Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 254 (2008) 2846–2865
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Uniqueness and disjointness of Klyachko models

Omer Offen [a,*], Eitan Sayag [b]

[a] *Humboldt-Universität zu Berlin, Mathematich-Naturwissenschaftliche Fakultät II, Institut für Mathematik, Sitz: Rudower Chausee 25, D-10099 Berlin, Germany*

[b] *Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem 91904, Israel*

Received 25 October 2007; accepted 2 January 2008

Available online 11 February 2008

Communicated by P. Delorme

Abstract

We show the uniqueness and disjointness of Klyachko models for GL_n over a non-Archimedean local field. This completes, in particular, the study of Klyachko models on the unitary dual. Our local results imply a global rigidity property for the discrete automorphic spectrum of GL_n.

Keywords: Multiplicity one theorem; Invariant distributions on l-spaces

1. Introduction

In this work we show that over a local non-Archimedean field, the mixed (symplectic-Whittaker) models introduced by Klyachko in [7] are disjoint and that multiplicity one is satisfied. In [8] we showed, over a p-adic field (a finite extension of $\mathbb{Q}_p$), the existence of Klyachko models for unitarizable representations. The upshot is then that for every irreducible, unitarizable representation of GL_n over a p-adic field there is a unique Klyachko model where it appears and it appears there with multiplicity one.

* Corresponding author.
E-mail addresses: offen@mathematik.hu-berlin.de (O. Offen), sayag@math.huji.ac.il (E. Sayag).

doi:10.1016/j.jfa.2008.01.004

To formulate the main result more precisely we introduce some notation. Let F be a non-Archimedean local field. For a positive integer r, denote by U_r the subgroup of upper triangular unipotent matrices in GL_r and let

$$Sp_{2k} = \{g \in GL_{2k}\colon\ {}^t g J_{2k} g = J_{2k}\}$$

where

$$J_{2k} = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix} \tag{1}$$

and $w_k \in GL_k(F)$ is the matrix with (i, j)th entry equal to $\delta_{i,n+1-j}$. Whenever $n = r + 2k$ we consider the subgroup $H_{r,2k}$ of GL_n defined by

$$H_{r,2k} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} \colon\ u \in U_r,\ X \in M_{r\times 2k},\ h \in Sp_{2k} \right\}.$$

Let ψ be a non-trivial character of F. For $u = (u_{i,j}) \in U_r(F)$ we set

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}). \tag{2}$$

Let $\psi_{r,2k}$ be the character of $H_{r,2k}(F)$ defined by

$$\psi_{r,2k}\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u). \tag{3}$$

When $n = r + 2k$ the space

$$\mathcal{M}_{r,2k} = \mathrm{Ind}_{H_{r,2k}(F)}^{GL_n(F)}(\psi_r)$$

is called a mixed model. Here Ind denotes the functor of non-compact smooth induction. Representations of $GL_n(F)$ are always assumed to be smooth. When we say that the representation π of $GL_n(F)$ is unitary we really mean that π is a smooth representation that has a unitary structure. We say that a representation π of $GL_n(F)$ admits the mixed model $\mathcal{M}_{r,2k}$ if $\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r,2k}) \neq 0$. The space

$$\mathcal{M} = \bigoplus_{k=0}^{[n/2]} \mathcal{M}_{n-2k,2k}$$

is referred to as the Klyachko model. The main result of this paper is the following.

Theorem 1. *Let F be a non-Archimedean local field and let π be an irreducible representation of $GL_n(F)$, then*

$$m_\pi = \dim_{\mathbb{C}}\big(\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M})\big) \leqslant 1. \tag{4}$$

When F is a finite field, it is proved in [6] that $m_\pi = 1$ for every irreducible representation π of $GL_n(F)$. When F is a non-Archimedean local field it is shown in [5] that there exists an irreducible representation π of $GL_3(F)$ so that $m_\pi = 0$. Thus, we cannot expect in general for the inequality (4) to be an equality. However, in [8] we showed that if F is a p-adic field then $m_\pi \geqslant 1$ for every irreducible, unitary representation π of $GL_n(F)$. We therefore have the following.

Corollary 1. *Let F be a p-adic field and let π be an irreducible, unitary representation of $GL_n(F)$, then $m_\pi = 1$.*

By Frobenius receiprocity [1, §2.28] for a representation π of $GL_n(F)$ we have

$$\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r,2k}) = \mathrm{Hom}_{H_{r,2k}(F)}(\pi, \psi_r). \tag{5}$$

It follows that for an irreducible, unitary representation π of $GL_n(F)$ there is a unique integer $0 \leqslant \kappa(\pi) \leqslant [\frac{n}{2}]$ such that

$$\mathrm{Hom}_{H_{n-2\kappa(\pi),2\kappa(\pi)}(F)}(\pi, \psi_{n-2\kappa(\pi),2\kappa(\pi)}) \cong \mathbb{C},$$

i.e. such that π is $(H_{n-2\kappa(\pi),2\kappa(\pi)}, \psi_{n-2\kappa(\pi),2\kappa(\pi)})$-*distinguished* and that the space of such functionals is one-dimensional. Moreover, $\kappa(\pi)$ is the explicit value assigned in [8, Theorem 8] in terms of Tadic's classification of the unitary dual.

The *if* direction of the following corollary was proved in [9, Theorem 1]. The other implication is straightforward from Theorem 1. Since it will not serve us further in this work the corollary is formulated using the notation of [9] without recalling it.

Corollary 2. *Let F be a p-adic field and let π be an irreducible, unitary representation of $GL_{2n}(F)$. Then π is distinguished by $Sp_{2n}(F)$ if and only if*

$$\pi \cong U(\delta_1, 2n_1) \times \cdots \times U(\delta_r, 2n_r) \times \pi\big(U\big(\delta'_1, 2n'_1\big), \alpha_1\big) \times \cdots \times \pi\big(U\big(\delta'_s, 2n'_s\big), \alpha_s\big)$$

for some discrete series representations $\delta_1, \ldots, \delta_r$, $\delta'_1, \ldots \delta'_s$, some positive integers $n_1, \ldots, n_r$, $n'_1, \ldots, n'_s$ and some real numbers $\alpha_1, \ldots, \alpha_s$ such that $-\frac{1}{2} < \alpha_i < \frac{1}{2}$.

In [8] we also studied globally over a number field, the mixed (symplectic-Whittaker) periods on the discrete automorphic spectrum of GL_n. Let F be a number field and let ψ be a non-trivial character of $F \setminus \mathbb{A}_F$. We use (2) to view ψ_r as a character of $U_r(\mathbb{A}_F)$ and (3) to view $\psi_{r,2k}$ as a character of $H_{r,2k}(\mathbb{A}_F)$. For an automorphic form ϕ in the discrete automorphic spectrum of $GL_n(\mathbb{A}_F)$ and a decomposition $n = r + 2k$ we consider the mixed period integral

$$P_{r,2k}(\phi) = \int\limits_{H_{r,2k}(F)\backslash H_{r,2k}(\mathbb{A}_F)} \phi(h)\psi_{r,2k}(h)\,dh. \tag{6}$$

We say that an irreducible, discrete spectrum automorphic representation π of $GL_n(\mathbb{A}_F)$ is $(H_{r,2k}, \psi_{r,2k})$-*distinguished* if $P_{r,2k}$ is not identically zero on the space of π. In [8] we provided an explicit integer $0 \leqslant \kappa(\pi) \leqslant [\frac{n}{2}]$ such that π is distinguished by

$$(H_{n-2\kappa(\pi),2\kappa(\pi)}, \psi_{n-2\kappa(\pi),2\kappa(\pi)}).$$

Furthermore, we showed that this period integral is factorizable. Corollary 1 (particularly, the disjointness of Klyachko models) then shows that $\kappa(\pi)$ is the unique such integer. Furthermore, it implies an interesting rigidity property of the discrete automorphic spectrum of GL_n.

