the elements $\mathrm{diag}\,(a_1, \dots, a_n)$, $a_i \in E_1$ where

$$E_1 = \{x \in E^\times \colon \ \mathrm{Nm}\,(x) = 1\}.$$

We fix a decomposable Haar measure on $(E_1)_{\mathbb{A}}$ and let $d_{(H_\eta^\xi)_{\mathbb{A}}} = \otimes_v d_{H_{\eta v}^{\xi v}}$ be the Haar measure on $(H_\eta^\xi)_{\mathbb{A}}$ determined by the isomorphism $H_\eta^\xi \simeq (E_1)^n$.

Next we define the local proportionality constant that appears in Theorem 3. Fix measures on T and on U so that $d_B = d_U d_T$ and similarly let $d_{B'} = d_{U'} d_{T'}$. We set

$$[d_B \colon\ d_{B'} \times d_{H_e^{w_n}}] = [d_T \colon\ d_{T'} \times d_{H_e^{w_n}}][d_U : d_{U'}]$$

where the terms on the right are defined as follows. There is an exact sequence

$$1 \to H_e^{w_n} \to T \overset{\mathrm{Nm}}{\to} T'$$

and a positive number $[d_T \colon\ d_{T'} \times d_e^{w_n}]$ such that

$$\int_T f(t)\, d_T t = [d_T \colon\ d_{T'} \times d_e^{w_n}] \int_{\mathrm{Nm}\,(T)} F(x)\, d_{T'} x$$

where

$$F(\,\mathrm{Nm}\, t) = \int_{H_e^{w_n}} f(yt)\, d_{H_e^{w_n}} y.$$

If F is nonarchimedean we set $U_0' = U' \cap K'$. If F is real we let U_0' be the set of all $u = (x_{j,k}) \in U'$ such that for all $j < k$ we have $x_{j,k} \in [0, 1]$ and if F is complex we let U_0' be the set of all $u = (z_{j,k}) \in U'$ such that for all $j < k$ we have $z_{j,k} = x_{j,k} + iy_{j,k}$ with $x_{j,k} \in [0, \frac{1}{2}]$ and $y_{j,k} \in [0, 1]$. Let U_0 be the analogue subgroup of U. We set

$$[d_U : d_{U'}] = \frac{d_U(U_0)}{d_{U'}(U_0')}.$$

3. Local periods and stabilization. Fix a unitary character χ of T which is a base change from a character of T'. Denote by $\mathcal{B}(\chi)$ the set of characters ν of

T' that base change to χ, i.e. that satisfy $\nu \circ \mathrm{Nm} = \chi$. For $\nu = (\nu_1, \ldots, \nu_n) \in \mathcal{B}(\chi)$ let

$$[\nu] = \{\nu, \omega\nu\}$$

where $\omega\nu = (\omega\nu_1, \ldots, \omega\nu_n)$. Let $\mathcal{C} \in G\backslash Y$ and let $\xi \in \mathcal{C}$. Let $a \in A_{\mathcal{C}}$, $t \in a$ and $\eta \in G$ be such that

$$\eta \cdot \xi = tw_n.$$

It is shown in [LR03] that the integral

$$J^{\xi}(\eta, \varphi, \lambda) = \int_{H^{\xi}_{\eta}\backslash H^{\xi}} e^{\langle \lambda, H(\eta\dot{h})\rangle} \varphi(\eta\dot{h})\, d_{H^{\xi}_{\eta}\backslash H^{\xi}} \dot{h},$$

where $\varphi \in I(\chi, \lambda)$, converges for $\mathrm{Re}\,\lambda$ sufficiently positive. The character ν_ω was defined in (4). Note that $\nu \mapsto \nu_\omega\nu$ permutes $\mathcal{B}(\chi)$. For $\nu \in \mathcal{B}(\chi)$ and $a \in A_{\mathcal{C}}$ let

$$\Delta^{\xi}_{\nu,\eta}(\lambda) = (\nu_\omega\nu)(t) e^{\frac{1}{2}\langle \lambda+\rho, H(t)\rangle}$$

and define the normalized period integral by

$$\tilde{J}^{\xi}_{\nu}(a, \varphi, \lambda) = \Delta^{\xi}_{\nu,\eta}(\lambda)^{-1} J^{\xi}(\eta, \varphi, \lambda). \tag{9}$$

The normalized period is independent of our choices of $t \in a$ and η as above. We may keep track of the dependence on ξ as follows. If $\xi' = g \cdot \xi \in \mathcal{C}$, we may choose $\eta' = \eta g^{-1}$, then $\eta' \cdot \xi' = tw_n$, $H^{\xi'} = gH^{\xi}g^{-1}$ and $H^{\xi'}_{\eta'} = gH^{\xi}_{\eta}g^{-1}$. We then see that

$$\tilde{J}^{g\cdot\xi}_{\nu}(a, I(g, \chi, \lambda)\varphi, \lambda) = \tilde{J}^{\xi}_{\nu}(a, \varphi, \lambda). \tag{10}$$

We will come back to the dependence on a choice of a representative $\xi \in \mathcal{C}$ later, when we define the stabilization of the relative Bessel distributions. When $a \notin A_{\mathcal{C}}$ we set $\tilde{J}^{\xi}_{\nu}(a, \varphi, \lambda) = 0$. We define the stable local period integral

$$J^{st,\xi}(\nu, \varphi, \lambda) = \sum_{a \in A} \tilde{J}^{\xi}_{\nu}(a, \varphi, \lambda).$$

Note that

$$J^{st,\xi}(\omega\nu, \varphi, \lambda) = \omega(\det(\xi w_n)) J^{st,\xi}(\nu, \varphi, \lambda). \tag{11}$$

In [LR00], the meromorphic continuation of the stable period is proved in the nonarchemidean case for the case $n = 3$ using the principle of Bernstein. The proof carries over verbatim for any n. Let E/F be a quadratic extension of p-adic fields and let $q = q_F$ be the cardinality of the residual field of F.

PROPOSITION 1. *The stable period $J^{st,\xi}(\nu,\varphi,\lambda)$ admits a meromorphic continuation to a rational function in q^{λ}.*

In §4 we explain how the meromorphic continuation follows also for the archimedean case (cf. Remark 1).

3.1. The split period. If $E = F \oplus F$ then $G = G' \times G'$. We have $\theta(g_1, g_2) = (\vartheta(g_2), \vartheta(g_1))$, where $\vartheta(g) = w_n {}^t g^{-1} w_n^{-1}$, $g \in G'$. In this case

$$Y = \{(g, \vartheta(g)^{-1}) \mid g \in G'\}$$

is a unique G-orbit, and the B-orbits in Y are in bijection with the Weyl group W via the Bruhat decomposition of G'. The stabilization is then trivial. For any $\xi = (g_0, \vartheta(g_0)^{-1}) \in Y$ we have

$$H^{\xi} = \{(g, \vartheta_{g_0^{-1}}(g)) \mid g \in G'\}$$

where $\vartheta_{g'}(g) = \vartheta(g' g g'^{-1})$. We take $\eta = (1, w_n \vartheta(g_0)) \in G$. Thus, $\eta \cdot \xi = (w_n, w_n)$ and

$$H_{\eta}^{\xi} = \{(t, \vartheta_{g_0^{-1}}(t)) \mid t \in T'\}.$$

Up to a ratio of certain measures, the local period can be expressed in terms of an intertwining operator. Note that the isomorphism $g \mapsto (g, {}^t g^{-1})$ from G' to H^{w_n} maps T' to $H_e^{w_n}$. Let χ be a character of $T = T' \times T'$, which is base change from T'. Thus it has the form $\chi = (\nu, \nu)$ and $\mathcal{B}(\chi) = \{\nu\}$.

PROPOSITION 2. *For $\varphi = \varphi_1 \otimes \varphi_2 \in I(\chi) = I'(\nu) \otimes I'(\nu)$ we have*

$$\begin{aligned} J^{st,\xi}(\nu, \varphi_1 \otimes \varphi_2, \lambda) &= (d_{H_e^{w_n} \backslash H^{w_n}} : d_{T' \backslash G'})(d_{B'} : d_{U'}\, d_{T'}) \\ &\quad \times \int_{B' \backslash G'} \varphi_1(\dot g)(M'(w_n, \lambda) I'(g_0, \nu, \lambda)\varphi_2)(\vartheta(\dot g))\, d_{B' \backslash G'} \dot g. \end{aligned}$$

Proof. With our convention on compatible measures we have,

$$\begin{aligned} &(d_{T' \backslash G'} : d_{H_e^{w_n} \backslash H^{w_n}}) J^{st,\xi}(\nu, \varphi_1 \otimes \varphi_2, \lambda) \\ &\quad = \int_{T' \backslash G'} e^{\langle \lambda, H(\dot g) + H(w_n \vartheta(g_0) \vartheta_{g_0^{-1}}(\dot g)\rangle} \varphi_1(\dot g)\varphi_2(w_n \vartheta(g_0)\vartheta_{g_0^{-1}}(\dot g))\, d_{T' \backslash G'} \dot g \\ &\quad = \int_{T' \backslash G'} e^{\langle \lambda, H(\dot g) + H(w_n \vartheta(\dot g))\rangle} \varphi_1(\dot g)(I'(g_0, \nu, \lambda)\varphi_2)(w_n \vartheta(\dot g))\, d_{T' \backslash G'} \dot g \\ &\quad = (d_{B'} : d_{U'}\, d_{T'}) \\ &\qquad \times \int_{B' \backslash G'} \varphi_1(\dot g) \int_{U'} e^{\langle \lambda, H(w_n \vartheta(u))\rangle} (I'(g_0, \nu, \lambda)\varphi_2)(w_n \vartheta(u)\vartheta(\dot g))\, d_{U'} u\, d_{B' \backslash G'} \dot g \end{aligned}$$

The proposition follows, by making the change of variables $u \mapsto \vartheta(u)$. □

If χ is unramified, $\xi \in K$ and $\varphi_0 = \varphi_0' \otimes \varphi_0' \in I(\chi, \lambda)$ is the K-invariant section so that $\varphi_0(e) = 1$, then

$$M'(w_n, \lambda)\varphi_0'(k) = d_{U'}(U_0') \prod_{i<j} \frac{L(\nu_i\nu_j^{-1}, \lambda_i - \lambda_j)}{L(\nu_i\nu_j^{-1}, \lambda_i - \lambda_j + 1)}, k \in K. \tag{12}$$

Since ω is trivial in the split case we get that

$$J^{st,\xi}(\nu, \varphi_0, \lambda) = \upsilon \prod_{i<j} \frac{L(\nu_i\nu_j^{-1}\omega, \lambda_i - \lambda_j)}{L(\nu_i\nu_j^{-1}, \lambda_i - \lambda_j + 1)} \tag{13}$$

where $\upsilon = (d_{H_e^{w_n}\backslash H^{w_n}} : d_{T'\backslash G'})(d_{B'} : d_{U'}\, d_{T'})d_{U'}(U_0')(\varphi_0', \varphi_0')$. In the p-adic case it is easy to see that

$$\upsilon = \frac{d_{H^{w_n}}(H^{w_n} \cap K))}{d_{H_e^{w_n}}(H_e^{w_n} \cap K)}.$$

The constant υ can also be expressed as a ratio of measures in the archimedean case.

3.2. The unramified inert period. In this subsection E/F is a quadratic extension of p-adic fields. Assume that χ is an unramified character and let φ_0 be the K-invariant element in $I(\chi, \lambda)$ such that $\varphi_0(e) = 1$. Our goal in this subsection is the following.

PROPOSITION 3. *If E/F is unramified and $\xi \in Y \cap K$ then,*

$$J^{st,\xi}(\nu, \varphi_0, \lambda) = \frac{d_{H^{w_n}}(H^{w_n} \cap K))}{d_{H_e^{w_n}}(H_e^{w_n})} \prod_{i<j} \frac{L(\nu_i\nu_j^{-1}\omega, \lambda_i - \lambda_j)}{L(\nu_i\nu_j^{-1}, \lambda_i - \lambda_j + 1)}.$$

The rest of this section is devoted to the proof of Proposition 3. It is enough to prove the proposition when χ is trivial and we assume that $\chi = 1_T$ is the trivial character of T throughout the section. The period integrals $J^{st,\xi}(\nu, \varphi_0, \lambda)$ are interpreted in terms of Hironaka's spherical functions on Hermitian symmetric spaces introduced in [Hir88]. In the unramified case, Hironaka provides in [Hir99] explicit formulas for the spherical functions, which we use to obtain Proposition 3. The asymptotic formula obtained in §3.2 as well as our interpretation of the periods in Lemma 5 are provided for a general quadratic extension of p-adic fields. The partial results in this subsection are therefore formulated for a general quadratic extension unless otherwise is specified. In order to interpret the period integrals as Hironaka's spherical functions, a certain compatibility condition, of measures induced from a homogenous space, is required. We start with clarifying this issue.

