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Group theory

Vanishing of local symplectic periods for cuspidal representations of the unitary group

Nullité des périodes symplectiques locales pour les représentations cuspidales du groupe unitaire

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A R T I C L E I N F O

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A B S T R A C T

We show that, for a quadratic extension of p-adic fields, no cuspidal representation of the quasi-split unitary group admits a non-trivial linear form invariant by the symplectic subgroup. Our proof is purely local.

R É S U M É

Nous montrons que, pour une extension quadratique de corps p-adiques, aucune représentation cuspidale du groupe unitaire semi-déployé n'admet de forme linéaire non nulle invariante par l'action du sous-groupe symplectique. Notre preuve est purement locale.

1. Introduction

Let G be a reductive p-adic group and H a closed subgroup. A smooth, complex valued representation (π, V) of G is called H-distinguished if there exists a non-zero linear form ℓ on V such that $\ell(\pi(h)v) = \ell(v)$ for all $h \in H$ and $v \in V$.

The class of H-distinguished representations plays an important role in the harmonic analysis of the homogeneous space G/H (see [2]). Furthermore, distinguished representations are crucial for the global theory of period integrals of automorphic forms, have applications to the study of special values of L-functions and to the description of the image of functorial lifts in the sense of Langlands functoriality conjectures.

We say that (G, H) is a vanishing pair if no irreducible cuspidal representation of G is H-distinguished. The terminology is borrowed from [1] where the global analogue is defined.

This note provides a new family of vanishing pairs. The groups concerned are as follows. Let E/F be a quadratic extension of non-Archimedean local fields of characteristic different than two. The symplectic group $\mathrm{Sp}_{2n}(F)$ naturally embeds in the quasi-split unitary group $U_{n,n}(F)$ on $2n$ variables associated with the extension E/F.

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Theorem 1.1. *(See Theorem 3.1.) The pair $(U_{n,n}(F), \mathrm{Sp}_{2n}(F))$ is a vanishing pair.*

In [1], Ash, Ginzburg and Rallis provide many examples of global vanishing pairs. The proof of Theorem 1.1 is inspired by their proof, but is purely local. We sometimes refer to an $\mathrm{Sp}_{2n}(F)$-invariant linear form as a symplectic period. The vanishing of symplectic periods for *tame* cuspidal representations was obtained in [10, Theorem 1.3] using the explicit construction of such representations due to J.K. Yu. Our proof of the generalization to all cuspidal representations is elementary in comparison.

We also remark that following [1] more directly, the techniques of this paper can also show the global analogue of our result.

Theorem 1.2. *Let k be a global field of characteristic different than two. Then $(U_{n,n}, \mathrm{Sp}_{2n})$ is a global vanishing pair. That is,*

$$\int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_k)} \phi(h)\, \mathrm{d}h = 0$$

for any cuspidal automorphic form on $U_{n,n}(k)\backslash U_{n,n}(\mathbb{A}_k)$.

Here $\mathbb{A}_k$ is the ring of adeles of k. As the global approach follows [1] closely, we omit the proof here. The novelty of this work is that the same techniques can be adapted in a local setting.

The discrete series representations of the quasi-split unitary groups are completely classified by Mœglin and Tadíc modulo cuspidal representations [8]. The classification in [8] is then extended to the class of tempered representations of these groups in [9] and [5]. The results in this paper are a first step in an ongoing project by the authors aiming at the classification of $\mathrm{Sp}_{2n}(F)$-distinguished tempered representations of $U_{n,n}(F)$ and in particular, at determining if distinguished discrete series representations exist.

Remark 1.3. After completion of this work, it came to our attention that Dijols and Prasad independently obtained the main result of this paper, which will be published in an upcoming work [3].

2. Notation and preliminaries

Let E/F be a quadratic extension of non-Archimedean local fields of characteristic different from two with Galois action $a \mapsto \bar{a}$. Fix a trace zero element $\iota \in E^*$.

In this note, we will be concerned with two families of reductive groups, the quasi-split unitary groups and the symplectic groups.

Let $J = J_n = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix}$, where $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n(F)$. Let $G = G_n$ be the quasi-split unitary group with respect to the Hermitian matrix ιJ and $H = H_n$ the symplectic group defined by the symplectic matrix J. Explicitly,

$$G = \{g \in \mathrm{GL}_{2n}(E) \mid {}^{\mathrm{t}}\bar{g} J g = J\}$$

and

$$H = \{g \in \mathrm{GL}_{2n}(F) \mid {}^{\mathrm{t}}g J g = J\}.$$

Thus, H is the subgroup of G of the fixed points under the involution $g \mapsto \bar{g} = J\, {}^{\mathrm{t}}g^{-1} J^{-1}$.