Corollary 3. *Let F be a number field and let $\pi = \bigotimes_v \pi_v$ be an irreducible, discrete spectrum automorphic representation of $G(\mathbb{A}_F)$. Then there exists a unique integer $k = \kappa(\pi)$ such that π is $(H_{n-2k,2k}, \psi_{n-2k})$-distinguished. Moreover the following are equivalent*:

(1) *π is $(H_{r,2k}, \psi_{r,2k})$-distinguished*;
(2) *π_v is $(H_{r,2k}, \psi_{r,2k})$-distinguished for all places v of F*;
(3) *π_{v_0} is $(H_{r,2k}, \psi_{r,2k})$-distinguished for some finite place v_0 of F.*

Remark 1. This rigidity property is best understood when Klyachko models are read off the Arthur type defined in [2]. This interpretation will be the subject of a forthcoming note.

The rest of this work is devoted to the proof of Theorem 1. It is organized as follows. After setting up the notation in Section 2, in Sections 3–4 we reduce Theorem 1 to a statement about invariant distributions on orbits. This statement is made more explicit in Section 5 and is then proved by induction in Section 6.

2. Notation

Let F be a non-Archimedean local field and for any positive integer r let $G_r = GL_r(F)$. We denote by I_r the identity matrix in G_r. We also set $G_0 = \{1\}$. Throughout, we fix a positive integer n and let $G = G_n$. For a partition $(n_1, \dots, n_t)$ of n we denote by $P_{(n_1,\dots,n_t)}$ the standard parabolic subgroup of G of type $(n_1, \dots, n_t)$. It consists of matrices in upper triangular block form. If $P = P_{(n_1,\dots,n_t)}$ we denote by $\overline{P}$ the parabolic opposite to P. It consists of matrices in lower triangular block form. When we say that $P = MU$ is the standard Levi decomposition of P we mean that U is its unipotent radical, and $M = P \cap \overline{P} = \{\mathrm{diag}(g_1, \dots, g_t)\colon\ g_i \in G_{n_i}\}$. We then denote by $\overline{U}$ the unipotent radical of $\overline{P}$. We denote by $a^{(r)}$ the r-tuple $(a, \dots, a)$, thus for example $P_{1^{(n)}}$ is the subgroup of upper triangular matrices in G. For any standard Levi subgroup M of G denote by W_M the Weyl group of M and let $W = W_G$. If M' is another standard Levi subgroup then any double coset in $W_M \backslash W / W_{M'}$ has a unique element of minimal length which we refer to as a left W_M and right $W_{M'}$ reduced Weyl element. We denote by ${}_M W_{M'}$ the set of all left W_M and right $W_{M'}$ reduced Weyl elements. For integers a and b we set $[a, b] = \{x \in \mathbb{Z}\colon\ a \leqslant x \leqslant b\}$. For any subset $A \subseteq [1, n]$ we denote by SA the permutation group in the elements of A. It will be convenient to identify W with $S[1, n]$. If $P = MU$ and $P' = M'U'$ are standard parabolic subgroups of G with their standard Levi decompositions, the Bruhat decomposition of G gives the disjoint union

$$G = \bigsqcup_{w \in {}_M W_{M'}} P w \overline{P'}. \tag{7}$$

For any matrix X let ${}^t X$ denote the transpose matrix. For a skew-symmetric matrix $\mathcal{I} = -{}^t\mathcal{I} \in G_{2k}$ let

$$Sp(\mathcal{I}) = \left\{ g \in G_{2k}\colon\ {}^t g \mathcal{I} g = \mathcal{I} \right\}$$

and let

$$J_{2k} = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix}$$

where $w_k \in G_k$ is the matrix with ijth entry $\delta_{i,n+1-j}$. Denote by U_r the subgroup of upper triangular unipotent matrices and by $\overline{U}_r$ the subgroup of lower triangular unipotent matrices in G_r. For non-negative integers r and k let

$$H_{r,2k} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} \colon u \in U_r,\ X \in M_{r\times 2k}(F),\ h \in Sp(J_{2k}) \right\}$$

and let

$$\overline{H}_{r,2k} = \left\{ \begin{pmatrix} u & 0 \\ X & h \end{pmatrix} \colon u \in \overline{U}_r,\ X \in M_{2k\times r}(F),\ h \in Sp(J_{2k}) \right\}.$$

Note that $\overline{H}_{r,2k}$ is the image of $H_{r,2k}$ under transpose. For $g \in G$ let

$$g^\tau = {}^t g^{-1}.$$

The restriction to $H_{r,2k}$ of the involution $\tau : G \to G$ defines a group isomorphism from $H_{r,2k}$ to $\overline{H}_{r,2k}$. Let $n = r + 2k = r' + 2k'$ and let $\mathcal{H}^{r,r'} = \mathcal{H}_n^{r,r'} = H_{r,2k} \times \overline{H}_{r',2k'}$. Thus

$$\mathcal{H}^{r,r'} = \left\{ \left(h_1, h_2^\tau\right) \colon h_1 \in H_{r,2k},\ h_2 \in H_{r',2k'} \right\}.$$

We denote by $e_{\mathcal{H}^{r,r'}}$ the identity element of $\mathcal{H}^{r,r'}$. It will also be useful to consider the map $\xi : \mathcal{H}^{r,r'} \to \mathcal{H}^{r',r}$ defined by

$$\xi\left(h_1, h_2^\tau\right) = \left(h_2, h_1^\tau\right).$$

The group $\mathcal{H}^{r,r'}$ acts on G by

$$h \cdot g = h_1 g\, {}^t h_2, \quad h = \left(h_1, h_2^\tau\right) \in \mathcal{H}^{r,r'},\ g \in G.$$

We observe that

$${}^t(h \cdot g) = \xi(h) \cdot {}^t g, \quad h \in \mathcal{H}^{r,r'},\ g \in G. \tag{8}$$

When $r = r'$ the map ξ is an involution of $\mathcal{H}^{r,r}$. The formula (8) allows us then to define the semi-direct product

$$\widetilde{\mathcal{H}}^{r,r} = \mathcal{H}^{r,r} \rtimes \{\pm 1\}$$

with multiplication rule

$$(h, \epsilon)(h', \epsilon') = \left(h\xi_\epsilon(h'), \epsilon\epsilon'\right) \quad \text{where } \xi_\epsilon(h) = \begin{cases} h, & \epsilon = 1, \\ \xi(h), & \epsilon = -1. \end{cases}$$

Here $h, h' \in \mathcal{H}^{r,r}$ and $\epsilon, \epsilon' \in \{\pm 1\}$. The group $\widetilde{\mathcal{H}}^{r,r}$ acts on G by

$$(h, \epsilon) \cdot g = h \cdot T_\epsilon(g) \quad \text{where } T_\epsilon(g) = \begin{cases} g, & \epsilon = 1, \\ {}^t g, & \epsilon = -1. \end{cases}$$

In order to unify notation, when $r \neq r'$ we shall set $\widetilde{\mathcal{H}}^{r,r'} = \mathcal{H}^{r,r'} \times \{1\}$.