3.2.1. Compatibility of measures. Let $\overline{B} = T\overline{U} = \overline{U}T$ be the group of lower triangular matrices in G, with unipotent radical $\overline{U}$, and let $\overline{\mathbf{B}}$ be the associated Borel subgroup of $\mathbf{G}$ defined over F. For $\xi \in Y$ let $\mathcal{C}^\xi = G\cdot\xi$ and let $\mathbf{Q}^\xi = \overline{\mathbf{B}} \times \mathbf{H}^\xi$. There is a right action of $\mathbf{Q}^\xi$ on $\mathbf{G}$ given by $g^{(b,h)} = b^{-1}gh$, $b \in \overline{\mathbf{B}}$, $h \in \mathbf{H}^\xi$, $g \in \mathbf{G}$. Let d_i: $\mathbf{Y} \to \mathbf{G_a}$ be the algebraic map, so that $d_i(x)$ is the upper left $i \times i$ minor of xw_n if $x \in Y$ and let

$$\mathbf{X}^\xi = \{g \in \mathbf{G} \mid d_i(g \cdot \xi) \neq 0,\ i = 1, \dots, n\}.$$

Then $\mathbf{X}^\xi$ is an open and dense set in $\mathbf{G}$ in the Zarisky topology. It is also a $\mathbf{Q}^\xi$-homogenous space. Recall that $A_{\mathcal{C}^\xi} = \{a \in A \mid \det a \in \det(\xi w_n)\,\mathrm{Nm}\,(E^\times)\}$. The Q^ξ-orbits in X^ξ are parameterized by $A_{\mathcal{C}^\xi}$. To $a \in A_{\mathcal{C}^\xi}$ we associate the orbit

$$X_a^\xi = \{\eta \in X^\xi \mid \exists q \in Q^\xi,\ (\eta^q)\cdot\xi \in aw_n\}.$$

Then X^ξ is the disjoint union of the Q^ξ-orbits X_a^ξ, $a \in A_{\mathcal{C}^\xi}$. We remark that whenever $\eta \in G$ is such that $\eta\cdot\xi \in aw_n$ we have $\mathbf{H}_\eta^\xi = \mathbf{H}^\xi \cap \eta^{-1}\overline{\mathbf{B}}\eta$.

Let $\mathbf{Q}$ and $\mathbf{Q}'$ be algebraic groups defined over F. Let $\mathbf{X}'$ be a $\mathbf{Q}'$-homogenous space and $\mathbf{X}$ any $\mathbf{Q}$-space, both defined over F. Let φ: $\mathbf{Q}' \to \mathbf{Q}$ be an isomorphism and let ϕ: $\mathbf{X}' \to \mathbf{X}$ be an open imbedding such that $\phi(x^q) = \phi(x)^{\varphi(q)}$ (both maps are algebraic but we do not assume that they are defined over F). A Q-invariant measure d_Xx on X is determined uniquely by a top degree invariant differential form on $\mathbf{X}$, which can be pulled back via ϕ to a top degree invariant differential form on $\mathbf{X}'$. Since such a form is unique up to a scalar, it can be shown that (even if not defined over F) it determines uniquely a Q'-invariant measure $d_{X'}x'$ on X'. If in addition we assume that $\phi(X') \subset X$ then for an integrable function f on X' we have

$$\int_{X'} f(x')\,d_{X'}x' = \int_{\phi(X')} f(\phi^{-1}(x))\,d_X(x).$$

The argument is similar to that given in §15 of [JL70] for the special case where $\mathbf{Q} = \mathbf{X}$ and $\mathbf{Q}' = \mathbf{X}'$ are inner forms, and we omit it here. We will denote $d_{X'}x'$ by $\phi^*(d_Xx)$ and refer to it as the pull back of d_Xx via ϕ.

Let $d_{X^\xi}x$ be the restriction of the Haar measure d_G on G to X^ξ. It is a Q^ξ-invariant measure, which is also the pull back of d_G via the imbedding of $\mathbf{X}^\xi$ into $\mathbf{G}$. We take the right Haar measure $d_{\overline{B}}$ on $\overline{B}$ normalized so that $d_{\overline{B}}(\overline{B}\cap K) = 1$ and let $d_{Q^\xi}q = d_{\overline{B}}d_{H^\xi}$ be the associated right Haar measure on Q^ξ. We also denote by $d^l_{\overline{B}}b = d_{\overline{B}}(b^{-1})$ the left Haar measure on $\overline{B}$ determined by $d_{\overline{B}}$. For $\eta \in X^\xi$, its stabilizer in $\mathbf{Q}^\xi$ is

$$\mathbf{Q}_\eta^\xi = \{(\eta h\eta^{-1}, h) \mid h \in \mathbf{H}_\eta^\xi\}.$$

For $\eta \in X^\xi$ such that $\eta\cdot\xi \in T'w_n$ the map $h \mapsto (\eta h\eta^{-1}, h)$ is an isomorphism from

H^ξ_η to Q^ξ_η and we take the Haar measure $d_{Q^\xi_\eta}$ on Q^ξ_η so that $(d_{H^\xi_\eta} : d_{Q^\xi_\eta}) = 1$. We construct, in two different ways, Q^ξ-invariant measures on $Q^\xi_\eta \backslash Q^\xi$. The first—$d_{Q^\xi_\eta \backslash Q^\xi}$—is the measure that satisfies

$$d_{Q^\xi} = d_{Q^\xi_\eta} \times d_{Q^\xi_\eta \backslash Q^\xi}.$$

The second—$\tilde{d}_{Q^\xi_\eta \backslash Q^\xi}$—is the pull back of d_{X^ξ} via the isomorphism

$$\phi^\xi_\eta(\dot{q}) = \eta^{\dot{q}}$$

from $\mathbf{Q}^\xi_\eta \backslash \mathbf{Q}^\xi$ to $\mathbf{X}^\xi$. To be more precise, we identify $Q^\xi_\eta \backslash Q^\xi$ with the Q^ξ-orbit of the identity in $\mathbf{Q}^\xi_\eta \backslash \mathbf{Q}^\xi$ and take the restriction of $(\phi^\xi_\eta)^*(d_{X^\xi} x)$ to this orbit.

LEMMA 2. *There is a constant $c \in \mathbb{R}^\times_+$ so that for all $\xi \in Y$ and $\eta \in X^\xi$ such that $\eta \cdot \xi = tw_n$ for some $t \in T'$ we have*

$$\tilde{d}_{Q^\xi_\eta \backslash Q^\xi} = c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t) d_{Q^\xi_\eta \backslash Q^\xi}.$$

Proof. We denote by $|\cdot|$ the canonical extension of the standard absolute value on $F^\times$ to a multiplicative map from $\overline{F}^\times$ to $\mathbb{R}^\times_+$. There exists an algebraic map $\Delta_{\mathbf{Q}^\xi}$ from $\mathbf{Q}^\xi$ to $\overline{F}^\times$ such that $\delta_{Q^\xi}(q) = \left|\Delta_{\mathbf{Q}^\xi}(q)\right|$ for $q \in Q^\xi$. Thus, δ_{Q^ξ} extends to a homomorphism from $\mathbf{Q}^\xi$ to $\mathbb{R}^\times_+$.

Up to a scalar, there is a unique Q^ξ-invariant measure on $Q^\xi_\eta \backslash Q^\xi$. Fix ξ, η and t as in the statement of the lemma and let c be the constant such that

$$\tilde{d}_{Q^\xi_\eta \backslash Q^\xi} = c \delta_{\overline{B}}^{-\frac{1}{2}}(t) d_{Q^\xi_\eta \backslash Q^\xi}.$$

Let ξ', t' and η' be another such triple. Let $g' \in \mathbf{G}$ be such that $g' \cdot \xi' = \xi$. Note then that $\mathbf{X}^{\xi'} = \mathbf{X}^\xi g'$ and $\mathbf{H}^\xi = g' \mathbf{H}^{\xi'} g'^{-1}$. By abuse of notation we also denote by $\mathrm{ad}(g')$ the map $(b, h) \mapsto (b, g' h g'^{-1})$ from $\mathbf{Q}^{\xi'}$ to $\mathbf{Q}^\xi$. Let $q' = (b', h') \in \mathbf{Q}^{\xi'}$ be such that $\eta' = (\eta g')^{q'} = b'^{-1} \eta g' h'$. Note that $t' w_n = b'^{-1} \cdot t w_n$ and therefore that

$$\delta_{Q^{\xi'}}(q') = \delta_{\overline{B}}^{\frac{1}{2}}(t t'^{-1}). \tag{14}$$

We have the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{Q}^{\xi'}_{\eta'} \backslash \mathbf{Q}^{\xi'} & \xrightarrow{l_{q'}} & \mathbf{Q}^{\xi'}_{\eta \mathbf{g}'} \backslash \mathbf{Q}^{\xi'} & \xrightarrow{\phi^{\xi'}_{\eta g'}} & \mathbf{X}^{\xi'} \\
 & & \downarrow & & \downarrow \\
 & & \mathbf{Q}^\xi_\eta \backslash \mathbf{Q}^\xi & \xrightarrow{\phi^\xi_\eta} & \mathbf{X}^\xi,
\end{array}$$

where the left vertical map is $\mathrm{ad}\,(g')$, the right is $x \mapsto xg'$ and $l_{q'}$ is left multiplication by q'. Note also that $\phi^{\xi'}_{\eta'} = \phi^{\xi'}_{\eta g'} \circ l_{q'}$. We then have

$$\tilde{d}_{Q^{\xi'}_{\eta'} \backslash Q^{\xi'}} = l^*_{q'}\, \mathrm{ad}\,(g')^* \tilde{d}_{Q^{\xi}_{\eta} \backslash Q^{\xi}} = \mathrm{ad}\,(q')^*\, \mathrm{ad}\,(g')^* \tilde{d}_{Q^{\xi}_{\eta} \backslash Q^{\xi}},$$

where the second equality comes from the $Q^{\xi'}$-invariance. Recall that the measures on the compact groups $Q^{\xi'}_{\eta'}$ and Q^{ξ}_{η} are defined via their isomorphism with E^n_1 and note that $\mathrm{ad}\,(g') \circ \mathrm{ad}\,(q')$ defines an isomorphism from $Q^{\xi'}_{\eta'}$ to Q^{ξ}_{η}. We therefore also have

$$d_{Q^{\xi'}_{\eta'}} = \mathrm{ad}\,(q')^*\, \mathrm{ad}\,(g')^* d_{Q^{\xi}_{\eta}}$$

and therefore,

$$\begin{aligned} \tilde{d}_{Q^{\xi'}_{\eta'} \backslash Q^{\xi'}} \times d_{Q^{\xi'}_{\eta'}} &= \mathrm{ad}\,(q')^*\, \mathrm{ad}\,(g')^* (\tilde{d}_{Q^{\xi}_{\eta} \backslash Q^{\xi}} \times d_{Q^{\xi}_{\eta}}) \\ &= c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t)\, \mathrm{ad}\,(q')^*\, \mathrm{ad}\,(g')^* d_{Q^{\xi}}. \end{aligned}$$

But by our conventions on measures $\mathrm{ad}\,(g')^* d_{Q^{\xi}} = d_{Q^{\xi'}}$. This implies that

$$\begin{aligned} \tilde{d}_{Q^{\xi'}_{\eta'} \backslash Q^{\xi'}} \times d_{Q^{\xi'}_{\eta'}} &= c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t)\, \mathrm{ad}\,(q')^* d_{Q^{\xi'}} \\ &= c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t) \delta_{Q^{\xi'}}(q') d_{Q^{\xi'}} \\ &= c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t) \delta_{Q^{\xi'}}(q') (d_{Q^{\xi'}_{\eta'} \backslash Q^{\xi'}} \times d_{Q^{\xi'}_{\eta'}}). \end{aligned}$$

Taking (14) into consideration the lemma follows. □

For every $a \in A_{\mathcal{C}^{\xi}}$ we choose representatives $\eta \in X^{\xi}$ and $t \in a$ such that $\eta \cdot \xi = tw_n$.

LEMMA 3. *There is a constant $c \in \mathbb{R}^{\times}_{+}$ such that for $\xi \in Y$ and an integrable function f on X^{ξ} we have,*

$$\int_{X^{\xi}} f(x)\, d_{X^{\xi}} x = c \sum_{a \in A_{\mathcal{C}^{\xi}}} \delta_{\overline{B}}^{-\frac{1}{2}}(t) \int_{H^{\xi}_{\eta} \backslash H^{\xi}} \int_{\overline{B}} f(b\eta \dot{h})\, d^l_{\overline{B}} b\, d_{H^{\xi}_{\eta} \backslash H^{\xi}} \dot{h}. \tag{15}$$

Proof. We have

$$\int_{X^{\xi}} f(x)\, d_{X^{\xi}} x = \sum_{a \in A_{\mathcal{C}^{\xi}}} \int_{X^{\xi}_a} f(x)\, d_{X^{\xi}} x = \sum_{a \in A_{\mathcal{C}^{\xi}}} \int_{Q^{\xi}_{\eta} \backslash Q^{\xi}} f(\eta^{\dot{q}})\, \tilde{d}_{Q^{\xi}_{\eta} \backslash Q^{\xi}} \dot{q}.$$

From Lemma 2 it follows that

$$\int_{Q^\xi_\eta\backslash Q^\xi} f(\eta^{\dot q})\,\tilde d_{Q^\xi_\eta\backslash Q^\xi}\dot q = c\,\delta_{\overline{B}}^{-\frac12}(t)\int_{Q^\xi_\eta\backslash Q^\xi} f(\eta^{\dot q})\,d_{Q^\xi_\eta\backslash Q^\xi}\dot q$$

for some constant c independent of ξ, η and t. We now integrate over $Q^\xi_\eta\backslash Q^\xi$ in stages, first over

$$Q^\xi_\eta\backslash(\overline{B}\times H^\xi_\eta)\simeq\overline{B}$$

and then over

$$(\overline{B}\times H^\xi_\eta)\backslash Q^\xi\simeq H^\xi_\eta\backslash H^\xi.$$

With these isomorphisms, transforming the measure $d_{\overline{B}}$ from $\overline{B}$ to $Q^\xi_\eta\backslash(\overline{B}\times H^\xi_\eta)$ and the measure $d_{H^\xi_\eta\backslash H^\xi}$ from $H^\xi_\eta\backslash H^\xi$ to $(\overline{B}\times H^\xi_\eta)\backslash Q^\xi$, the integration in stages gives precisely the measure $d_{Q^\xi_\eta\backslash Q^\xi}\dot q$ on $Q^\xi_\eta\backslash Q^\xi$. We therefore obtain,

$$\int_{Q^\xi_\eta\backslash Q^\xi} f(\eta^{\dot q})\,d_{Q^\xi_\eta\backslash Q^\xi}\dot q = \int_{H^\xi_\eta\backslash H^\xi}\int_{\overline{B}} f(b^{-1}\eta\dot h)\,d_{\overline{B}}b\,d_{H^\xi_\eta\backslash H^\xi}\dot h.$$

Making the change of variables $b\mapsto b^{-1}$ and summing over $A_{\mathcal{C}_\xi}$ the lemma follows. □

3.2.2. Asymptotic of the period. For a variable $\lambda\in\mathfrak{a}_0^*$, by $\lim_{\lambda\to\infty}$ we will mean the limit as $\lambda_i-\lambda_{i+1}\to\infty$ for all $i=1,\dots,n-1$. Let δ_1 denote the delta function of the trivial class in A.