When G' is either G or H, we call a parabolic subgroup of G' standard if it contains the subgroup of upper-triangular matrices in G'. The map $Q \mapsto Q \cap H$ is a bijection between standard parabolic subgroups of G and of H.

The standard parabolic subgroups of G are in bijection with compositions $(n_1, \dots, n_r; m)$ of n with $m, r \geq 0$, $n_1, \dots, n_r \geq 1$ and $n_1 + \dots + n_r + m = n$. Denote by

$$Q_{(n_1,\dots,n_r;m)} = L_{(n_1,\dots,n_r;m)} V_{(n_1,\dots,n_r;m)}$$

the standard parabolic subgroup with unipotent radical $V_{(n_1,\dots,n_r;m)}$ and standard Levi subgroup

$$L_{(n_1,\dots,n_r;m)} = \{\mathrm{diag}(g_1, \dots, g_r, h, g_r^*, \dots, g_1^*) : h \in G_m,\ g_i \in \mathrm{GL}_{n_i}(E),\ i = 1, \dots, r\}$$

isomorphic to $\mathrm{GL}_{n_1}(E) \times \dots \times \mathrm{GL}_{n_r}(E) \times G_m$. The involution $g \mapsto g^*$ on $\mathrm{GL}_k(E)$ (for any k) is defined by $g^* = w_k {}^{\mathrm{t}}\bar{g}^{-1} w_k^{-1}$.

We also set

$$P_{(n_1,\dots,n_r;m)} = M_{(n_1,\dots,n_r;m)} U_{(n_1,\dots,n_r;m)} = Q_{(n_1,\dots,n_r;m)} \cap H$$

where

$$M_{(n_1,\dots,n_r;m)} = L_{(n_1,\dots,n_r;m)} \cap H \simeq \mathrm{GL}_{n_1}(F) \times \dots \times \mathrm{GL}_{n_r}(F) \times H_m$$

and

$$U_{(n_1,\dots,n_r;m)} = V_{(n_1,\dots,n_r;m)} \cap H.$$

Of particular interest to us will be the parabolic subgroups $Q_{(1;n-1)}$ and $P_{(1;n-1)}$ and their Levi decompositions. Set $V = V_{(1;n-1)}$ and $U = U_{(1;n-1)}$. Explicitly,

$$V = \left\{ \begin{pmatrix} 1 & x & z \\ & I_{2(n-1)} & x' \\ & & 1 \end{pmatrix} : x \in E^{2(n-1)}, x' = -J_{n-1}{}^{\mathrm{t}}\bar{x},\ z \in E,\ z - \bar{z} = xJ_{n-1}{}^{\mathrm{t}}\bar{x} = -xx' \right\}$$

and

$$U = \left\{ \begin{pmatrix} 1 & x & z \\ & I_{2(n-1)} & x' \\ & & 1 \end{pmatrix} : x \in F^{2(n-1)}, x' = -J_{n-1}{}^{\mathrm{t}}x,\ z \in F \right\}.$$

It is straightforward that U is a normal subgroup of V and that $F^{2(n-1)} \simeq U\backslash V$. An explicit isomorphism is given by $x \mapsto Uv(x)$, where

$$v(x) = \begin{pmatrix} 1 & \imath x & 0 \\ & I_{2(n-1)} & (\imath x)' \\ & & 1 \end{pmatrix}, \quad x \in F^{2(n-1)}.$$

Fix a non-trivial character ψ of F. For any k and $a = (a_1, \dots, a_k) \in F^k$, let ψ_a be the character of F^k defined by $\psi_a(x_1, \dots, x_k) = \psi(a_1x_1 + \dots + a_kx_k)$.

The dual group is therefore

$$\widehat{U\backslash V} = \{\chi_a : a \in F^{2(n-1)}\}$$

where for $a = (a_1, \dots, a_{2(n-1)})$, we set

$$\chi_a \begin{pmatrix} 1 & x & z \\ & I_{2(n-1)} & x' \\ & & 1 \end{pmatrix} = \psi_a((2\imath)^{-1}(x - \bar{x})).$$

For any $k \le n$, denote by $\iota_n = \iota_{n,k} : G_k \to G$ the embedding defined by

$$\iota_n(g) = \mathrm{diag}(I_{n-k}, g, I_{n-k}).$$

For a subgroup S of G_k, let $S^{(n)}$ denote its image under ι_n (a subgroup of $L_{(n-k;k)}$).