For a non-trivial character ψ of F we define as in Section 1 the generic character ψ_r of U_r by (2) and the character $\psi_{r,2k}$ of $H_{r,2k}$ by (3). Let $\theta^{r,r'}$ be the character of $\mathcal{H}^{r,r'}$ defined by

$$\theta^{r,r'}\big(h_1, h_2^\tau\big) = \psi_{r,2k}(h_1)\psi_{r',2k'}(h_2).$$

We also extend $\theta^{r,r'}$ to the character $\widetilde{\theta}^{r,r'}$ of $\widetilde{\mathcal{H}}^{r,r'}$ defined by

$$\widetilde{\theta}^{r,r'}(h, \epsilon) = \epsilon\theta^{r,r'}(h).$$

3. Reduction to invariant distributions

Let $n = r + 2k = r' + 2k'$ be two decompositions of n. Let $\mathcal{H} = \mathcal{H}^{r,r'}$ and $\theta = \theta^{r,r'}$. The action of $\widetilde{\mathcal{H}}$ on G defines an action on $C_c^\infty(G)$ and on the space $\mathfrak{D}(G) = C_c^\infty(G)^*$ of distributions on G by

$$(h \cdot \phi)(g) = \phi\big(h^{-1} \cdot g\big) \quad \text{and} \quad (h \cdot D)(\phi) = D\big(h^{-1} \cdot \phi\big)$$

for $h \in \widetilde{\mathcal{H}}$, $g \in G$, $\phi \in C_c^\infty(G)$ and $D \in \mathfrak{D}(G)$. In this section we show that Theorem 1 reduces to the following.

Proposition 1. *If $D \in \mathfrak{D}(G)$ is such that $h \cdot D = \tilde{\theta}(h)D$ for all $h \in \widetilde{\mathcal{H}}$ then $D = 0$, i.e.*

$$\operatorname{Hom}_{\widetilde{\mathcal{H}}}\big(C_c^\infty(G), \tilde{\theta}\big) = 0. \tag{9}$$

3.1. Proposition 1 implies Theorem 1

Let π be an irreducible representation of G. Set $H = H_{r,2k}$, $H' = H_{r',2k'}$, $\psi = \psi_{r,2k}$ (forgive the abuse of notation) and $\psi' = \psi_{r',2k'}$. Denote by $\overline{H}$ (respectively $\overline{H'}$) the image of H (respectively H') under τ. Let $\ell \in \operatorname{Hom}_H(\pi, \psi)$ and $\ell' \in \operatorname{Hom}_{H'}(\pi, \psi')$. The representation $\pi^\tau(g) = \pi(g^\tau)$ realizes the contragradient representation $\tilde{\pi}$ on the space V_π of π [3] (see also [1, Theorem 7.3]). Note that $\ell' \in \operatorname{Hom}_{\overline{H'}}(\pi^\tau, (\psi')^\tau)$ defines a functional $\tilde{\ell}'$ on the space $V_{\tilde{\pi}}$ of $\tilde{\pi}$ and that $\tilde{\ell}' \in \operatorname{Hom}_{\overline{H'}}(\tilde{\pi}, (\psi')^\tau)$. Note further that $\ell \circ \pi(\phi)$ is a smooth vector in $V_{\tilde{\pi}}$. Define the distribution D on G by

$$D(\phi) = \tilde{\ell}'\big(\ell \circ \pi(\phi)\big), \quad \phi \in C_c^\infty(G). \tag{10}$$

For $h \in H$ and $h' \in H'$ we have $\pi((h^{-1}, {}^t h') \cdot \phi) = \pi(h) \circ \pi(\phi) \circ \pi({}^t h')$ and therefore

$$\big(\big(h, (h')^\tau\big) \cdot D\big)(\phi) = \tilde{\ell}'\big(\ell \circ \pi(h) \circ \pi(\phi) \circ \pi\big({}^t h'\big)\big).$$

By our assumption on ℓ and ℓ' we have, $\ell \circ \pi(h) = \psi(h)\ell$ and $\tilde{\ell}' \circ \tilde{\pi}((h')^\tau) = \psi'(h')\tilde{\ell}'$, $h \in H$, $h' \in H'$. Also note that for any $\tilde{v} \in V_{\tilde{\pi}}$ viewed as a smooth functional on π the composition $\tilde{v} \circ \pi(g)$ is again a smooth functional on π and in fact

$$\big(\tilde{v} \circ \pi(g)\big)(v) = \tilde{v}\big(\pi(g)v\big) = \big(\tilde{\pi}\big(g^{-1}\big)\tilde{v}\big)(v)$$

i.e.,

$$\tilde{v} \circ \pi(g) = \tilde{\pi}\big(g^{-1}\big)\tilde{v}.$$

Applying this to $\tilde{v} = \ell \circ \pi(\phi)$ and $g = {}^t h'$ we get that

$$\begin{aligned}\big(\big(h, (h')^\tau\big) \cdot D\big)(\phi) &= \psi(h)\, \tilde{\ell}'\big(\big(\ell \circ \pi(\phi)\big) \circ \pi\big({}^t h'\big)\big) \\ &= \psi(h)\, \tilde{\ell}'\big(\tilde{\pi}\big((h')^\tau\big)\big(\ell \circ \pi(\phi)\big)\big) = \theta\big(h, (h')^\tau\big)\, D(\phi).\end{aligned}$$

We see that D is $(\mathcal{H}, \theta)$-equivariant. If $r \neq r'$ it follows from Proposition 1 that $D = 0$. If we assume further that ℓ is non-zero then the vectors $\ell \circ \pi(\phi)$, $\phi \in C_c^\infty(G)$ span $V_{\tilde{\pi}}$. We conclude that $\tilde{\ell}'$ must vanish identically on $V_{\tilde{\pi}}$ and hence also $\ell' = 0$. This shows that

$$\dim_{\mathbb{C}}\big(\mathrm{Hom}_{H_{r,2k}}(\pi, \psi_{r,2k})\big) \dim_{\mathbb{C}}\big(\mathrm{Hom}_{H_{r',2k'}}(\pi, \psi_{r',2k'})\big) = 0 \quad \text{whenever } r \neq r'. \tag{11}$$

Assume now that $r = r'$. Recall that $e_{\mathcal{H}}$ is the unit element of $\mathcal{H}$. Note that $(e_{\mathcal{H}}, -1) \cdot \phi = {}^t\phi$ where ${}^t\phi(g) = \phi({}^t g)$, $\phi \in C_c^\infty(G)$, $g \in G$. Note further that for every $h \in \mathcal{H}$ we have

$$(h, 1)(e_{\mathcal{H}}, -1) = (e_{\mathcal{H}}, -1)\big(\xi(h), 1\big)$$

and that $\theta(\xi(h)) = \theta(h)$. Since $D \in \mathrm{Hom}_{\mathcal{H}}(C_c^\infty(G), \theta)$, it also follows that

$$D_1 = D - (e_{\mathcal{H}}, -1) \cdot D \in \mathrm{Hom}_{\mathcal{H}}\big(C_c^\infty(G), \theta\big).$$