LEMMA 4. *Let $\xi\in K\cdot w_n$ then for all $a\in A$ and $\nu\in\mathcal{B}(1_T)$ we have*

$$\frac{d_{H_e^{w_n}}(H_e^{w_n})}{d_{H^{w_n}}(H^{w_n}\cap K)}\lim_{\lambda\to\infty}\tilde J^\xi_\nu(a,\varphi_0,\lambda)$$

equals $\delta_1(a)$ if E/F is unramified and equals $(\nu\nu_\omega)^{-1}(a)\,\mathrm{ch}_{A_{\mathcal{C}^{w_n}}}(a)$ if E/F is ramified.

Proof. For λ positive enough, the integrand in the period $J^\xi_\nu(a,\varphi_0,\lambda)$ is bounded uniformly by an integrable function. Indeed the more positive λ is, the smaller the integrand is and convergence for a fixed λ_0 follows from the global convergence proved in [LR03]. We may therefore apply Lebesgue's dominant convergence theorem and compute the limit inside the integral. Since φ_0 is K-invariant, it follows from (10) that it is enough to prove the lemma when $\xi=w_n$. Let $a\in A_{\mathcal{C}^{w_n}}$, $t\in a$ and $\eta\in G$ be such that

$$\eta\cdot w_n = tw_n.$$

We have

$$\tilde{J}_{\nu}^{w_n}(a,\varphi_0,\lambda)=\nu\nu_{\omega}(a^{-1})e^{-\frac{1}{2}\langle\lambda+\rho,H(t)\rangle}\int_{H_{\eta}^{w_n}\backslash H^{w_n}}e^{\langle\lambda+\rho,H(\eta\dot{h})\rangle}\,d_{H_{\eta}^{w_n}\backslash H^{w_n}}\dot{h}.$$

Let $x\in G$ be any element such that $x\cdot w_n=tw_n$. Decompose $x=\alpha uk$, using the Iwasawa decomposition $G=TUK$. Note that $xw_n=tw_n\theta(x)$ and therefore

$$H(x)=H(t)+H(w_n\theta(\alpha)w_n^{-1})+H(w_n\theta(u))=H(t)-H(x)+H(w_n\theta(u)).$$

We see that

$$H(x)=\frac{1}{2}[H(t)+H(w_n\theta(u))].$$

It is well known and easy to prove that $H(w_nu')$ is in the negative obtuse Weyl chamber for any $u'\in U$ and that it is strictly negative unless $u'\in K$, i.e.,

$$\lim_{\lambda\to\infty}e^{\langle\lambda,H(w_nu')\rangle}=\begin{cases}0 & u'\notin K\\ 1 & u'\in K\end{cases}$$

and therefore,

$$\lim_{\lambda\to\infty}e^{-\frac{1}{2}\langle\lambda,H(t)\rangle}e^{\langle\lambda,H(x)\rangle}=\begin{cases}0 & x\notin TK\\ 1 & x\in TK.\end{cases}$$

Applying this to $x=\eta\dot{h}$ we obtain

$$\lim_{\lambda\to\infty}\tilde{J}_{\nu}^{w_n}(a,\varphi_0,\lambda)=\nu\nu_{\omega}(a^{-1})\mathrm{vol}(V_a)$$

where $V_a=\{\dot{h}\in H_{\eta}^{w_n}\backslash H^{w_n}\mid\eta\dot{h}\subset TK\}$. Assume first that E/F is unramified. If a is not the trivial case $\mathrm{Nm}(T)$ and $\dot{h}\in V_a$ then, $\eta h=\alpha k$ for some $\alpha\in T$ and $k\in K$. We then have $\alpha k\cdot w_n=tw_n$ and hence $k\cdot w_n=\alpha^{-1}\cdot(tw_n)=\mathrm{Nm}\,(\alpha)^{-1}tw_n\in K$. Since t is not a norm, this is a contradiction. Thus V_a is empty. For $a=1$ we may assume that $t=e$ and take $\eta=e$. We then observe that if $\eta h=\alpha k$ the above argument implies that $\alpha\in K$. We therefore get that $h\in K$. Thus $V_a=H_e^{w_n}\backslash(H^{w_n}\cap K)$. This completes the lemma in the unramified case. If E/F is ramified then for every $a\in A$ we may choose a representative $t\in K\cap T$ whose diagonal entries are either 1 or a fixed non square unit in F. If $a\in A_{\mathcal{C}^{w_n}}$ then the number of non square units in the diagonal entries of t is even. For any unit $u\in\mathcal{O}_F^{\times}$ there exists $k\in GL_2(\mathcal{O}_E)$ such that $k\cdot w_2=\mathrm{diag}\,(u,u)w_2$. This fact follows from [Jac62] (see Proposition 8.1 for the non 2-adic case and Proposition 9.2.c for the 2-adic case). It follows that there exists $\eta\in K$ such that $\eta\cdot w_n=tw_n$. The same line of argument now shows that $V_a=H_{\eta}^{w_n}\backslash(H^{w_n}\cap K)$ for every $a\in A_{\mathcal{C}^{w_n}}$. □

COROLLARY 2. *Let $\xi \in K \cdot w_n$ then for all $\nu \in \mathcal{B}(1_T)$ we have,*

$$\frac{d_{H_e^{w_n}}(H_e^{w_n})}{d_{H^{w_n}}(H^{w_n} \cap K)} \lim_{\lambda \to \infty} J^{st,\xi}(\nu, \varphi_0, \lambda)$$

equals 1 *if E/F is unramified and equals $2^{n-1} \operatorname{ch}_{\{\nu_\omega, \omega\nu_\omega\}}(\nu)$ if E/F is ramified.*

Proof. For the unramified case, since

$$\delta_1(a) = \frac{d_{H_e^{w_n}}(H_e^{w_n})}{d_{H^{w_n}}(H^{w_n} \cap K))} \lim_{\lambda \to \infty} \tilde{J}^{\xi}_{\nu_\omega}(a, \varphi_0, \lambda)$$

we also have

$$\hat{\delta}_1(\nu^{-1}) = \frac{d_{H_e^{w_n}}(H_e^{w_n})}{d_{H^{w_n}}(H^{w_n} \cap K))} \lim_{\lambda \to \infty} J^{st,\xi}(\nu, \varphi_0, \lambda).$$

But the Fourier transform of the delta function at $a = 1$, is the constant function 1. For the ramified case we have similarly, the Fourier transform of the function $a \mapsto \nu_\omega^{-1}(a) \operatorname{ch}_{A_{\mathcal{C}w_n}}(a)$ evaluated at ν^{-1} is $2^{n-1} \operatorname{ch}_{\{\nu_\omega, \omega\nu_\omega\}}(\nu^{-1}) = 2^{n-1} \operatorname{ch}_{\{\nu_\omega, \omega\nu_\omega\}}(\nu)$. □

3.2.3. Hironaka's spherical functions on Hermitian forms. For $a \in A$ let $\mathcal{O}_a$ denote the $\overline{B}$-orbit containing aw_n. For $s = (s_1, \dots, s_n) \in \mathbb{C}^n$ and $a \in A$, Hironaka defined the following spherical function on Y,

$$\omega_a(y; s) = \int_K \operatorname{ch}_{\mathcal{O}_a}(k \cdot y) \prod_{i=1}^{n} |d_i(y \cdot x)|_F^{s_i} \, d_K k,$$

where the Haar measure satisfies $d_K(K) = 1$. The integral converges whenever $\operatorname{Re} s_i > 0$ for all $i < n$ and $\omega_a(y; s)$ admits a meromorphic continuation to a rational function of q^s. For a quadratic character χ of T, Hironaka also considered the spherical functions $L(y, \chi; s)$. The way Hironaka defined those spherical functions, they depend on the restriction of χ to T' but not on χ. Furthermore, for a ramified quadratic extension, the Hecke eigenvalue of the spherical functions $L(y, \chi; s)$ is different for different χ. It is therefore, more natural to define $L(y, \tau; s)$ for $\tau \in \mathcal{B}(1_T)$ using Hironaka's formulas. Thus, for $\tau = (\tau_1, \dots, \tau_n) \in \mathcal{B}(1_T)$ let

$$L(y, \tau; s) = \sum_{a \in A} \left(\prod_{i=1}^{n} \tau_i(d_i(a)) \right) \omega_a(y; s).$$

These are spherical functions for the symmetric space Y with a Hecke eigenfunction independent of τ. If E/F is unramified then the spherical functions $L(y, \tau; s)$

were computed explicitly in [Hir99]. In particular, if $\xi \in Y \cap K$ then

$$L(\xi, \tau; s) = \left(\prod_{i=1}^{n} \frac{L(\omega^{i+1}, i)}{L(1_{F^\times}, 1)}\right) \prod_{i<j} \frac{L(\nu_i \nu_j^{-1} \omega, \lambda_i - \lambda_j)}{L(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j + 1)} \tag{16}$$

where $\nu = (\nu_1, \dots, \nu_n) \in \mathcal{B}(1_T)$ is related to τ by

$$\nu_i = \omega^{n+1-i} \prod_{j=n+1-i}^{n} \tau_j \tag{17}$$

and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is related to s by

$$\lambda_i = \frac{n+1}{2} - i - (s_{n+1-i} + \cdots + s_n). \tag{18}$$

To obtain (16) from the results of ([Hir99]; pp. 569–571) we recall that $L(\mu, z + \sqrt{-1}\frac{\pi}{\log q}) = L(\mu\,\omega, z)$ for any character μ of $F^\times$ and $z \in \mathbb{C}$.

3.2.4. Proof of Proposition 3. The next lemma relates the stable intertwining operator to Hironaka's spherical functions. This will, in particular, imply Proposition 3.

LEMMA 5. *There is a constant c such that*

$$L(\xi, \tau; s) = c\, J^{st,\xi}(\nu, \varphi_0, \lambda) \tag{19}$$

for all $\xi \in Y$, $\nu \in \mathcal{B}(1_T)$ and $\lambda \in \mathbb{C}^n$, where ν and τ are related by (17) and λ and s are related by (18). If E/F is unramified then

$$c = \frac{d_{H_e^{w_n}}(H_e^{w_n})}{d_{H^{w_n}}(H^{w_n} \cap K))} \prod_{i=1}^{n} \frac{L(\omega^{i+1}, i)}{L(1_{F^\times}, 1)}. \tag{20}$$

Proof. We normalize d_G so that $d_G(K) = 1$. Let $\xi \in Y$ and let $\mathcal{C}^\xi$ denote the G-orbit of ξ. We recall that d_{X^ξ} is the restriction of d_G to X^ξ. For $a \in A_{\mathcal{C}^\xi}$ let $t \in a$ and $\eta' \in G$ be such that $\eta' \cdot \xi = tw_n$. Let ${}^{w_n}t = w_n t w_n^{-1}$, ${}^{w_n}a = w_n a w_n^{-1}$ and set $\eta = w_n \eta'$. Then, $\eta \cdot \xi = ({}^{w_n}t)w_n \in ({}^{w_n}a)w_n$. Since

$$\mathrm{ch}_{\mathcal{O}_a}(g \cdot \xi) = \mathrm{ch}_{X_a^\xi}(g),\ g \in G,$$

we have

$$\omega_a(\xi; s) = \int_{X_a^\xi} \mathrm{ch}_K(x) \prod_{i=1}^{n} |d_i(x \cdot \xi)|_F^{s_i}\, d_{X^\xi} x.$$

It follows from Lemma 3, that there is a positive constant c, independent of ξ and of a, such that

$$\omega_a(\xi; s) = c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t) \int_{H^{\xi}_{\eta'}\backslash H^{\xi}} \left[\int_{\overline{B}} \mathrm{ch}_K\,(b\eta'\dot{h}) \prod_{i=1}^{n} |d_i(b \cdot w_n t)|_F^{s_i}\, d^l_{\overline{B}} b \right] d_{H^{\xi}_{\eta'}\backslash H^{\xi}}\dot{h}. \tag{21}$$