The group $S = H_{n-1}^{(n)}$ normalizes both U and V and therefore acts on $U\backslash V$ via conjugation. For $h \in H_{n-1}$ and $s = \iota_n(h) \in S$, we observe that $sv(x)s^{-1} = v(xh^{-1})$ and therefore, for the dual action

$$(s \cdot \chi)(v) = \chi(s^{-1}vs), \quad s \in S,\ v \in U\backslash V,\ \chi \in \widehat{U\backslash V}$$

we have

$$s \cdot \chi_a = \chi_{a\,{}^{\mathrm{t}}h}.$$

Since H_k acts transitively on $F^{2k} \setminus \{0\}$, it follows that

$$\widehat{U\backslash V} = \{\chi_0\} \sqcup S \cdot \chi_{e_1}$$

consists of two S-orbits, where $e_1 = (1, 0, \dots, 0) \in F^{2(n-1)}$. We further note that the stabilizer S_1 of χ_{e_1} satisfies $S_1 = R_{n-1}^{(n)}$, where for any k we set

$$R_k = \{h \in P_{(1;k-1)} : h_{1,1} = 1\}.$$

Note that R_{n-1} and H_{n-1} are both unimodular and there is therefore a unique H_{n-1}-invariant measure $\mathrm{d}h$ on $R_{n-1}\backslash H_{n-1} \simeq S_1\backslash S$. Since the Haar measure on $F^{2(n-1)}$ is H_{n-1}-invariant, by restricting to the open dense subset $F^{2(n-1)} \setminus \{0\}$ it induces $\mathrm{d}h$ on $R_{n-1}\backslash H_{n-1}$ up to a positive constant. With the appropriate normalization of measures, by Pontryagin duality, for every $f \in C_c^\infty(U\backslash V)$, we therefore have ($I$ is the identity matrix in V)

$$f(I) = \int\limits_{R_{n-1}^{(n)}\backslash H_{n-1}^{(n)}} \int\limits_{U\backslash V} f(v)\chi_{e_1}(h^{-1}vh)\,\mathrm{d}v\,\mathrm{d}h. \tag{1}$$

3. The main result

Let σ be a representation of G. For a linear form ℓ on σ and a vector w in the space of σ, let

$$c_{\ell,w}(g) = \ell(\sigma(g)w), \ g \in G$$

be the associated generalized matrix coefficient.

Theorem 3.1. *Let σ be an irreducible cuspidal representation of G. Then*

$$\mathrm{Hom}_H(\sigma, 1) = 0.$$

Proof. We first remark that, since σ is cuspidal, we have

$$\mathrm{Hom}_V(\sigma, 1) = 0 \tag{2}$$

whenever V is the unipotent radical of a proper parabolic subgroup of G.

Let $\ell \in \mathrm{Hom}_H(\sigma, 1)$. It follows from [6, Proposition 8.1] (see also [7] for related results) that $c_{\ell,w} \in C_c^\infty(H\backslash G)$ for every $w \in \sigma$ (note that G has compact center).

For $k = 0, 1, \ldots, n$ let

$$\mathcal{V}_k = V_{(1^{(k)};n-k)}, \ V_k = V^{(n)}_{(1;n-k)}, \ \mathcal{U}_k = \mathcal{V}_k \cap H \text{ and } U_k = V_k \cap H.$$

(By $a^{(k)}$ we mean the composition $(a, \ldots, a)$ of ak.) Recall that $\{\iota_n(v(x)) : x \in F^{2(n-k)}\}$ is a complete set of representatives for $U_k\backslash V_k$.

Note that $\mathcal{V}_0$ is the trivial group and $\mathcal{V}_n$ is the unipotent radical of the standard Borel subgroup of G. For $k = 0, 1, \ldots, n-1$, let μ_k be the (degenerate) character of $\mathcal{V}_n$ defined by

$$\mu_k(\nu) = \psi((2\iota)^{-1}\sum_{i=1}^{k}(\nu_{i,i+1} - \bar{\nu}_{i,i+1})).$$

Then, μ_k is trivial on $\mathcal{U}_n$ as well as on the unipotent radical $V_{(n;0)}$ of the Siegel parabolic subgroup of G. Let

$$\ell_k(w) = \int_{\mathcal{U}_k\backslash\mathcal{V}_k} c_{\ell,w}(\nu)\mu_k(\nu)\,\mathrm{d}\nu.$$

We show by (reverse) induction on k that $\ell_k = 0$ for $k = 0, 1, \ldots, n-1$. Since $\ell_0 = \ell$, the theorem will follow.