Furthermore, since $\tilde{\theta}(e_{\mathcal{H}}, -1) = -1$ and $(e_{\mathcal{H}}, -1) \cdot D_1 = -D_1$ we conclude that $D_1 \in \mathrm{Hom}_{\tilde{\mathcal{H}}}(C_c^\infty(G), \tilde{\theta})$. Proposition 1 now implies that

$$D = (e_{\mathcal{H}}, -1) \cdot D. \tag{12}$$

Let $B : C_c^\infty(G) \times C_c^\infty(G) \to \mathbb{C}$ be the bilinear form defined by

$$B(\phi_1, \phi_2) = D(\phi_1 * \phi_2) \tag{13}$$

where

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(x)\phi_2\big(x^{-1}g\big)\, dx.$$

Note that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2) \quad \text{and} \quad {}^t(\phi_1 * \phi_2) = {}^t\phi_2 * {}^t\phi_1, \phi_1, \phi_2 \in C_c^\infty(G).$$

Thus, (12) implies that

$$B(\phi_1, \phi_2) = B\big((e_{\mathcal{H}}, -1) \cdot \phi_2, (e_{\mathcal{H}}, -1) \cdot \phi_1\big).$$

This implies that $R_B = (e_{\mathcal{H}}, -1) \cdot L_B$ where

$$L_B = \big\{\phi \in C_c^\infty(G)\colon\ B(\phi, \cdot) \equiv 0\big\} \quad \text{and} \quad R_B = \big\{\phi \in C_c^\infty(G)\colon\ B(\cdot, \phi) \equiv 0\big\}$$

are respectively the left and right kernels of B. In other words

$$R_B = \big\{{}^t\phi\colon\ \phi \in L_B\big\}. \tag{14}$$

For a functional λ on V_π let

$$\mathfrak{K}(\lambda, \pi) = \big\{\phi \in C_c^\infty(G)\colon\ \lambda \circ \pi(\phi) = 0\big\}.$$

Note that

$$B(\phi_1, \phi_2) = \big(\tilde{\ell}' \circ \tilde{\pi}\big(\phi_2^\vee\big)\big)\big(\ell \circ \pi(\phi_1)\big)$$

where

$$\phi^\vee(g) = \phi\big(g^{-1}\big)$$

and therefore assuming further that both ℓ and ℓ' are not zero we have

$$L_B = \mathfrak{K}(\ell, \pi) \quad \text{and} \quad R_B = \big\{\phi^\vee\colon\ \phi \in \mathfrak{K}(\tilde{\ell}', \tilde{\pi})\big\}.$$

By our definitions we have

$$\mathfrak{K}(\tilde{\ell}', \tilde{\pi}) = \mathfrak{K}\big(\ell', \pi^\tau\big) = \big\{\big({}^t\phi\big)^\vee\colon\ \phi \in \mathfrak{K}(\ell', \pi)\big\}$$

and therefore

$$R_B = \big\{{}^t\phi\colon\ \phi \in \mathfrak{K}(\ell', \pi)\big\}.$$

It now follows from (14) that

$$\mathfrak{K}(\ell, \pi) = \mathfrak{K}(\ell', \pi).$$

Since π is irreducible we get that $\ker \ell = \ker \ell'$ and therefore that ℓ and ℓ' are proportional. We therefore proved that

$$\dim_{\mathbb{C}}\big(\mathrm{Hom}_{H_{r,2k}}(\pi, \psi_{r,2k})\big) \leqslant 1 \quad \text{for all } 0 \leqslant k \leqslant \left[\frac{n}{2}\right]. \tag{15}$$

Theorem 1 is now a straightforward consequence of (5), (11) and (15).

4. Reduction to $\mathcal{H}$-orbits

We keep the notation introduced in Section 3. For every $g \in G$ we denote by $\mathcal{H}_g$ the stabilizer of g in $\mathcal{H}$ and by $\widetilde{\mathcal{H}}_g$ the stabilizer of g in $\widetilde{\mathcal{H}}$. The purpose of this section is to reduce Proposition 1 to the following.

Proposition 2. *For every $g \in G$ the character $\widetilde{\theta}$ is non-trivial on $\widetilde{\mathcal{H}}_g$.*

Remark 2. The objects involved and the statement of Proposition 2 make sense over any field F and in fact, our proof is valid in this generality. In particular, using Mackey theory, it gives an alternative proof of the uniqueness and disjointness of Klyachko models over a finite field.

4.1. Proposition 2 implies Proposition 1

Assume now that Proposition 2 holds. We deduce that Proposition 1 also holds. Let $\mathbf{1}_{\widetilde{\mathcal{H}}_g}$ denote the trivial character of $\widetilde{\mathcal{H}}_g$. Note that $h \cdot g \mapsto \widetilde{\mathcal{H}}_g h^{-1}$ is a homeomorphism of $\widetilde{\mathcal{H}}$-spaces $\widetilde{\mathcal{H}} \cdot g \simeq \widetilde{\mathcal{H}}_g \backslash \widetilde{\mathcal{H}}$ that induces an $\widetilde{\mathcal{H}}$-isomorphism

$$C_c^\infty(\widetilde{\mathcal{H}} \cdot g) \simeq \operatorname{ind}_{\widetilde{\mathcal{H}}_g}^{\widetilde{\mathcal{H}}}(\mathbf{1}_{\widetilde{\mathcal{H}}_g}) \tag{16}$$

where ind denotes smooth induction with compact support. Therefore, by Frobenius reciprocity [1, §2.29]

$$\operatorname{Hom}_{\widetilde{\mathcal{H}}}\big(C_c^\infty(\widetilde{\mathcal{H}} \cdot g), \tilde{\theta}\big) = \operatorname{Hom}_{\widetilde{\mathcal{H}}_g}(\delta_{\widetilde{\mathcal{H}}_g}, \theta_{|\widetilde{\mathcal{H}}_g}) \tag{17}$$

where $\delta_{\widetilde{\mathcal{H}}_g}$ is the modulus function of $\widetilde{\mathcal{H}}_g$. Since the image of $\tilde{\theta}$ lies in the unit circle (in fact, the image of θ lies in the group of p-powered roots of unity where p is the residual characteristic of F) and since $\delta_{\widetilde{\mathcal{H}}_g}$ is positive, we get that whenever $\tilde{\theta}_{|\widetilde{\mathcal{H}}_g}$ is non-trivial we also have

$$\tilde{\theta}_{|\widetilde{\mathcal{H}}_g} \neq \delta_{\widetilde{\mathcal{H}}_g}. \tag{18}$$

It follows from Proposition 2 that (18) holds for every $g \in G$ and therefore by (16) that

$$\operatorname{Hom}_{\widetilde{\mathcal{H}}}\big(C_c^\infty(\widetilde{\mathcal{H}} \cdot g), \tilde{\theta}\big) = 0, \quad g \in G. \tag{19}$$