If $t = \mathrm{diag}\,(t_1, \ldots, t_n)$ and $b = t_b u_b$, with $t_b = \mathrm{diag}\,(b_1, \ldots, b_n)$ and u_b lower triangular unipotent then,

$$d_i(b \cdot t w_n) = t_1 \cdots t_i\ \mathrm{Nm}\,(b_1 \cdots b_i).$$

The inner integral of (21) is then equal to,

$$e^{\frac{1}{2}\langle \rho - \lambda, H({}^{w_n}t)\rangle} \int_{\overline{B}} \mathrm{ch}_K\,(b\eta'\dot{h}) e^{\langle \rho - \lambda, H(w_n b w_n^{-1})\rangle}\, d^l_{\overline{B}} b.$$

After a change of variables $b \mapsto w_n^{-1} b w_n$ it becomes

$$e^{\frac{1}{2}\langle \rho - \lambda, H({}^{w_n}t)\rangle} \int_{B} \mathrm{ch}_K\,(b\eta\dot{h}) e^{\langle \rho - \lambda, H(b)\rangle}\, d^l_B b, \tag{22}$$

where d^l_B is now the left Haar measure on B normalized so that $d^l_B(B \cap K) = 1$. There is a projection $\mathcal{P}_\lambda$: $C_c^\infty(G) \to I(\lambda)$ defined by

$$\mathcal{P}_\lambda(\phi)(g) = e^{-\langle \lambda, H(g)\rangle} \int_B \phi(bg) e^{\langle \rho - \lambda, H(b)\rangle}\, d^l_B b$$

and it satisfies $\varphi_0 = \mathcal{P}_\lambda(\mathrm{ch}_K)$. Plugging this into (22) and (22) into (21) and observing that $H^{\xi}_{\eta} = H^{\xi}_{\eta'}$ we get,

$$\omega_a(\xi; s) = c\, \delta_{\overline{B}}^{-\frac{1}{2}}(t) e^{\frac{1}{2}\langle \rho - \lambda, H({}^{w_n}t)\rangle} \int_{H^{\xi}_{\eta}\backslash H^{\xi}} e^{\langle \lambda, H(\eta\dot{h})\rangle} \varphi_0(\eta\dot{h})\, d_{H^{\xi}_{\eta}\backslash H^{\xi}}\dot{h}.$$

Since $\delta_{\overline{B}}^{-\frac{1}{2}}(t) = e^{\langle \rho, H(t)\rangle} = e^{-\langle \rho, H({}^{w_n}t)\rangle}$ we get that

$$\omega_a(\xi; s) = c\tilde{J}^{\xi}_{\nu_\omega}({}^{w_n}a, \varphi_0, \lambda). \tag{23}$$

Taking into account the fact that $\nu = \nu^{-1}$ for $\nu \in \mathcal{B}(1_T)$ we get (19). We now assume that E/F is unramified. Since (19) holds for all ξ, to evaluate c we may as well assume that $\xi \in K$. From (16) and (19) we get that

$$J^{st,\xi}(\nu, \varphi_0, \lambda)\ =\ c^{-1} \left(\prod_{i=1}^{n} \frac{L(\omega^{i+1}, i)}{L(1_F, 1)} \right) \prod_{i<j} \frac{L(\nu_i \nu_j^{-1}\omega, \lambda_i - \lambda_j)}{L(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j + 1)}. \tag{24}$$

We now take the limit as $\lambda \to \infty$ on both sides of (24). Since

$$\lim_{s\to\infty} L(\mu, s+d) = 1$$

for any character μ of $F^\times$, (20) follows from Corollary 2. $\square$

Proposition 3 readily follows. $\square$

4. Global stabilization—Proof of Theorem 1. Fix a unitary character χ of $T\backslash T_{\mathbb{A}}$ and let $\nu \in \mathcal{B}(\chi)$. Let $\mathcal{C} \in G\backslash Y$ and choose $\xi \in \mathcal{C}$. The intertwining period is defined for λ with $\operatorname{Re}\lambda$ sufficiently large, by the integral

$$J^{\xi}(a, \varphi, \lambda) = \int_{(H^{\xi}_{\eta})_{\mathbb{A}}\backslash H^{\xi}_{\mathbb{A}}} e^{\langle \lambda, H(\eta h)\rangle} \varphi(\eta h)\, d_{H^{\xi}_{\eta}\backslash H^{\xi}} \dot{h},$$

where $a \in A_{\mathcal{C}}$ and $\eta \in G$ is such that $\eta \cdot \xi \in a w_n$. The integral is independent of the choice of η and the dependence on the choice of ξ is given by the global analogue of (10). It is proved in [LR03] that the integral converges for $\operatorname{Re}\lambda$ sufficiently large and that with our conventions on measures

$$\Pi^{H^{\xi}}(E(\varphi, \lambda)) = (d_{E_1\backslash(E_1)_{\mathbb{A}}}(E_1\backslash(E_1)_{\mathbb{A}})^n \sum_{a\in A_{\mathcal{C}}} J^{\xi}(a, \varphi, \lambda).$$

We recall that the individual summands on the right-hand side are not expected to have a meromorphic continuation ([LR00]; Remark 3).

For $\varphi = \otimes_v \varphi_v \in I(\chi)$, $\mathcal{C} \in G_{\mathbb{A}}\backslash Y_{\mathbb{A}}$, $\xi \in \mathcal{C}$ and λ with $\operatorname{Re}\lambda$ sufficiently large define,

$$J^{st,\xi}(\nu, \varphi, \lambda) = \prod_v J^{st,\xi_v}(\nu_v, \varphi_v, \lambda). \tag{25}$$

By Proposition 3 and (13) the product converges and the Fourier inversion of §2.2 can be applied. For every place v of F define the function

$$g_v(a_v) = \operatorname{ch}_{\mathcal{C}_v}(a_v w_n) \tilde{J}^{\xi_v}_{\nu_v}(a_v, \varphi_v, \lambda)$$

on A_v and set $g(a) = \prod_v g_v(a_v)$, $a \in A_{\mathbb{A}}$. By (8) we have

$$g(a) = \operatorname{ch}_{\mathcal{C}}(a w_n) J^{\xi}(a, \varphi, \lambda)$$

for $a \in A$. We have

$$2^n \sum_{a\in A} g(a) = \sum_{\kappa\in(A_{\mathbb{A}}/A)^*} \hat{g}(\kappa) = \sum_{\kappa\in(A_{\mathbb{A}}/A)^*} \prod_v \hat{g}_v(\kappa_v).$$

Note that

$$\hat{g}_v(\kappa_v^{-1}) = J^{st,\xi_v}(\kappa_v \nu_v, \varphi_v, \lambda)$$

and that $\kappa\nu$ ranges over $\mathcal{B}(\chi)$ as κ ranges over the characters of $A_{\mathbb{A}}/A$. We obtain

$$\Pi^{H^\xi}(E(\varphi,\lambda)) = 2^{-n}(d_{E_1\backslash(E_1)_{\mathbb{A}}}(E_1\backslash(E_1)_{\mathbb{A}})^n \sum_{\nu\in\mathcal{B}(\chi)} J^{st,\xi}(\nu,\varphi,\lambda). \tag{26}$$

To complete the proof of Theorem 1 we need the following:

LEMMA 6. *The global stable intertwining periods $J^{st,\xi}(\nu,\varphi,\lambda)$ admit a meromorphic continuation in λ.*

Proof. We fix an auxiliary p-adic inert place w of F. For convenience we choose w such that $\omega_w(\det(\xi_w w_n)) = 1$. For $\nu_w \in \mathcal{B}(\chi_w)$ there exist $\varphi_w \in I(\chi_w)$ so that $J^{st,\xi_w}(\nu'_w, \varphi_w, \lambda) = \delta_{[\nu_w],[\nu'_w]}$. To see this, we look back at the definition of the local stable periods. Fix representatives $\{\eta\}$ as in §3. As observed in the proof of Lemma 6 of [LR00], the periods

$$(J^{\xi_w}(\eta,\varphi_w,\lambda))_\eta \tag{27}$$

are integrals over the 2^{n-1} disjoint open $H_w^{\xi_w}$-orbits in $B_w\backslash G_w$ and are linearly independent. The vector (27) can therefore be arbitrary in $\mathbb{C}^{2^{n-1}}$ for different choices of φ_w. Write $\mathcal{B}(\chi_w) = \mathcal{B}_1(\chi_w)\cup\mathcal{B}_2(\chi_w)$ so that for all $\nu_w \in \mathcal{B}(\chi_w)$ the set $[\nu_w]\cap\mathcal{B}_i(\chi_w)$ contains a unique element for $i = 1,2$. It is also observed in [loc. cit.] that the $2^{n-1}\times 2^{n-1}$ matrix $\Delta = (\Delta^{\xi_w}_{\nu_w,\eta}(\lambda)^{-1})_{\nu_w\in\mathcal{B}_i(\chi_w),\eta}$ is invertible (and it is independent of i by our assumption on w). Since

$$(J^{st,\xi_w}(\nu_w,\varphi_w,\lambda))_{\nu_w\in\mathcal{B}_i(\chi_w)} = \Delta(J^{\xi_w}(\eta,\varphi_w,\lambda))_\eta$$

we may choose φ_w as desired. From (26), we now get that whenever $\varphi = \varphi_w\otimes\varphi^w$ with $\varphi^w \in \otimes'_{v\neq w} I(\chi_v)$ we have

$$\Pi^{H^\xi}(E(\varphi,\lambda)) = 2^{1-n}(d_{E_1\backslash(E_1)_{\mathbb{A}}}(E_1\backslash(E_1)_{\mathbb{A}})^n J^{st,\xi}(\nu,\varphi,\lambda).$$

This gives the meromorphic continuation of the functional on $\otimes'_{v\neq w} I(\chi_v)$ defined for decomposable elements by

$$\otimes_{v\neq w}\varphi_v \mapsto \prod_{v\neq w} J^{st,\xi_v}(\nu_v,\varphi_v,\lambda).$$

The lemma now follows from Proposition 1. □

This completes the proof of Theorem 1. □

Corollary 1 now follows from Proposition 3 and (13). □

Remark 1. From Lemma 6, we also get the meromorphic continuation of the local stable periods. Indeed, the meromorphic continuation is given by Proposition 1 for the p-adic inert places and by Proposition 2 in the split places. To obtain the meromorphic continuation in the inert archimedean case, we now apply Lemma 6 to a global quadratic extension with a single inert archimedean place, say $\mathbb{Q}[i]/\mathbb{Q}$.

We make another observation that will be useful later. For $\nu \in \mathcal{B}(\chi)$ and $\xi \in Y_{\mathbb{A}}$ it follows from (11) that,

$$J^{st,\xi}(\omega\nu, \varphi, \lambda) = \omega(\det \xi) J^{st,\xi}(\nu, \varphi, \lambda).$$

In the notation of §2.1, it follows that whenever $\det \xi \notin X$ we have,

$$J^{st,\xi}(\nu, \varphi, \lambda) + J^{st,\xi}(\omega\nu, \varphi, \lambda) = 0. \tag{28}$$

5. Local stable relative Bessel distributions. The local Whittaker functional is defined by the integral

$$\mathcal{W}(\varphi, \lambda) = \int_U e^{\langle \lambda, H(w_n u)\rangle} \varphi(w_n u) \overline{\psi_U(u)}\, du \tag{29}$$

for $\varphi \in I(\chi, \lambda)$. It converges absolutely for $\operatorname{Re}\lambda$ sufficiently large and admits an analytic continuation. Let $\Phi \in C_c^\infty(Y)$ and $\nu \in \mathcal{B}(\chi)$. The local stable relative Bessel distribution is defined by

$$\tilde{B}^{st}(\Phi, \nu, \lambda) = \sum_{\varphi} \left[\int_Y \Phi(y) J^{st,y}(\nu, \varphi, \lambda)\, d_Y y \right] \overline{\mathcal{W}}(\varphi, -\bar{\lambda}), \tag{30}$$

where the sum is over an orthonormal basis of $I(\chi)$. It is meromorphic in λ. For $\xi \in Y$ we also define the ξ-th local relative Bessel distribution of G by

$$\tilde{B}^{st,\xi}(f, \nu, \lambda) = \sum_{\varphi} J^{st,\xi}(\nu, I(f, \chi, \lambda)\varphi, \lambda) \overline{\mathcal{W}}(\varphi, -\bar{\lambda}).$$

Choose a set of representatives $\{\xi\}$ in Y for $G\backslash Y$ and a family of functions $\{f^\xi\}$ in $C_c^\infty(G)$ such that

$$\Phi(g^{-1} \cdot \xi) = \int_{H^\xi} f^\xi(hg)\, d_{H^\xi} h. \tag{31}$$

LEMMA 7.

$$\tilde{B}^{st}(\Phi, \nu, \lambda) = \sum_{\xi} \tilde{B}^{st,\xi}(f^\xi, \nu, \lambda).$$

Proof. Fix an orbit $\mathcal{C} = G \cdot \xi \in G\backslash Y$. Let $a \in A_{\mathcal{C}}$ and let $\eta \cdot \xi = tw_n \in aw_n$. Then,

$$J^{\xi}(\eta, I(f^{\xi}, \chi, \lambda)\varphi, \lambda) = \int_{H^{\xi}_{\eta}\backslash H^{\xi}} \int_{G} f^{\xi}(x) e^{\langle \lambda, H(\eta \dot{h} x)\rangle} \varphi(\eta \dot{h} x)\, d_G x\, d_{H^{\xi}_{\eta}\backslash H^{\xi}} \dot{h}.$$

Making the change of variables $x \mapsto \dot{h}^{-1}x$, the last expression becomes

$$\int_{H^{\xi}_{\eta}\backslash H^{\xi}} \int_{G} f^{\xi}(\dot{h}^{-1}x) e^{\langle \lambda, H(\eta x)\rangle} \varphi(\eta x)\, d_G x\, d_{H^{\xi}_{\eta}\backslash H^{\xi}} \dot{h}.$$