We first make the following observations. Note that $V_{k+1} \subseteq G^{(n)}_{n-k} \subseteq L_{(1^{(k)};n-k)}$, and therefore V_{k+1} normalizes $\mathcal{V}_k$. Since also $\mathcal{V}_{k+1} = V_{k+1}\mathcal{V}_k$ and $V_{k+1} \cap \mathcal{V}_k = 1$, it follows that

$$\mathcal{V}_{k+1} = V_{k+1} \ltimes \mathcal{V}_k \quad \text{and similarly} \quad \mathcal{U}_{k+1} = U_{k+1} \ltimes \mathcal{U}_k. \tag{3}$$

With an appropriate normalization of invariant measures, we therefore have

$$\int_{\mathcal{U}_{k+1}\backslash\mathcal{V}_{k+1}} f(\nu)\,\mathrm{d}\nu = \int_{U_{k+1}\backslash V_{k+1}} \int_{\mathcal{U}_k\backslash\mathcal{V}_k} f(\nu v)\,\mathrm{d}\nu\,\mathrm{d}v, \quad f \in C_c^\infty(\mathcal{U}_{k+1}\backslash\mathcal{V}_{k+1}). \tag{4}$$

Note further that

$$\mu_k(v^{-1}\nu v) = \mu_k(\nu), \quad \nu \in \mathcal{V}_k, \ v \in V_{k+1}. \tag{5}$$

It follows from (5) that $c_{\ell_k,w}(ug) = c_{\ell_k,w}(g)$, $u \in U_{k+1}$, $g \in G$. Since $c_{\ell,w}$ has compact support modulo H, its restriction to $\mathcal{V}_{k+1}$ has compact support modulo $\mathcal{U}_{k+1}$. It therefore follows from (3) and the definition of ℓ_k that the restriction to V_{k+1} of $c_{\ell_k,w}$ lies in $C_c^\infty(U_{k+1}\backslash V_{k+1})$.

For $1 \le k < n$ let $e_1(k) = (1, 0, \ldots, 0) \in F^{2(n-k)}$ and let $\chi_k = \chi_{e_1(k)}$ be the associated character of $U_k\backslash V_k$. Note that $\chi_k = \mu_k|_{V_k}$.

Since μ_k and μ_{k+1} coincide on $\mathcal{V}_k$ for $k < n-1$, it follows that

$$\mu_{k+1}(\nu v) = \mu_k(\nu)\chi_{k+1}(v), \ v \in V_{k+1}, \ \nu \in \mathcal{V}_k. \tag{6}$$

Finally, we also observe that

$$\mu_k(h^{-1}\nu h) = \mu_k(\nu), \quad \nu \in \mathcal{V}_k, \ h \in H^{(n)}_{n-k-1}. \tag{7}$$

Note that $U_n = V_n$. It therefore follows from (4) that

$$\ell_{n-1}(w) = \int\limits_{\mathcal{U}_n \backslash \mathcal{V}_n} c_{\ell,w}(v)\mu_{n-1}(v)\, dv$$

and therefore that $\ell_{n-1} \in \mathrm{Hom}_{\mathcal{V}_n}(\sigma, \mu_{n-1}^{-1})$. Since μ_{n-1} is trivial on $V_{(n;0)}$, it follows from (2) applied to $V = V_{(n;0)}$ that $\ell_{n-1} = 0$.

Assume now that $k < n-1$ and that $\ell_{k+1} = 0$. We apply (1) to the restriction to V_{k+1} of $c_{\ell_k,w}$ for $w \in \sigma$. We get that

$$\ell_k(w) = \int\limits_{R^{(n)}_{n-k-1} \backslash H^{(n)}_{n-k-1}} \left[\int\limits_{U_{k+1} \backslash V_{k+1}} \int\limits_{\mathcal{U}_k \backslash \mathcal{V}_k} \ell(\sigma(\nu v) w)\mu_k(\nu)\, d\nu\, \chi_{k+1}(h^{-1} v h)\ dv \right] dh.$$

Note that $H^{(n)}_{n-k-1}$ normalizes each of the groups V_{k+1}, U_{k+1}, $\mathcal{V}_k$ and $\mathcal{U}_k$. Making the changes of variables $v \mapsto h v h^{-1}$ and $\nu \mapsto h \nu h^{-1}$ and applying (7) and the H-invariance of ℓ, we therefore have

$$\ell_k(w) = \int\limits_{R^{(n)}_{n-k-1} \backslash H^{(n)}_{n-k-1}} \left[\int\limits_{U_{k+1} \backslash V_{k+1}} \int\limits_{\mathcal{U}_k \backslash \mathcal{V}_k} \ell(\sigma(\nu v h^{-1}) w)\mu_k(\nu)\, d\nu\, \chi_{k+1}(v)\ dv \right] dh.$$

Applying (4) and (6) we get that

$$\ell_k(w) = \int\limits_{R^{(n)}_{n-k-1} \backslash H^{(n)}_{n-k-1}} \ell_{k+1}(\sigma(h^{-1}) w)\, dh$$

which is zero by the induction hypothesis. $\square$

Remark 3.2. In [4], Heumos and Rallis showed that $(\mathrm{GL}_{2n}(F), \mathrm{Sp}_{2n}(F))$ is a vanishing pair by disjointness of Whittaker and symplectic models. With a few minor modifications, our proof adapts to this pair and provides a new local proof that does not apply the fact that cuspidal representations of $\mathrm{GL}_m(F)$ are generic.

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