Proposition 1 follows from (19) using the Gelfand–Kazhdan theory [3]. Indeed, we apply [1, Theorem 6.9] to the following setting. We view $C_c^\infty(G)$ as a module over itself by convolution. By [1, Proposition 1.14] it uniquely defines a sheaf $\mathcal{F}$ over the l-space G. We let $\widetilde{\mathcal{H}}$ act on $C_c^\infty(G)$ by

$$h \cdot_{\tilde{\theta}} \phi = \tilde{\theta}(h) h \cdot \phi.$$

This defines an action of $\widetilde{\mathcal{H}}$ on the sheaf $\mathcal{F}$. The space of $\widetilde{\mathcal{H}}$-invariant distributions on $\mathcal{F}$ is then precisely $\operatorname{Hom}_{\widetilde{\mathcal{H}}}(C_c^\infty(G), \tilde{\theta})$. The action of $\widetilde{\mathcal{H}}$ on G is constructible by [1, §6.15, Theorem A]. The second assumption of [1, Theorem 6.9] is precisely (19). It follows that there are no $\widetilde{\mathcal{H}}$-invariant distributions on the sheaf $\mathcal{F}$, i.e. that (9) holds.

5. The property of $\mathcal{H}$-orbits made explicit

In order to prove Proposition 2 it will be convenient to reformulate it, by describing more explicitly the property of the $\widetilde{\mathcal{H}}$-orbits that we wish to prove. We begin with this reformulation.

5.1. The property $\mathcal{P}(g,r,r')$

For $g \in G$ let $\mathcal{P}(g,r,r') = \mathcal{P}_n(g,r,r')$ be the following property: either

$$\text{there exists } y \in H_{r,2k} \text{ such that } g^{-1}yg \in \overline{H}_{r',2k'} \text{ and } \theta^{r,r'}\big(y, g^{-1}yg\big) \neq 1 \tag{20}$$

or $r = r'$ and

$$\text{there exists } y \in H_{r,2k} \text{ such that } g^{-1}y\,{}^t g \in \overline{H}_{r,2k} \text{ and } \theta^{r,r}\big(y, g^{-1}y\,{}^t g\big) = 1. \tag{21}$$

Lemma 1. *For every $g \in G$, $\widetilde{\theta}^{r,r'}$ is non-trivial on $\widetilde{\mathcal{H}}_g^{r,r'}$ if and only if the property $\mathcal{P}(g,r,r')$ holds.*

Proof. Note that

$$\mathcal{H}_g^{r,r'} = \big\{\big(y, g^{-1}yg\big):\ y \in H_{r,2k} \cap g\overline{H}_{r',2k'}g^{-1}\big\}$$

and therefore (20) holds if and only if $\theta^{r,r'}$ is not trivial on $\mathcal{H}_g^{r,r'}$. If $r \neq r'$ this proves the lemma. If $r = r'$ it remains to show that when $\theta^{r,r}$ is trivial on $\mathcal{H}_g^{r,r}$ then $\widetilde{\theta}^{r,r}$ is not trivial on $\widetilde{\mathcal{H}}_g^{r,r}$ if and only if we have (21). Note that

$$\big\{h \in \mathcal{H}^{r,r}:\ (h,-1) \in \widetilde{\mathcal{H}}_g^{r,r}\big\} = \big\{\big(y, g^{-1}y\,{}^t g\big):\ y \in H_{r,2k} \cap g\overline{H}_{r,2k}g^{\tau}\big\}.$$

If $y \in H_{r,2k} \cap g\overline{H}_{r,2k}g^{\tau}$ then for $h = (y, g^{-1}y\,{}^t g) \in \mathcal{H}^{r,r}$ we have $h \cdot {}^t g = g$, i.e. $(h,-1) \in \widetilde{\mathcal{H}}_g^{r,r}$ and therefore by (8) we get that $h\,\xi(h) \in \mathcal{H}_g^{r,r}$ so that $\theta^{r,r}(h\,\xi(h)) = 1$. Since $\theta^{r,r} = \theta^{r,r} \circ \xi$ we have $\theta^{r,r}(h) \in \{\pm 1\}$. With this notation (21) is satisfied by y if and only if $\theta^{r,r}(h) = 1$ if and only if $\tilde{\theta}^{r,r}(h,-1) = -1$. The remaining of the lemma follows. $\square$

We make here another simple observation that will help to shorten some of the arguments in the proof of Proposition 2.

Lemma 2. *If the property $\mathcal{P}(g,r,r')$ holds then $\mathcal{P}(h \cdot g,r,r')$ holds for all $h \in \widetilde{\mathcal{H}}$ and $\mathcal{P}({}^t g,r',r)$ holds.*

Proof. Note that $\widetilde{\mathcal{H}}_{h\cdot g} = h\widetilde{\mathcal{H}}_g h^{-1}$ and that $\widetilde{\theta}$ is a character. Thus, the first statement is immediate from Lemma 1. If $r = r'$ this argument with $h = (e_{\mathcal{H}}, -1)$ also contains the second statement. If $r \neq r'$ the second statement follows from the fact that $\mathcal{H}_{{}^t g}^{r',r} = \xi(\mathcal{H}_g^{r,r'})$ (that follows from (8)) and the fact that $\theta \circ \xi = \theta$. $\square$

In light of Lemma 1 in order to show Proposition 2 we need to show that for every $r, r' \leqslant n$ such that $n - r \equiv n - r' \equiv 0 \pmod 2$ and for every $g \in G$ we have $\mathcal{P}(g,r,r')$. This will occupy the rest of this paper.

5.2. Two cases where $\mathcal{P}(g,r,r')$ is already known

There are two extremes that are already known. The first is a well-known fact concerned with the double coset space $U_n\backslash G/\overline{U}_n$. It can be found in the proof of [3, Lemma 4.3.8] (it is essentially the steps (a)–(d) verifying condition 4 of [3, Theorem 4.2.10]) and it is applied in order to prove the uniqueness of Whittaker models. We provide a proof here for the sake of completeness.

Lemma 3. *For every $g \in G$ the property $\mathcal{P}_n(g,n,n)$ holds.*

Proof. By the Bruhat decomposition every $\mathcal{H}^{n,n}$-orbit in G contains an element of the form aw where $w \in W$ and $a = \mathrm{diag}(a_1,\dots,a_n)$. We show that if $\theta^{n,n}$ is trivial on the stabilizer $\mathcal{H}^{n,n}_{aw}$, i.e. if (20) is not satisfied then

$$\begin{aligned} &\text{for all } i = 1,\dots,n-1, \quad \text{either } w^{-1}(i) < w^{-1}(i+1)\\ &\text{or } w^{-1}(i) = w^{-1}(i+1)+1 \text{ and } a_i = a_{i+1}. \end{aligned} \tag{22}$$