Integrating over G in stages, first over H^{ξ}_{η} and then over $H^{\xi}_{\eta}\backslash G$ and changing the order of integration we obtain

$$\begin{aligned}
&\int_{H^{\xi}_{\eta}\backslash G} \left[\int_{H^{\xi}} f^{\xi}(h\dot{x})\, d_{H^{\xi}} h\right] e^{\langle \lambda, H(\eta \dot{x})\rangle} \varphi(\eta \dot{x})\, d_{H^{\xi}_{\eta}\backslash G} \dot{x} \\
&\qquad = \int_{H^{\xi}_{\eta}\backslash G} \Phi(\dot{x}^{-1} \cdot \xi) e^{\langle \lambda, H(\eta \dot{x})\rangle} \varphi(\eta \dot{x})\, d_{H^{\xi}_{\eta}\backslash G} \dot{x} \\
&\qquad = \int_{H^{\xi}\backslash G} \Phi(\dot{x}^{-1} \cdot \xi) J^{\xi}(\eta, I(\dot{x}, \chi, \lambda)\varphi, \lambda)\, d_{H^{\xi}\backslash G} \dot{x}.
\end{aligned}$$

It now follows from (10) that

$$\begin{aligned}
\tilde{J}^{\xi}_{\nu}(a, I(f^{\xi}, \chi, \lambda)\varphi, \lambda) &= \int_{H^{\xi}\backslash G} \Phi(\dot{x}^{-1} \cdot \xi) \tilde{J}^{\dot{x}^{-1}\cdot\xi}_{\nu}(a, \varphi, \lambda)\, d_{H^{\xi}\backslash G} \dot{x} \\
&= \int_{\mathcal{C}} \Phi(y) \tilde{J}^{y}_{\nu}(a, \varphi, \lambda)\, d_Y y.
\end{aligned}$$

Note that this implies in particular that the period $\tilde{J}^{\xi}_{\nu}(a, I(f^{\xi}, \chi, \lambda)\varphi, \lambda)$ is independent of the choice of representative ξ and the choice of f^{ξ} representing Φ. Summing over $a \in A$ we obtain

$$J^{st,\xi}(\nu, I(f^{\xi}, \chi, \lambda)\varphi, \lambda) = \int_{\mathcal{C}} \Phi(y) J^{st,y}(\nu, \varphi, \lambda)\, d_Y y.$$

Summing over the representatives of the orbits in $G\backslash Y$ we obtain

$$\sum_{\xi} J^{st,\xi}(\nu, I(f^{\xi}, \chi, \lambda)\varphi, \lambda) = \int_{Y} \Phi(y) J^{st,y}(\nu, \varphi, \lambda)\, d_Y y. \tag{32}$$

The lemma follows. ∎

The local Bessel distribution on G' is defined by

$$B'(f',\nu,\lambda) = \sum_{\varphi'} \mathcal{W}'(I(f',\nu,\lambda)\varphi',\lambda)\overline{\mathcal{W}}'(\varphi',-\bar{\lambda}).$$

It is holomorphic in λ. Here $\mathcal{W}'(\varphi',\lambda)$ is the analogue of (29) for G' and the sum is over an orthonormal basis of $I'(\nu)$.

Remark 2. We wish to stress the dependence of the local distributions at hand on the choices of measures. Note that the choice of an orthonormal basis for $I(\chi)$ is inverse proportional to $d_{B\backslash G}$, that $\overline{\mathcal{W}}(\varphi,-\bar{\lambda})$ is proportional to d_U and that with our conventions on measures the stable intertwining period is proportional to $d_{H_e^{w_n}\backslash H^{w_n}}$ and d_Y is proportional to $d_{H^{w_n}\backslash G}$. Taking all this into account, the distribution $\tilde{B}^{st}(\Phi,\nu,\lambda)$ depends on the measures on U, B and $H_e^{w_n}$ but it is independent on the measures on Y on G and on the unitary groups. Replacing $(d_U, d_B, d_{H_e^{w_n}})$ by $(\alpha d_U, \beta d_B, \gamma d_{H_e^{w_n}})$ changes the relative stable Bessel distribution by the factor $\alpha\beta\gamma^{-1}$. Similarly, the distribution $B'(f',\nu,\lambda)$ depends only on the measures on U' and B'. Replacing $(d_{B'}, d_{U'})$ by $(\alpha d_{B'}, \beta d_{U'})$ changes the Bessel distribution by the factor $\alpha\beta^2$. Note also that the matching condition is proportional to d_U and homogenous of degree 2 in $d_{U'}$. All in all, this explains the proportionality constant $[d_B\colon\ d_{B'} \times d_{H_e^{w_n}}]$ that appears in Theorem 3.

5.1. Local Bessel identity—split case. As in [LR00], we obtain the Bessel identity in the split case. Recall that in this case ω is the trivial character. Let $\Phi \in C_c^\infty(Y)$ and let $f'(g) = (d_U : d_{U'}\, d_{U'})\Phi(\vartheta(g)^{-1}, g)$ for $g \in G'$. We then have $\Phi \overset{\delta}{\leftrightarrow} f'$ for $\delta \in \{0,1\}$.

PROPOSITION 4.

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = e(\psi)^{-\dim U'}[d_B\colon\ d_{B'} \times d_{H_e^{w_n}}]\gamma(\nu,\lambda,\psi)B'(f',\nu,\lambda).$$

Proof. It is enough to prove the proposition for $\lambda \in i\mathfrak{a}_0^*$ since all terms involved are meromorphic. As observed in [LR00], $\varphi \mapsto \vartheta(\varphi)(g) = \varphi(\vartheta(g))$ is a self-adjoint operator on $I'(\nu)$. By Proposition 2,

$$\begin{aligned} J^{st,y}(\nu,\varphi_1'\otimes\varphi_2',\lambda) \;=\; & (d_{H_e^{w_n}\backslash H^{w_n}} : d_{T'\backslash G'})(d_{B'} : d_{U'}\, d_{T'}) \\ & \times (\vartheta \circ M'(w,\lambda) \circ I'(g,\nu,\lambda)\varphi_2', \overline{\varphi}_1') \end{aligned}$$

for $y = (g, \vartheta(g)^{-1}) \in Y$. Let $f(g) = \Phi(g, \vartheta(g)^{-1})$. We have,

$$\begin{aligned} \int_Y \Phi(y) J^{st,y}(\nu,\varphi_1\otimes\varphi_2,\lambda)\, d_Y y \;=\; & (d_{H_e^{w_n}\backslash H^{w_n}} : d_{T'\backslash G'})(d_{B'} : d_{U'}\, d_{T'})(d_{H^{w_n}\backslash G} : d_{G'}) \\ & \times \int_{G'} f(g)(\vartheta \circ M'(w,\lambda) \circ I'(g,\nu,\lambda)\varphi_2', \overline{\varphi}_1')\, dg \end{aligned}$$

$$= \left(\frac{d_{H_e^{w_n}\backslash H^{w_n}}}{d_{T'\backslash G'}}\right)\left(\frac{d_{B'}}{d_{U'}\,d_{T'}}\right)\left(\frac{d_{H^{w_n}\backslash G}}{d_{G'}}\right) \times (\vartheta \circ M'(w,\lambda) \circ I'(f,\nu,\lambda)\varphi_2', \overline{\varphi}_1').$$

If $\{\varphi_i'\}$ is an orthonormal basis of $I'(\nu)$ then the set $\{(d_{B'\backslash G'} \times d_{B'\backslash G'} : d_{B\backslash G})^{\frac{1}{2}}\varphi_i' \otimes \varphi_j'\}$ is an orthonormal basis of $I(\chi)$. Thus,

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = \left(\frac{d_{H_e^{w_n}\backslash H^{w_n}}}{d_{T'\backslash G'}}\right)\left(\frac{d_{B'}}{d_{U'}\,d_{T'}}\right)\left(\frac{d_{H^{w_n}\backslash G}}{d_{G'}}\right)\left(\frac{d_{B'\backslash G'} \times d_{B'\backslash G'}}{d_{B\backslash G}}\right)\left(\frac{d_U}{d_{U'}\,d_{U'}}\right) \times \sum_{i,j}(\vartheta \circ M'(w,\lambda) \circ I'(f,\nu,\lambda)\varphi_i', \overline{\varphi}_j')\overline{\mathcal{W}}(\varphi_i,\lambda)\overline{\mathcal{W}}(\varphi_j,\lambda).$$

Note that

$$\left(\frac{d_{H_e^{w_n}\backslash H^{w_n}}}{d_{T'\backslash G'}}\right)\left(\frac{d_{B'}}{d_{U'}\,d_{T'}}\right)\left(\frac{d_{H^{w_n}\backslash G}}{d_{G'}}\right)\left(\frac{d_{B'\backslash G'} \times d_{B'\backslash G'}}{d_{B\backslash G}}\right)\left(\frac{d_U}{d_{U'}\,d_{U'}}\right) = \left(\frac{d_B}{d_{B'}\,d_{H_e^{w_n}}\,d_{U'}}\right)\left(\frac{d_U}{d_{U'}\,d_{U'}}\right).$$

Here the map $(b_1,b_2) \mapsto (b_1,(t_2,t_2^{-1}),u_2)$ is an isomorphism from $B = B' \times B'$ to $B' \times H_e^{w_n} \times U'$ where $b_2 = t_2u_2$ with $u_2 \in U'$. By a special case of a result of Shahidi we have the local functional equation

$$\mathcal{W}(M'(w,\lambda)\varphi', w\lambda) = e(\psi)^{-\dim U'} d_{U'}(U_0')\gamma(\nu,\lambda,\psi)\mathcal{W}(\varphi',\lambda)$$

for $\varphi' \in I'(\nu)$ [Sha81]. The same computation as that of Proposition 4 of [LR00] now gives

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = e(\psi)^{-\dim U'} d_{U'}(U_0')\left(\frac{d_B}{d_{B'}\,d_{H_e^{w_n}}\,d_{U'}}\right)\left(\frac{d_U}{d_{U'}\,d_{U'}}\right)\gamma(\nu,\lambda,\psi) \sum_i \mathcal{W}'(I'(\vartheta(f^\circ),\nu,\lambda)\varphi_i',\lambda)\overline{\mathcal{W}}'(\varphi_i',\lambda)$$

where $\vartheta(f^\circ)(g) = f(\vartheta(g)^{-1})$. Since $f' = (d_U : d_{U'}\,d_{U'})\vartheta(f^\circ)$ and $d_{U'}(U_0')(d_B : d_{B'}\,d_{H_e^{w_n}}\,d_{U'}) = [d_B\colon\, d_{B'} \times d_{H_e^{w_n}}]$, the proposition follows. □

5.2. Local Bessel identity—unramified case. Here E/F is an unramified quadratic extension of p-adic fields and ψ has conductor $\mathcal{O}_F$. The Hecke algebra $\mathcal{H}_G$ acts on the space of compactly supported K-invariant functions on Y by the convolution

$$f * \Phi(s) = \int_G f(g)\Phi(g^{-1} \cdot s)\, d_G g.$$

Let Φ_0 be the characteristic function of $K \cap Y$, let $f \in \mathcal{H}_G$ and let

$$f' = \frac{d_G(K) d_U(U \cap K)}{d_{U'}(U' \cap K')^2}\,\mathrm{bc}\,(f).$$

By the fundamental lemma of Jacquet, in the case of odd residual characteristic we have

$$f^\vee * \Phi_0 \overset{\delta}{\leftrightarrow} f'$$

for $\delta \in \{0, 1\}$ where $f^\vee(g) = f(g^{-1})$ [Jac05]. Note that the matching is independent of δ since f' is supported in $\ker(\omega \circ \det)$ whenever f' is in the image of base change.

Proposition 5. *Let ν be an unramified, unitary character of B. Then,*

$$\tilde{B}^{st}(f^\vee * \Phi_0, \nu, \lambda) = [d_B\colon\ d_{B'} \times d_{H_e^{w_n}}]\gamma(\nu, \lambda, \psi) B'(f', \nu, \lambda). \tag{33}$$

Proof. The set $Y \cap K = K \cdot w_n$ is a unique K-orbit, and therefore the function $\Phi = f^\vee * \Phi_0$ is supported in $G \cdot w_n$. If $\xi \in Y$ is such that $\{w_n, \xi\}$ is a set of representatives for the two G-orbits in Y, we choose functions f^{w_n} and f^ξ as in (31). We may take f^ξ=0. It is noted by Jacquet in [Jac05] that we may also take $f^{w_n} = d_G(K) d_{H^{w_n}}(H^{w_n} \cap K)^{-1} f$. By Lemma 7, $\tilde{B}^{st}(f^\vee * \Phi_0, \nu, \lambda)$ equals

$$\begin{aligned} &d_G(K) d_{H^{w_n}}(H^{w_n} \cap K))^{-1}(\varphi_0, \varphi_0)^{-1} J^{st, w_n}(\nu, I(f, \chi, \lambda)\varphi_0, \lambda)\, \overline{\mathcal{W}}(\varphi_0, -\bar{\lambda}) \\ &\quad = d_G(K)^2 d_{H^{w_n}}(H^{w_n} \cap K))^{-1}(\varphi_0, \varphi_0)^{-1} \hat{f}(\chi, \lambda) J^{st, w_n}(\nu, \varphi_0, \lambda)\, \overline{\mathcal{W}}(\varphi_0, -\bar{\lambda}). \end{aligned}$$

Note that $(\varphi_0, \varphi_0)^{-1} d_G(K) = d_B(B \cap K)$. By Proposition 3 we therefore obtain,

$$\tilde{B}^{st}(f^\vee * \Phi_0, \nu, \lambda) = d_G(K) \frac{d_B(B \cap K)}{d_{H_e^{w_n}}(H_e^{w_n})} \hat{f}(\chi, \lambda) \overline{\mathcal{W}}(\varphi_0, -\bar{\lambda}) \prod_{i<j} \frac{L(\nu_i \nu_j^{-1} \omega, \lambda_i - \lambda_j)}{L(\nu_i \nu_j^{-1}, \lambda_i - \lambda_j + 1)}.$$

Note that

$$\frac{d_B(B \cap K)}{d_{B'}(B' \cap K') d_{H_e^{w_n}}(H_e^{w_n})} = [d_B\colon\ d_{B'} \times d_{H_e^{w_n}}].$$

The rest of the proof applies the formula of [CS80] for the spherical Whittaker function and proceeds the same way as in Proposition 5 of [LR00]. □

6. Global Bessel distributions. We now turn to the global setting. Fix a set of representatives $\{\xi\}$ in $Y_{\mathbb{A}}$ for $G_{\mathbb{A}}\backslash Y_{\mathbb{A}}$ so that $\xi \in Y$ whenever it is the representative of an orbit in $i(G\backslash Y)$. For a function Φ on $Y_{\mathbb{A}}$ let $\{f^{\xi}\}$ be smooth functions of compact support on $G_{\mathbb{A}}$ such that

$$\Phi(g^{-1}\cdot\xi) = \int_{H^{\xi}_{\mathbb{A}}} f^{\xi}(hg)\, d_{H^{\xi}}h.$$

The global relative Bessel distribution is defined by

$$\tilde{B}(\Phi,\chi,\lambda) = \sum_{\varphi}\left[\sum_{\xi\in i(G\backslash Y)} \Pi^{H^{\xi}}(E(I(f^{\xi},\chi,\lambda)\varphi,\lambda))\right]\overline{\mathcal{W}}(\varphi,-\bar{\lambda}),$$

where the outer sum is over an orthonormal basis $\{\varphi\}$ of $I(\chi)$. It is meromorphic in λ. The sum is independent of the choice of basis and we will see later that it is independent of the choice of representatives $\{\xi\}$ and functions $\{f^{\xi}\}$ representing Φ. For $\nu \in \mathcal{B}(\chi)$ we also define the global stable relative Bessel distribution on $Y_{\mathbb{A}}$

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = \sum_{\varphi}\left[\int_{Y_{\mathbb{A}}} \Phi(y) J^{st,y}(\nu,\varphi,\lambda)\, d_Y y\right]\overline{\mathcal{W}}(\varphi,-\bar{\lambda}) \tag{34}$$

and the ξ-th relative Bessel distribution on $G_{\mathbb{A}}$

$$\tilde{B}^{st,\xi}(f,\nu,\lambda) = \sum_{\varphi} J^{st,\xi}(\nu, I(f,\chi,\lambda)\varphi,\lambda)\overline{\mathcal{W}}(\varphi,-\bar{\lambda}).$$

They are meromorphic in λ. Set

$$A(\chi,\lambda) = \prod_{i<j} L(\chi_i^{-1}\chi_j, \lambda_j - \lambda_i + 1).$$

It follows from [CS80] and our local unramified computation that

$$\prod_{v} A_v(\chi_v,\lambda)\tilde{B}^{st}_v(\Phi_v,\nu_v,\lambda)$$

converges for $\operatorname{Re}\lambda$ positive enough and we get that

$$\tilde{B}^{st}(\Phi,\nu,\lambda) = \frac{1}{A(\chi,\lambda)}\prod_{v} A_v(\chi_v,\lambda)\tilde{B}^{st}_v(\Phi_v,\nu_v,\lambda).$$

From Lemma 7 we therefore obtain also globally,

$$\tilde{B}^{st}(\Phi,\nu,\lambda)=\sum_{\xi\in G_{\mathbb{A}}\backslash Y_{\mathbb{A}}}\tilde{B}^{st,\xi}(f^{\xi},\nu,\lambda).$$

From (28) we get that

$$\sum_{\nu\in\mathcal{B}(\chi)}\tilde{B}^{st}(\Phi,\nu,\lambda)=\sum_{\nu\in\mathcal{B}(\chi)}\sum_{\xi\in i(G\backslash Y)}\tilde{B}^{st,\xi}(f^{\xi},\nu,\lambda),$$

where the inner sum is now only over orbits with rational representatives. Therefore, combined with (26) we obtain

$$\tilde{B}(\Phi,\chi,\lambda)=2^{-n}(\operatorname{vol}(E_1\backslash(E_1)_{\mathbb{A}})^n\sum_{\nu\in\mathcal{B}(\chi)}\tilde{B}^{st}(\Phi,\nu,\lambda). \tag{35}$$

The identity of trace formulas of Jacquet [Jac05], compares between the RTF for Y and the KTF for G'. For $\delta\in\{0,1\}$ and for δ-matching functions $\Phi\overset{\delta}{\leftrightarrow}f'$ we have

$$\mathrm{RTF}(\Phi)=\mathrm{KTF}(f') \tag{36}$$

where

$$\mathrm{RTF}(\Phi)=\int_{U\backslash U_{\mathbb{A}}}K_{\Phi}(u)\psi_U(u)\,d_Uu,$$

$$\mathrm{KTF}(f')=\int_{U'\backslash U'_{\mathbb{A}}}\int_{U'\backslash U'_{\mathbb{A}}}K_{f'}(u_1^{-1},u_2)\psi_{U'}(u_1u_2)\,d_{U'}u_1\,d_{U'}u_2,$$

$$K_{\Phi}(g)=\sum_{\xi\in Y}\Phi(g^{-1}\cdot\xi)\text{ and }K_{f'}(x,y)=\sum_{\gamma\in G'}f'(x^{-1}\gamma y).$$

The fine spectral expansion of (36), is now available thanks to Lapid [Lap06]. The most continuous contribution to the left-hand side is the sum over all unitary characters χ of $T\backslash T_{\mathbb{A}}$ that are base change, of

$$\frac{d_{H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}}(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}})}{n!2^n d_{B\backslash B^1_{\mathbb{A}}}(B\backslash B^1_{\mathbb{A}})}\int_{i\mathfrak{a}_0^*}\sum_{\nu\in\mathcal{B}(\chi)}\tilde{B}^{st}(\Phi,\nu,\lambda)\,d\lambda$$

whereas on the right-hand side it is the sum over the same set, of

$$\frac{1}{n!d_{B'\backslash(B'_{\mathbb{A}})^1}(B'\backslash(B'_{\mathbb{A}})^1)}\int_{i\mathfrak{a}_0^*}\sum_{\nu\in\mathcal{B}(\chi)}B'(f',\nu,\lambda)\,d\lambda.$$

Here we assume that

$$d_{B_{\mathbb{A}}} = d_{B^1_{\mathbb{A}}\backslash B_{\mathbb{A}}} \times d_{B^1_{\mathbb{A}}} \text{ and } d_{B'_{\mathbb{A}}} = d_{B'^1_{\mathbb{A}}\backslash B'_{\mathbb{A}}} \times d_{B'^1_{\mathbb{A}}},$$

where

$$(d_{B^1_{\mathbb{A}}\backslash B_{\mathbb{A}}} : d_{(\mathbb{R}_+^\times)^n}) = (d_{B'^1_{\mathbb{A}}\backslash B'_{\mathbb{A}}} : d_{(\mathbb{R}_+^\times)^n}) = 1.$$

We now compare the fine spectral expansion of each side of the identity and apply Lemma 4 of [LR00] in the same way Proposition 6 of [LR00] applies it. This is now possible, in our more general setting, thanks to the fundamental lemma of Jacquet, [Jac04] and [Jac05]. We obtain

$$\sum_{\nu\in\mathcal{B}(\chi)} \tilde{B}^{st}(\Phi,\nu,\lambda) = \frac{2^n d_{B\backslash B^1_{\mathbb{A}}}(B\backslash B^1_{\mathbb{A}})}{d_{H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}}(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}})d_{B'\backslash(B'_{\mathbb{A}})^1}(B'\backslash(B'_{\mathbb{A}})^1)} \times \sum_{\nu\in\mathcal{B}(\chi)} B'(f',\nu,\lambda) \tag{37}$$

for $\lambda \in i\mathfrak{a}_0^*$ and hence for all λ (both sides are meromorphic).

LEMMA 8. *For a p-adic inert place v of F the local distributions $\tilde{B}^{st}_v(\Phi,\nu,\lambda)$, $\nu \in \mathcal{B}(\chi)$ are linearly independent for λ generic.*

Proof. Let $\xi \in Y$. We write the set $\mathcal{B}(\chi)$ as the disjoint union of two sets $\mathcal{B}_1(\chi)$ and $\mathcal{B}_2(\chi)$ so that $\mathcal{B}_2(\chi) = \omega\mathcal{B}_1(\chi)$. It can be shown as in Lemma 6 of [LR00] that the distributions $\{\tilde{B}^{st,\xi}_v(f,\nu,\lambda)\}_{\nu\in B_i(\chi)}$ are linearly independent for $i = 1,2$. Indeed, the argument in [loc. cit.] reduces the linear independence of the distributions to the linear independence of the stable period integrals $\{J^{st,\xi}_v(\nu,\varphi,\lambda)\}_{\nu\in B_i(\chi)}$. The latter linear independence follows from the fact that the matrix $\Delta_i = (\Delta^{\xi}_{\nu,\eta})_{\nu\in\mathcal{B}_i(\chi),\eta}$ with representatives $\{\eta\}$ as in §3, is invertible as in the proof of Lemma 6. Note that

$$\tilde{B}^{st,\xi}_v(f,\omega\nu,\lambda) = \omega(\det(\xi w_n))\tilde{B}^{st,\xi}_v(f,\nu,\lambda).$$

Let ξ_1 and ξ_2 be representatives of the two G-orbits in Y, i.e., such that $\omega(\det(\xi_1\xi_2)) = -1$. Assume that the distribution $\sum_{\nu\in\mathcal{B}(\chi)}\alpha_\nu\tilde{B}^{st}_v(\Phi,\nu,\lambda)$ is identically zero for some constants $\alpha_\nu \in \mathbb{C}$. Applying Lemma 7 we get that the sum over all ν in, say $\mathcal{B}_1(\chi)$, of

$$(\alpha_\nu + \omega(\det(\xi_1 w_n))\alpha_{\omega\nu})B^{st,\xi_1}_v(f_1,\nu,\lambda) + (\alpha_\nu + \omega(\det(\xi_2 w_n))\alpha_{\omega\nu})B^{st,\xi_2}_v(f_2,\nu,\lambda)$$

is identically zero for any two functions f_1 and f_2 on G. We therefore get that $\alpha_\nu + \omega(\det(\xi_1 w_n))\alpha_{\omega\nu} = \alpha_\nu + \omega(\det(\xi_2 w_n))\alpha_{\omega\nu} = 0$ for all $\nu \in \mathcal{B}_1(\chi)$ and hence that $\alpha_\nu = 0$ for all $\nu \in \mathcal{B}(\chi)$. □

We recall Lemma 5 of [LR00].

LEMMA 9. *Let V_1, V_2, V_3 be vector spaces. Consider vectors $\{x_i^j\}_{i=1}^m$ and $\{y_i^j\}_{i=1}^m$ in V_j, $j = 1, 2, 3$ such that*

$$\sum_{i=1}^m x_i^1 \otimes x_i^2 \otimes x_i^3 = \sum_{i=1}^m y_i^1 \otimes y_i^2 \otimes y_i^3.$$

If $\{x_i^j\}_{i=1}^m$ is linearly independent in V_j, $j = 1, 2, 3$, then there exists a permutation σ of $\{1, 2, \ldots, m\}$ such that for all i

$$y_i^1 \otimes y_i^2 \otimes y_i^3 = x_{\sigma(i)}^1 \otimes x_{\sigma(i)}^2 \otimes x_{\sigma(i)}^3.$$

COROLLARY 3. *Assume that E/F is split at all real places of F. There exists a permutation τ_χ^δ on $\mathcal{B}(\chi)$ such that whenever $\Phi \overset{\delta}{\leftrightarrow} f'$ we have*

$$\tilde{B}^{st}(\Phi, \nu, \lambda) = \frac{2^n d_{B\backslash B_{\mathbb{A}}^1}(B\backslash B_{\mathbb{A}}^1)}{d_{H_e^{w_n}\backslash (H_e^{w_n})_{\mathbb{A}}}(H_e^{w_n}\backslash (H_e^{w_n})_{\mathbb{A}}) d_{B'\backslash (B'_{\mathbb{A}})^1}(B'\backslash (B'_{\mathbb{A}})^1)} B'(f', \tau_\chi^\delta(\nu), \lambda).$$

Proof. Using the localization principle of [GK75], if v is a p-adic inert place then $B'(f'_v, \nu_v, \lambda)$ depends only on the relevant orbital integrals for f'_v and therefore depends only on Φ_v. This is an observation of Jacquet ([Jac01];§4). Since every Φ has a δ-matching f', we may consider $B'(f', \nu, \lambda)$ as a distribution of $Y_{\mathbb{A}}$. As in ([LR00], Corollary 1) the corollary follows in this case by applying Lemma 9 to (37) using two auxiliary p-adic inert places. It follows from Lemma 8 that Lemma 9 may be applied. □

6.1. Independence on ψ. So far, we suppressed from our notation the dependence of our objects on the additive character ψ. We will now justify this, by showing that Theorem 3 (and therefore also Theorem 2) is compatible with a change of additive character ψ. This will also correct some inaccuracies in [LR00]. In order to explain the dependence of the objects on the additive character, we now append it to our notation. We even assume that the invariant measures depend on ψ.