When (22) is satisfied then there is a partition $(n_1,\dots,n_t)$ of n such that $w = \mathrm{diag}(w_{n_1},\dots,w_{n_t})$ is the longest element of W_M where M is the standard Levi subgroup of G of type $(n_1,\dots,n_t)$ and a lies in the center of M. In particular, we then have $aw = {}^t(aw)$ and therefore aw satisfies (21) with $y = I_n$. Assume now that $\theta^{n,n}$ is trivial on $\mathcal{H}^{n,n}_{aw}$. Let $u,\ v \in U_n$ be such that $(u,v^\tau) \in \mathcal{H}^{n,n}_{aw}$. Thus, $w^{-1}a^{-1}uaw = v^\tau \in \overline{U}_n$ and therefore for any $i<j$ if $w^{-1}(i) < w^{-1}(j)$ then $u_{i,j} = 0$ and if $w^{-1}(i) > w^{-1}(j)$ then $(v^\tau)_{w^{-1}(i),w^{-1}(j)} = a_i^{-1}a_j u_{i,j}$. Let $E_{i,j} \in M_{n\times n}(F)$ be the matrix with (b,c)th entry equal to $\delta_{(i,j),(b,c)}$ and let $u_{i,j}(s) = I_n + s\,E_{i,j}$, $s \in F$. If $w^{-1}(i) > w^{-1}(i+1)$ then for any $s \in F$ if $u = u_{i,i+1}(s)$ and $v = (w^{-1}a^{-1}uaw)^\tau$ we have $(u,v^\tau) \in \mathcal{H}^{n,n}_{aw}$. If $w^{-1}(i) > w^{-1}(i+1)+1$ then $\theta(u,v^\tau) = \psi(s)$ and since s may be chosen arbitrarily this leads to a contradiction. Thus $w^{-1}(i) = w^{-1}(i+1)+1$ and $1 = \theta(u,v^\tau) = \psi(s(1-a_i^{-1}a_{i+1}))$. It follows that $a_i = a_{i+1}$. The property (22) is therefore satisfied. □

The second extreme is with respect to the symplectic group. It was proved by Heumus and Rallis [5, Proposition 2.3.1] based on results of Klyachko [7, Corollary 5.6]. Recently, Goldstein and Guralnick essentially provided an independent proof over any field [4, Proposition 3.1].

Lemma 4. *When n is even for every $g \in G$ the property $\mathcal{P}_n(g,0,0)$ holds.*

Proof. We show that when $r = r' = 0$ (21) holds for every $g \in G$. That is, we show that for every $g \in G$ we have ${}^tg \in Sp(J_n)g\,Sp(J_n)$. As observed in the proof of Lemma 2, it is enough to prove that there exists $y \in Sp(J_n)g\,Sp(J_n)$ such that ${}^ty \in Sp(J_n)g\,Sp(J_n)$. Let $n = 2k$ and let

$$J_n' = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} = {}^t\sigma J_n \sigma \quad \text{where } \sigma = \begin{pmatrix} w_k & 0 \\ 0 & I_k \end{pmatrix}.$$

Thus,

$$Sp(J_n') = \sigma^{-1} Sp(J_n)\sigma.$$

It follows from [4, Proposition 3.1] that there exists $g' \in G_k$ such that $\mathrm{diag}(I_k, g') \in Sp(J_n')\sigma^{-1} \times g\sigma Sp(J_n')$, i.e. that $y = \sigma\,\mathrm{diag}(I_k, g')\sigma^{-1} \in Sp(J_n)gSp(J_n)$. Since every matrix in G_k is conjugate to its transpose and since $\mathrm{diag}(x, {}^tx) \in Sp(J_n')$ for every $x \in G_k$ we see that $\mathrm{diag}(I_k, {}^tg') \in Sp(J_n')\sigma g\sigma Sp(J_n')$, i.e. that ${}^ty = \sigma\,\mathrm{diag}(I_k, {}^tg')\sigma^{-1} \in Sp(J_n)gSp(J_n)$. □

6. Proof by induction of $\mathcal{P}_n(g, r, r')$

Fix two decompositions $n = r + 2k = r' + 2k'$. We prove by induction on n that for every $g \in G$ we have $\mathcal{P}_n(g, r, r')$. If $r = r' = 0$ then this is Lemma 4. We assume from now on that $r + r' > 0$. The induction hypothesis is that for all $n_1 < n$, all $r_1, r_1' \leqslant n_1$ such that $n_1 - r_1 \equiv n_1 - r_1' \equiv 0 \pmod 2$ and all $g' \in G_{n_1}$ we have $\mathcal{P}_{n_1}(g', r_1, r_1')$. Set $H = H_{r,2k}$, $H' = H_{r',2k'}$, $\mathcal{H} = H \times \overline{H}'$ and $\theta = \theta^{r,r'}$. Let $P = P_{(1^{(r)},2k)}$ and $P' = P_{(1^{(r')},2k')}$. For $w \in W$ viewed as a permutation in $S[1, n]$ let

$$I_w = \{i \in [1, r]:\ w^{-1}(i) \in [1, r']\}.$$

6.1. A simple proof for most Bruhat cells

Lemma 5. *Let $w \in {}_MW_{M'}$ be such that I_w is not empty then the property $\mathcal{P}(g, r, r')$ holds for every $g \in Pw\overline{P'}$.*

Proof. Note that $U \times \overline{U}' \subseteq \mathcal{H}$ and therefore that every $\mathcal{H}$-orbit in $Pw\overline{P'}$ contains an element of MwM'. In light of Lemma 2 we may assume without loss of generality that $g \in MwM'$.

Assume first that there exists an integer i such that $1 \leqslant i \leqslant \min\{r, r'\}$ and $I_w = w^{-1}(I_w) = [1, i]$. We can then write $w = \mathrm{diag}(w_1, w_2)$ for some $w_1 \in S[1, i]$ and $w_2 \in S[i + 1, n]$. Thus for $g \in MwM'$ there exist $g_1, g_2 \in G_{n-i}$, and $a = \mathrm{diag}(a_1, \ldots, a_i) \in G_i$ such that $g = \mathrm{diag}(I_i, g_1)w\,\mathrm{diag}(a, g_2) = \mathrm{diag}(w_1a, g')$ for $g' = g_1w_2g_2 \in G_{n-i}$. Let $(u_1, u_2^\tau, \epsilon) \in (\widetilde{\mathcal{H}}_i^{i,i})_{w_1a}$ be such that $\widetilde{\theta}^{i,i}(u_1, u_2^\tau, \epsilon) \neq 1$ and let $(h_1, h_2^\tau, \epsilon') \in (\widetilde{\mathcal{H}}_{n-i}^{r-i,r'-i})_{g'}$ be such that $\widetilde{\theta}^{r-i,r'-i}(h_1, h_2^\tau, \epsilon') \neq 1$. The first exists by Lemma 3. For the second we apply the induction hypothesis to have $\mathcal{P}_{n-i}(g', r - i, r' - i)$. If $\epsilon = 1$ then

$$h = (\mathrm{diag}(u_1, I_{n-i}), \mathrm{diag}(u_2, I_{n-i})^\tau, 1) \in \widetilde{\mathcal{H}}_g \quad \text{and} \quad \widetilde{\theta}(h) = \widetilde{\theta}^{i,i}(u_1, u_2^\tau, 1) \neq 1.$$

Similarly, if $\epsilon' = 1$ then

$$h = (\mathrm{diag}(I_i, h_1), \mathrm{diag}(I_i, h_2)^\tau, 1) \in \widetilde{\mathcal{H}}_g \quad \text{and} \quad \widetilde{\theta}(h) = \widetilde{\theta}^{r-i,r'-i}(h_1, h_2^\tau, 1) \neq 1.$$