Let $a \in F^\times$ and let $\psi' = \psi(a\cdot)$. Following the definition of matching, it is not hard to see that

$$\text{if } \Phi \overset{\psi',\delta}{\leftrightarrow} f' \text{ then } \Phi_a \overset{\psi,\delta}{\leftrightarrow} f'_a,$$

where

$$\Phi_a(x) = (d_U^{\psi'} : d_U^{\psi})\Phi(t^{-1} \cdot x), f'_a(g) = (d_{U'}^{\psi'} : d_{U'}^{\psi})^2 f'(a^{n-1}t^{-1}gt)$$

and $t = \operatorname{diag}(a^{n-1}, \ldots, a, 1)$. For $\varphi \in I(\chi, \lambda)$ we have

$$\mathcal{W}^{\psi'}(\varphi, \lambda) = (d_U^{\psi'} : d_U^{\psi})\chi({}^{w_n}t^{-1})e^{-\langle \rho + w\lambda, H(t)\rangle}\mathcal{W}^{\psi}(I(t, \chi, \lambda)\varphi, \lambda)$$

and

$$J^{st,\xi,\psi'}(\nu, \varphi, \lambda) = (d^{\psi'}_{H_e^{w_n}\backslash H^{w_n}} : d^{\psi}_{H_e^{w_n}\backslash H^{w_n}})J^{st,\xi,\psi}(\nu, \varphi, \lambda).$$

Similarly for $\varphi' \in I'(\nu, \lambda)$ we have

$$\mathcal{W'}^{\psi'}(\varphi', \lambda) = (d_{U'}^{\psi'} : d_{U'}^{\psi})\nu({}^{w_n}t^{-1})e^{-\langle \rho + w\lambda, H'(t)\rangle}\mathcal{W'}^{\psi}(I(t, \chi, \lambda)\varphi, \lambda).$$

The gamma factor satisfies

$$\gamma(\nu, \lambda, \psi') = \omega(a)^{\dim U'}\,|a|^{\frac{\dim U'}{2}}\left[\prod_{i<j}|a|^{\lambda_j - \lambda_i}\,\nu_j\nu_i^{-1}(a)\right]\gamma(\nu, \lambda, \psi).$$

Note also that

$$I^{\psi'}(f, \chi, \lambda)\varphi = (d_G^{\psi'} : d_G^{\psi})I^{\psi}(f, \chi, \lambda)\varphi$$

and similarly

$$I^{\psi'}(f', \nu, \lambda)\varphi' = (d_{G'}^{\psi'} : d_{G'}^{\psi})I^{\psi}(f', \nu, \lambda)\varphi'.$$

We also observe that

$$\begin{aligned}(38)\ & I(t, \nu, \lambda)I^{\psi}(f', \nu, \lambda)I(t^{-1}, \nu, \lambda)\varphi' \\ &= (d_{U'}^{\psi} : d_{U'}^{\psi'})^2(\nu_1 \cdots \nu_n)(a^{n-1})\,|a|^{(n-1)(\lambda_1 + \cdots \lambda_n)}\,I^{\psi}(f'_a, \nu, \lambda)\varphi'.\end{aligned}$$

We obtain

$$\begin{aligned}\mathcal{B'}^{\psi'}(f', \nu, \lambda) &= (d_{U'}^{\psi'} : d_{U'}^{\psi})^2(d_{G'}^{\psi'} : d_{G'}^{\psi})e^{-\langle \rho, H(t)\rangle} \\ &\qquad \times \sum_{\varphi'}\mathcal{W'}^{\psi}(I'(t, \nu, \lambda)I'^{\psi}(f', \nu, \lambda)\varphi', \lambda)\overline{\mathcal{W}'}^{\psi}(I'(t, \nu, \lambda)\varphi, \lambda) \\ &= (d_{B'}^{\psi'} : d_{B'}^{\psi})e^{-\langle \rho, H(t)\rangle}(\nu_1 \cdots \nu_n)(a^{n-1})\,|a|^{(n-1)(\lambda_1 + \cdots \lambda_n)}\,B^{\psi}(f'_a, \nu, \lambda).\end{aligned}$$

The first equality is obtained by converting ψ' to ψ according to the recipe provided above. The second equality is obtained by changing the orthonormal basis $\{\varphi'\}$ with respect to ψ' to the orthonormal basis $\{(d^{\psi'}_{B'\backslash G'} : d^{\psi}_{B'\backslash G'})^{\frac{1}{2}}I(t^{-1}, \nu, \lambda)\varphi'\}$

with respect to ψ and applying (38). Similarly, we obtain

$$\tilde{B}^{st,\psi'}(\Phi,\nu,\lambda)=(d_B^{\psi'}:d_B^{\psi})(d^{\psi}_{H_e^{w_n}}:d^{\psi'}_{H_e^{w_n}})e^{\langle w\lambda-\rho,H(t)\rangle}\nu^2({}^{w_n}t)\tilde{B}^{st,\psi}(\Phi_a,\nu,\lambda).$$

To see this, note that for $\chi=\nu\circ\mathrm{Nm}$ and $t\in T'$ we have $\chi({}^{w_n}t)=\nu^2({}^{w_n}t)$ and after an appropriate change of orthonormal basis apply (10). We now note that

$$e^{\langle w\lambda,H(t)\rangle}\nu^2({}^{w_n}t)=(\nu_1\cdots\nu_n)(a^{n-1})\,|a|^{(n-1)(\lambda_1+\cdots+\lambda_n)}\prod_{i<j}|a|^{\lambda_j-\lambda_i}\,\nu_j\nu_i^{-1}(a).$$

It follows that if κ is a root of unity, such that

$$\tilde{B}^{st,\psi'}(\Phi,\nu,\lambda)=\kappa\, e(\psi')^{-\dim U'}[d_B^{\psi'}:\ d_{B'}^{\psi'}\times d^{\psi'}_{H_e^{w_n}}]\,\gamma(\nu,\lambda,\psi')B'^{\,\psi'}(f',\nu,\lambda);$$

then we also have

$$\begin{aligned}\tilde{B}^{st,\psi}(\Phi_a,\nu,\lambda)\ &=\ (d_{B'}^{\psi'}:d_{B'}^{\psi})(d^{\psi'}_{H_e^{w_n}}:d^{\psi}_{H_e^{w_n}})(d_B^{\psi}:d_B^{\psi'})\\&\quad\times\kappa\,(\omega(a)\,|a|^{\frac12})^{\dim U'}e(\psi')^{-\dim U'}\\&\quad\times[d_B^{\psi'}:\ d_{B'}^{\psi'}\times d^{\psi'}_{H_e^{w_n}}]\gamma(\nu,\lambda,\psi)B'^{\,\psi}(f'_a,\nu,\lambda)\\&=\ (\kappa\,\omega(a)^{\dim U'})e(\psi)^{-\dim U'}\\&\quad\times[d_B^{\psi}:\ d_{B'}^{\psi}\times d^{\psi}_{H_e^{w_n}}]\,\gamma(\nu,\lambda,\psi)B'^{\,\psi}(f'_a,\nu,\lambda).\end{aligned}$$

7. The Bessel identities. In what follows, we normalize the relevant measures in a convenient way depending on ψ and prove Theorem 3. That the theorem holds for any choice of measures and any ψ will then follow from Remark 2 and the discussion in §6.1.

The measures on the local groups will be determined by a nontrivial character ψ of F as follows. If F is a local field we put on F the measure d_F^{ψ} which is self-dual with respect to ψ. If $\psi_a=\psi(a\cdot)$, $a\in F^\times$ then $d_F^{\psi_a}=|a|^{\frac12}\,d_F^{\psi}$. Set

$$\mathfrak{d}_F=\mathfrak{d}_F^{\psi}=\begin{cases}d_F^{\psi}(\mathcal{O}_F) & F\text{ nonarchimedean},\\ d_F^{\psi}([0,1]) & F\text{ real},\\ d_F^{\psi}(\{x+iy:\ 0\le x\le\frac12,\ 0\le y\le 1\}) & F\text{ complex}.\end{cases}$$

If F is nonarchimedean and ψ has conductor $\mathcal{O}_F$ then $\mathfrak{d}_F^{\psi}=1$. The same is true if F is archimedean and $\psi(x)=e^{2\pi i\,\mathrm{Tr}_{F/\mathbb{R}}x}$. We have $\mathfrak{d}_F^{\psi_a}=|a|^{\frac12}\,\mathfrak{d}_F^{\psi}$. On $F^\times$ we take the measure $d^{\psi}_{F^\times}=L(1,1_{F^\times})\frac{d_F^{\psi}x}{|x|}$. On B' we define the measure $d^{\psi}_{B'}=d^{\psi}_{U'}d^{\psi}_{T'}$ where $d^{\psi}_{U'}x=\otimes_{i<j}d_F^{\psi}x_{i,j}$ and $(d^{\psi}_{T'}:d^{\psi}_{(F^\times)^n})=1$.

Globally, we fix a nontrivial character ψ of $F\backslash\mathbb{A}$. On $\mathbb{A}$ we take the self-dual measure $d_{\mathbb{A}}$ with respect to ψ. It is also given by $\otimes_v\, d_{F_v}^{\psi_v}$. This does not depend on the choice of ψ, and we have $d_{F\backslash\mathbb{A}}(F\backslash\mathbb{A}) = 1$. Similarly, $\mathfrak{d}_F := \prod_v \mathfrak{d}_{F_v}(\psi_v)$ does not depend on ψ and in fact $\mathfrak{d}_F = |\Delta_F|^{-\frac{1}{2}}$ where Δ_F is the discriminant of F. On the group of idèles $\mathbb{I}_F$, we put the measure $d_{\mathbb{I}_F} = \otimes_v\, d_{F_v^\times}^{\psi_v}$. On $\mathbb{I}_F^1$, the kernel of the norm map, we take the measure $d_{\mathbb{I}_F^1}$ so that $d_{\mathbb{I}_F^1\backslash\mathbb{I}_F}$ is the pull back of $\frac{dt}{t}$ under the isomorphism $|\cdot|\colon \mathbb{I}_F^1\backslash\mathbb{I}_F \to \mathbb{R}_+$. Then $d_{F^\times\backslash\mathbb{I}_F^1}(F^\times\backslash\mathbb{I}_F^1) = \mathrm{Res}_{s=1}L(s, 1_{F^\times})$ where $L(s, 1_{F^\times})$ is the completed Dedekind ζ function for F. We also set $d_{B'_{\mathbb{A}}}^{\psi} = \otimes_v d_{B'_v}^{\psi_v}$.

Locally, if E is either a quadratic extension of F or $F \oplus F$, denote $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. Let $d_{E_1}^{\psi}$ be the measure on E_1 defined by the relation

$$\int_{E^\times} f(z)\ d_{E^\times}^{\psi_E} z = \int_{\mathrm{Nm}\,(E^\times)} F(x)\, d_{F^\times}^{\psi} x \quad \text{where} \quad F(\,\mathrm{Nm}\, t) = \int_{E_1} f(yt)\, d_{E_1}^{\psi} y.$$

The measure on $H_e^{w_n}$ is such that $(d_{H_e^{w_n}}^{\psi} : d_{(E_1)^n}^{\psi}) = 1$. Globally, for a nontrivial character ψ of $F\backslash\mathbb{A}$ we set $d_{(H_e^{w_n})_{\mathbb{A}}} = \otimes_v d_{(H_e^{w_n})_v}^{\psi_v}$.

Note that locally, by our definitions, we have

$$e(\psi)^{-\dim U'}[d_B^{\psi_E} : d_{B'}^{\psi} \times d_{H_e^{w_n}}^{\psi}] = 1.$$

Globally, we also have

$$\frac{2^n d_{B\backslash B_{\mathbb{A}}^1}(B\backslash B_{\mathbb{A}}^1)}{d_{H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}}(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}) d_{B'\backslash(B'_{\mathbb{A}})^1)}(B'\backslash(B'_{\mathbb{A}})^1)} = 1.$$

Indeed, by Ono's formula for the Tamagawa number of a torus [Ono66] we have

$$d_{H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}}(H_e^{w_n}\backslash(H_e^{w_n})_{\mathbb{A}}) = (2L(1,\omega))^n,$$

whereas

$$d_{B'\backslash(B'_{\mathbb{A}})^1}(B'\backslash(B'_{\mathbb{A}})^1) = (\mathrm{Res}_{s=1}L(s, 1_{F^\times}))^n$$

and

$$d_{B\backslash B_{\mathbb{A}}^1}(B\backslash B_{\mathbb{A}}^1) = (\mathrm{Res}_{s=1}L(s, 1_{E^\times}))^n.$$

We recall a lemma from [LR00] (Corollary 2) that is used to deduce the Bessel identities. For any finite set S of finite places, denote by $\mathcal{U}_S$ the compact group of unramified unitary characters of $T'_S = \prod_{v\in S} T'_v$. Let S_∞ be the set of archimedean places in F.