If on the other hand $\epsilon = \epsilon' = -1$ then

$$h = (\mathrm{diag}(u_1, h_1), \mathrm{diag}(u_2, h_2)^\tau, -1) \in \widetilde{\mathcal{H}}_g \quad \text{and} \quad \widetilde{\theta}(h) = -1.$$

We are now left with the case that either I_w or $w^{-1}(I_w)$ is not of the form $[1, i]$ as above. Note that if $g \in Pw\overline{P'}$ then ${}^tg \in P'w^{-1}\overline{P}$ and that $w^{-1} \in {}_{M'}W_M$. It follows from Lemma 2 that it is enough to prove our lemma either for g or for ${}^tg$. We may therefore assume, without loss of generality, that I_w is not of the form $[1, i]$ for any $1 \leqslant i \leqslant \min\{r, r'\}$. Since we assume that $g \in MwM'$ there exist $g_1 \in G_{2k}$, $g_2 \in G_{2k'}$ and $a = \mathrm{diag}(a_1, \ldots, a_{r'})$ a diagonal matrix in $G_{r'}$ such that $g = \mathrm{diag}(I_r, g_1)w\,\mathrm{diag}(a, g_2)$. By our assumption on w we have that $[1, r] \setminus I_w$

is not empty. Let $\ell = \min([1,r] \setminus I_w)$. Since $[1, \ell - 1]$ is contained but does not equal I_w the set $[\ell + 1, r] \cap I_w$ is not empty. Let $q = \min([\ell + 1, r] \cap I_w)$. Then $q - 1 \notin I_w$ and $q \in I_w$. In particular, $w^{-1}(q-1) > r'$ and $w^{-1}(q) \leqslant r'$. Let $E_{i,j} \in M_{n\times n}(F)$ be the matrix with (b,c)th entry equal to $\delta_{(i,j),(b,c)}$ and let $u_{i,j}(s) = I_n + s\ E_{i,j},\ s \in F$. Note that $u_{q-1,q}(s) \in U \subseteq H_{r,2k}$ and that $\psi_{r,2k}(u_{q-1,q}(s)) = \psi(s)$. Thus, there exists $s \in F$ such that $\psi_{r,2k}(u_{q-1,q}(s)) \neq 1$. On the other hand,

$$g^{-1}u_{q-1,q}(s)g = \begin{pmatrix} a^{-1} & 0 \\ 0 & g_2^{-1} \end{pmatrix} u_{w^{-1}(q-1),w^{-1}(q)}(s) \begin{pmatrix} a & 0 \\ 0 & g_2 \end{pmatrix} = \begin{pmatrix} I_{r'} & 0 \\ * & I_{2k'} \end{pmatrix} \in \overline{H}_{r',2k'}$$

and $\psi_{r',2k'}(g^{-1}u_{q-1,q}(s)g) = 1$. It follows that $h_s = (u_{q-1,q}(s), g^{-1}u_{q-1,q}(s)g) \in \mathcal{H}_g$ and if s is such that $\psi_{r,2k}(u_{q-1,q}(s)) \neq 1$ then $\theta(h_s) \neq 1$. □

6.2. *The closed Bruhat cell*

We are now left with the case that I_w is empty. Since this means that w^{-1} maps $[1,r]$ into $[r'+1,n]$ we must have, in particular, $n \geqslant r + r'$. It is not difficult to see that there is then a unique such element in ${}_M W_{M'}$, namely,

$$w = w^{r,r'} = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix}.$$

Note then that $Pw\overline{P'}$ is the closed Bruhat cell. We remark further that this contains the case that either r or r' is 0. Let $g \in MwM'$. Note that there exist $g_1 \in G_{2k}$ and $g_2 \in G_{2k'}$ such that

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix}.$$

Indeed, for $t \in G_r$, $t' \in G_{r'}$ (and in particular when t and t' are diagonal) if $g_1' \in G_{2k}$ and $g_2' \in G_{2k'}$ we have

$$\begin{pmatrix} t & \\ & g_1' \end{pmatrix} w \begin{pmatrix} t' & \\ & g_2' \end{pmatrix} = \begin{pmatrix} I_r & \\ & g_1' \end{pmatrix} \begin{pmatrix} 0 & t & 0 \\ t' & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} I_r' & \\ & g_2' \end{pmatrix} = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix}$$

where $g_1 = g_1' \operatorname{diag}(t', I_{2k-r'})$ and $g_2 = \operatorname{diag}(t, I_{2k'-r}) g_2'$.

In order to show $\mathcal{P}(g,r,r')$ we distinguish between two cases. We denote by $\langle v_1, \ldots, v_i \rangle$ the subspace of a vector space V spanned by $v_1, \ldots, v_i \in V$. Let V be a subspace of the vector space $M_{\ell\times 1}(F)$ for some positive integer ℓ. We say that a skew symmetric matrix $\mathcal{I} \in M_{\ell\times\ell}(F)$ is totally isotropic on V if ${}^t v\mathcal{I}v' = 0$ for all $v,\ v' \in V$. Denote by e_i the column vector with 1 in the ith row and 0 in each other row. Thus $e_i \in M_{\ell\times 1}(F)$ for an integer ℓ which is implicit in our notation. Let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \quad \text{and} \quad \mathcal{I}_2 = g_2^\tau J_{2k'} g_2^{-1}.$$

We say that g belongs to the totally isotropic case if both $\mathcal{I}_1^{-1}$ is totally isotropic on $\langle e_1, \dots, e_{r'}\rangle$ and $\mathcal{I}_2$ is totally isotropic on $\langle e_1, \dots, e_r\rangle$. Otherwise we say that g does not belong to the totally isotropic case. It is easy to verify that this property indeed depends only on g and not on g_1 and g_2. Note that if g belongs to the totally isotropic case we must have $r' \leqslant k$ and $r \leqslant k'$. These inequalities are crucial for the proof of Lemma 11. They follow from the simple observation that a totally isotropic subspace for a nondegenerate symplectic form $\mathcal{I}$ is of dimension at most half of the rank of $\mathcal{I}$. We now prove $\mathcal{P}(g, r, r')$ separately in each of the two cases.

6.2.1. When g does not belong to the totally isotropic case

In this case we prove that g satisfies (20). It will be convenient to make this property more explicit. We say that the 2 skew-symmetric forms $\mathcal{I}_1 \in G_{2k}$ and $\mathcal{I}_2 \in G_{2k'}$ satisfy the property $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ if there exist $u \in U_r$ and $u' \in U_{r'}$ such that $\psi_r(u) \neq \psi_{r'}(u')$ and for some $X \in M_{r\times 2k'-r}(F)$, $Y \in M_{r'\times 2k-r'}(F)$ and $D \in G_{n-(r+r')}$ we have

$$\begin{pmatrix} u & X \\ 0 & D \end{pmatrix} \in Sp(\mathcal{I}_2) \quad \text{and} \quad \begin{pmatrix} {}^t u' & 0 \\ Y & D \end{pmatrix} \in Sp(\mathcal{I}_1).$$

Lemma 6. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

and let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \quad \textit{and} \quad \mathcal{I}_2 = g_2^\tau J_{2k'} g_2^{-1}.$$

Then g satisfies (20) *if and only if* $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$.