LEMMA 10. *Let $S = S_\infty \cup S_f$ be a finite set of places, containing the archimedean places. Given a place $w \notin S$, a unitary character $\eta = (\eta_v)_{v \in S_f}$ of T'_{S_f} and an open set $U \subset \mathcal{U}_{S_f}$, there exists a Hecke character ϱ of $T'_{\mathbb{A}}$ which is unramified outside $S \cup \{w\}$ such that $\varrho_{S_f}^{-1}\eta \in U$.*

To prove Theorem 3 in the nonarchimedean case, we choose a favorable global situation. Given a quadratic extension E^0/F^0 of p-adic fields, there is a quadratic extension of number fields E/F such that:

• There is a place v_0 of F such that $E_{v_0}/F_{v_0} \simeq E^0/F^0$.

• Every real place of F splits in E. If v is an even place of F and $E_v/F_v \not\simeq E^0/F^0$ then v splits in E.

• If $S'_1 = \{v_1, \dots, v_l\}$ is the set of places of F that ramify over E then $E_{v_i}/F_{v_i} \simeq E^0/F^0$ for $i = 1, \dots, l$.

Let S_1 be the set of places v in F such that $E_v/F_v \simeq E^0/F^0$. Note that S'_1 may be empty, but in any case S_1 contains S'_1 and v_0. Let μ be a unitary character of T'_{v_0} (and hence of T'_v for $v \in S_1$). Let w_1 be an auxiliary place of F, inert in E with residual characteristic $p \nmid 2q_{F^0}$ and let $S_2 = \{w_1, \dots, w_m\}$ be the set of all places of F of residual characteristic p. Let $\psi = \otimes_v \psi_v$ be a nontrivial additive character of $F\backslash\mathbb{A}$ so that ψ_v has conductor $\mathcal{O}_{F_v}$ for $v \in S_2$. For any Hecke character η set

$$L_p(\eta, s) = \prod_{i=1}^{m} L(\eta_{w_i}, s).$$

Let τ_χ^δ be the permutation on $\mathcal{B}(\chi)$ given by Corollary 3. For $\nu = (\nu_1, \dots, \nu_n) \in \mathcal{B}(\chi)$ let $[\nu] = \{\nu, \omega\nu\}$ where $\omega\nu = (\omega\nu_1, \dots, \omega\nu_n)$.

LEMMA 11. *There exists a nonempty open set $U_2 \subset \mathcal{U}_{S_2}$, such that whenever ν is a Hecke character of $T'_{\mathbb{A}}$ such that $\nu_{S_2} \in U_2$, we have $\tau_\chi^\delta(\nu) \in [\nu]$.*

Proof. Denote $\tau_\chi^\delta(\nu) = \nu' = (\nu'_1, \dots, \nu'_n)$. We must show that the condition $[\nu'] \neq [\nu]$ imposes a nontrivial closed condition on ν_{S_2}. Since $\nu \circ \mathrm{Nm} = \nu' \circ \mathrm{Nm}$, we must have $\nu'_i \in \{\nu_i, \omega\nu_i\}$ for all i. Assume that $[\nu'] \neq [\nu]$ and let

$$\mathcal{I} = \{(i,j)\colon\ 1 \le i < j \le n,\ \nu_i\nu_j^{-1} = \nu'_i\nu_j'^{-1}\omega\}.$$

Our assumption is equivalent to the fact that $\mathcal{I}$ is not empty. Let S be a finite set of places of F, containing S_1, the archimedean and the even places, disjoint from S_2 and large enough so that ψ_w has conductor $\mathcal{O}_{F_w}$ and ν_w is unramified for all $w \notin S$. Denote $S = S_\infty \cup S_f$. Let $f'_S = \prod_{v \in S} f'_v$ be such that $B'_S(f'_S, \nu'_S, \lambda)$ is not zero as a function of λ. It follows from Jacquet's smooth matching for the p-adic places [Jac03] and the fact that all archimedean places split in E/F, that we can find a function $\Phi_S = \otimes_{v \in S}\Phi_v$ such that $f'_S \overset{\delta}{\leftrightarrow} \Phi_S$. We set $f' = f'_S \otimes \mathrm{ch}_{K'^S}$ and $\Phi = \Phi_S \otimes \mathrm{ch}_{K^S \cap \mathbf{Y}(F^S)}$. Thus $\Phi \overset{\delta}{\leftrightarrow} f'$ by Jacquet's fundamental lemma, and we get the identity

$$\tilde{B}^{st}(\Phi,\nu,\lambda)=B'(f',\nu',\lambda) \tag{39}$$

of Corollary 3. We apply the local Bessel identities obtained in Proposition 4 for the split places and in Proposition 5 for the unramified places. From the same type of manipulations as in the proof of Lemma 8 of [LR00] we obtain from the above identity the relation

$$c_{S_f}(\lambda)\prod_{(i,j)\in\mathcal{I}}L^S(\nu_i\nu_j^{-1},\lambda_i-\lambda_j+1)L^S(\nu_i\nu_j^{-1},\lambda_i-\lambda_j)$$
$$=\prod_{(i,j)\in\mathcal{I}}L^S(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j+1)L^S(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j)$$

for some rational function $c_{S_f}(\lambda)$ in $\{q_v^\lambda \mid v\in S_f\}$. If $\mathrm{Re}\,\lambda$ is positive enough then the expression in each side is an absolutely convergent infinite product and can be expressed as a multiple Dirichlet series in the variables $\lambda_i-\lambda_{i+1}$, $i=1,\dots,n-1$. We can therefore compare the purely p-powered multi-coefficients to get

$$\prod_{(i,j)\in\mathcal{I}}L_p(\nu_i\nu_j^{-1},\lambda_i-\lambda_j+1)L_p(\nu_i\nu_j^{-1},\lambda_i-\lambda_j)$$
$$=\prod_{(i,j)\in\mathcal{I}}L_p(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j+1)L_p(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j).$$

This equality holds for all λ. We now fix once and for all $\lambda_2,\dots,\lambda_n\in i\mathbb{R}$ such that

$$L_p(\nu_i\nu_j^{-1},\lambda_i-\lambda_j)^{-1}L_p(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j)^{-1}\neq 0$$

for all $(i,j)\in\mathcal{I}$ such that $2\le i$. Note that there exist an index j_0 such that $(1,j_0)\in\mathcal{I}$. Indeed, otherwise $\nu_j'=\nu_1'\nu_1^{-1}\nu_j$ for all j and since $\nu_1'\nu_1^{-1}\in\{1_{F^\times},\omega\}$ this contradicts our assumption on ν. There is then a nonzero number c such that for all λ_1 we have

$$c\prod_{(1,j)\in\mathcal{I}}L_p(\nu_1\nu_j^{-1},\lambda_1-\lambda_j+1)L_p(\nu_1\nu_j^{-1},\lambda_1-\lambda_j)$$
$$=\prod_{(1,j)\in\mathcal{I}}L_p(\nu_1\nu_j^{-1}\omega,\lambda_1-\lambda_j+1)L_p(\nu_1\nu_j^{-1}\omega,\lambda_1-\lambda_j).$$

Denote $\alpha_{j,v}=\nu_1\nu_j^{-1}(\varpi_v)$, and let $q_v=p^{n_v}$ for $v\in S_2$. Let $x=p^{-\lambda_1}$. We have

$$c^{-1}\prod_{(1,j)\in\mathcal{I}}\prod_{v\in S_2}(1-\alpha_{j,v}p^{n_v(\lambda_j-1)}x^{n_v})(1-\alpha_{j,v}p^{n_v\lambda_j}x^{n_v})$$
$$=\prod_{(1,j)\in\mathcal{I}}\prod_{v\in S_2}(1+\alpha_{j,v}p^{n_v(\lambda_j-1)}x^{n_v})(1+\alpha_{j,v}p^{n_v\lambda_j}x^{n_v}).$$

This must hold as an equality of polynomials in x. Fix $v_0 \in S_2$, let $\zeta_0^{n_{v_0}} = \alpha_{j_0,v_0}$ and set $x_0 = p^{-\lambda_{j_0}}\zeta_0$. The left-hand side vanishes at x_0. It follows that there exist $(j, v) \neq (j_0, v_0)$ with $(1,j) \in \mathcal{I}$ and $v \in S_2$ such that

$$\alpha_{j,v}p^{n_v\lambda_j} = -\alpha_{j_0,v_0}p^{n_{v_0}\lambda_{j_0}}.$$

This is a nontrivial closed condition on ν_{S_2}. □

We now note that $B'(f'_\omega, \nu, \lambda) = B'(f', \omega\nu, \lambda)$ and therefore from Lemma 8 and (5) we get that

$$\tau_\chi^{1-\delta}(\nu) = \omega\tau_\chi^{\delta}(\nu). \tag{40}$$

It follows that for ν such that $\nu_{S_2} \in U_2$ as in Lemma 11 there exists $\delta(\nu) \in \{0,1\}$ such that $\tau_\chi^{\delta(\nu)}(\nu) = \nu$. Using the same argument as in ([LR00], pp. 346–347), applying Lemma 10, Corollary 3 and Lemma 11 we obtain locally in the nonarchimedean case, that for every unitary character ν of T' there exists $\delta(\nu) \in \{0,1\}$ such that

$$\text{if } \Phi \overset{\delta(\nu)}{\leftrightarrow} f' \text{ then } \tilde{B}^{st}(\Phi,\nu,\lambda) = \kappa_{E/F}\gamma(\nu,\lambda,\psi)B'(f',\nu,\lambda) \tag{41}$$

for a root of unity $\kappa_{E/F}$ as in the statement of Theorem 3. To complete the proof in the nonarchimedean case it remains to show that $\delta(\nu)$ depends only on n. Let E_i/F_i be a quadratic extension of p-adic fields and ν_i a unitary character of $\mathbf{T}'(F_i)$ for $i = 1, 2$. There exists a quadratic extension E/F of number fields, split at all real places and such that there are places v_i of F for which $E_{v_i}/F_{v_i} \simeq E_i/F_i$. There also exists a unitary character ν of $T'\backslash T'_{\mathbb{A}}$ such that $\nu_{v_i} = \nu_i$ for $i = 1, 2$. Let $\chi = \nu \circ \mathrm{Nm}$ and let τ_χ^δ be the permutation of $\mathcal{B}(\chi)$ given by Corollary 3. It follows from (41) and Lemma 8 that $\tau_\chi^{\delta(\nu_i)}(\nu) = \nu$ and therefore from (40) that $\delta(\nu_1) = \delta(\nu_2)$. This completes the proof of Theorem 3 in the nonarchimedean case.

To prove Theorem 3 in the archimedean case it is enough to consider the global quadratic extension $E/F = \mathbb{Q}[i]/\mathbb{Q}$. Let $\delta = \delta(n) \in \{0,1\}$ be the chosen δ for which Theorem 3 holds in the nonarchimedean case. Let

$$A^\infty(\chi^\infty,\lambda) = \frac{A_\infty(\chi_\infty,\lambda)}{A(\chi,\lambda)}$$

and set

$$\tilde{B}^{st,\infty}(\cdot,\nu^\infty,\lambda) = \frac{1}{A^\infty(\chi^\infty,\lambda)}\prod_p A_p(\chi_p,\lambda)\tilde{B}^{st}_p(\cdot,\nu_p,\lambda)$$

and

$$B'^{\infty}(\cdot, \nu^{\infty}, \lambda) = \frac{\gamma_{\infty}(\nu_{\infty}, \lambda, \psi_{\infty})}{A^{\infty}(\chi^{\infty}, \lambda)} \prod_{p} \gamma_p(\nu_p, \lambda, \psi_p) A_p(\chi_p, \lambda) B'_p(\cdot, \nu_p, \lambda),$$

the products being over all primes and are convergent for $\operatorname{Re}\lambda$ sufficiently large. As explained in the proof of Corollary 3, we may regard both as distributions of $Y^{\infty} = \prod'_p Y_p$ via the δ-matching. It follows from the local Bessel identity at the split and at the nonarchimedean places that

$$\tilde{B}^{st,\infty}(\cdot, \nu^{\infty}, \lambda) = \frac{\prod_p \kappa_{E_p/F_p}}{\gamma_{\infty}(\nu_{\infty}, \lambda, \psi_{\infty})} B'^{\infty}(\cdot, \nu^{\infty}, \lambda). \tag{42}$$

If $\Phi_{\infty} \overset{\delta}{\leftrightarrow} f'_{\infty}$ we denote,

$$\alpha_{\nu}(\lambda) = \tilde{B}^{st}_{\infty}(\Phi_{\infty}, \nu_{\infty}, \lambda) \text{ and } \beta_{\nu}(\lambda) = B'_{\infty}(f'_{\infty}, \nu_{\infty}, \lambda).$$

From (37) we have

$$\sum_{\nu \in \mathcal{B}(\chi)} \alpha_{\nu}(\lambda) \tilde{B}^{st,\infty}(\cdot, \nu^{\infty}, \lambda) = \sum_{\nu \in \mathcal{B}(\chi)} \beta_{\nu}(\lambda) B'^{\infty}(\cdot, \nu^{\infty}, \lambda).$$

It follows from Lemma 8 that the distributions $(\tilde{B}^{st,\infty}(\cdot, \nu^{\infty}, \lambda))_{\nu \in \mathcal{B}(\chi)}$ are linearly independent and therefore that

$$\alpha_{\nu}(\lambda) = \kappa_{\mathbb{C}/\mathbb{R}} \gamma_{\infty}(\nu_{\infty}, \lambda, \psi_{\infty}) \beta_{\nu}(\lambda)$$

where $\kappa_{\mathbb{C}/\mathbb{R}} = \prod_p \kappa_{E_p/F_p}^{-1}$. This completes the proof of Theorem 3. □

Theorem 2 is now immediate from (37) Lemma 8 and Theorem 3. □

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