Proof. Let

$$y = \begin{pmatrix} u & Z \\ & h \end{pmatrix} \in H$$

with $u \in U_r$, $h \in Sp(J_{2k})$ and $Z \in M_{r\times 2k}(F)$. To explicate condition (20) we compute $g^{-1}yg$. First note that we have

$$\begin{pmatrix} I_r & \\ & g_1^{-1} \end{pmatrix} \begin{pmatrix} u & Z \\ & h \end{pmatrix} \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} = \begin{pmatrix} u & Zg_1 \\ & g_1^{-1} h g_1 \end{pmatrix}.$$

We write

$$g_1^{-1} h g_1 = \begin{pmatrix} {}^t u' & B \\ Y & D \end{pmatrix} \quad \text{and} \quad Zg_1 = (Z_1, Z_2)$$

with $u' \in M_{r' \times r'}(F)$, $D \in M_{2k-r' \times 2k-r'}(F)$, $Z_1 \in M_{r \times r'}(F)$ and $Z_2 \in M_{r \times 2k-r'}(F)$. We then have

$$\begin{pmatrix} 0 & I_{r'} & 0 \\ I_r & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} u & Z_1 & Z_2 \\ 0 & {}^t u' & B \\ 0 & Y & D \end{pmatrix} \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} = \begin{pmatrix} {}^t u' & 0 & B \\ Z_1 & u & Z_2 \\ Y & 0 & D \end{pmatrix}.$$

Therefore,

$$g^{-1} y g = \begin{pmatrix} {}^t u' & (0, B) g_2 \\ g_2^{-1} \begin{pmatrix} Z_1 \\ Y \end{pmatrix} & g_2^{-1} \begin{pmatrix} u & Z_2 \\ 0 & D \end{pmatrix} g_2 \end{pmatrix}.$$

We see that $g^{-1} y g \in \overline{H'}$ if and only if $u' \in U_{r'}$, $B = 0$ and

$$g_2^{-1} \begin{pmatrix} u & Z_2 \\ 0 & D \end{pmatrix} g_2 \in Sp(J_{2k'}).$$

Recall also that

$$\begin{pmatrix} {}^t u' & 0 \\ Y & D \end{pmatrix} \in g_1^{-1} Sp(J_{2k}) g_1.$$

With this notation, when $g^{-1} y g \in \overline{H'}$ we have

$$\theta\big(y, g^{-1} y g\big) = \psi_r(u) \psi_{r'}\big((u')^{-1}\big).$$

Since

$$g_1^{-1} Sp(J_{2k}) g_1 = Sp(\mathcal{I}_1) \quad \text{and} \quad g_2 Sp(J_{2k'}) g_2^{-1} = Sp(\mathcal{I}_2),$$

the lemma is now immediate. □

In order to proceed we need the following lemma of Klyachko [7, §1.3, Step 3, p. 368].

Lemma 7. *Let $\mathcal{I} = -{}^t\mathcal{I} \in G_{2k}$ and let $r \leqslant 2k$ be such that $\mathcal{I}$ is not totally isotropic on $\langle e_1, \ldots, e_r \rangle$. Then there exist $u \in U_r$ with $\psi_r(u) \neq 1$ and $X \in M_{r \times 2k-r}(F)$ such that*

$$\begin{pmatrix} u & X \\ 0 & I_{2k-r} \end{pmatrix} \in Sp(\mathcal{I}). \tag{23}$$

Proof. Let $i \in [1, r-1]$ be maximal so that $\mathcal{I}$ is totally isotropic on $\langle e_1, \ldots, e_i \rangle$. There is therefore $v_0 \in \langle e_1, \ldots, e_i \rangle$ such that ${}^t v_0 \mathcal{I} e_{i+1} \neq 0$. We may further assume that $v_0 \in e_i + \langle e_1, \ldots, e_{i-1} \rangle$ since if ${}^t e_i \mathcal{I} e_{i+1} \neq 0$ then we may take $v_0 = e_i$ and otherwise, we may replace v_0 by its sum with any scalar multiple of e_i. Let $V = M_{2k \times 1}(F)$ and for every $s \in F$ define $\lambda_s \in \mathrm{Hom}_F(V, F)$ by $\lambda_s(v) = s\, {}^t v_0 \mathcal{I} v$. Note that the map $s \mapsto \lambda_s(e_{i+1})$, $s \in F$ is onto F. Identify $GL(V)$ with G_{2k} via the standard basis $\{e_1, \ldots, e_{2k}\}$ and define an element $h_s \in G_{2k}$ by

$$h_s(v) = v + \lambda_s(v) v_0.$$

Thus, $h_s \in Sp(\mathcal{I})$ is of the form (23) with $\psi_r(u) = \psi(\lambda_s(e_{i+1}))$. □

Lemma 8. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

not belong to the totally isotropic case and let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \quad \text{and} \quad \mathcal{I}_2 = g_2^\tau J_{2k'} g_2^{-1}.$$

Then $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ holds.

Proof. If $\mathcal{I}_2$ is not totally isotropic on $\langle e_1, \dots, e_r\rangle$ then by Lemma 7 there exist $u \in U_r$ and $X \in M_{r\times 2k'-r}$ such that $\psi_r(u) \neq 1$ and

$$\begin{pmatrix} u & X \\ 0 & I_{2k'-r} \end{pmatrix} \in Sp(\mathcal{I}_2).$$

Then $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ is satisfied with $Y = 0$, $u' = I_{r'}$ and $D = I_{n-(r+r')}$. Note further that $Sp(\mathcal{I}_1^{-1}) = \{{}^t g\colon\ g \in Sp(\mathcal{I}_1)\}$. Thus, if $\mathcal{I}_1^{-1}$ is not totally isotropic on $\langle e_1, \dots, e_{r'}\rangle$ then by Lemma 7 applied to $\mathcal{I}_1^{-1}$ there exist $u' \in U_{r'}$ and $Y \in M_{2k-r'\times r'}$ such that $\psi_{r'}(u') \neq 1$ and

$$\begin{pmatrix} {}^t u' & 0 \\ Y & I_{2k-r'} \end{pmatrix} \in Sp(\mathcal{I}_1).$$

Thus, $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ is satisfied with $X = 0$, $u = I_r$ and $D = I_{n-(r+r')}$. □

6.2.2. When g belongs to the totally isotropic case

Assume from now on that both $\mathcal{I}_2$ is totally isotropic on $\langle e_1, \dots, e_r\rangle$ and $\mathcal{I}_1^{-1}$ is totally isotropic on $\langle e_1, \dots, e_{r'}\rangle$. Recall that, in particular, we then have $r \leqslant k'$ and $r' \leqslant k$. In the case at hand $\mathcal{H} \cdot g$ contains an element of a rather simple form that will allow us the inductive argument. In order to bring g to this simpler form we need the following lemma.

Lemma 9. *Let $\ell \leqslant m$ and $Q = P_{(\ell, 2m-\ell)}$. Then*

$$Sp(J_{2m})Q = \{g \in G_{2m}\colon\ {}^t g J_{2m} g \text{ is totally isotropic on } \langle e_1, \dots, e_\ell\rangle\}.$$

Proof. If $h \in Sp(J_{2m})$ and $q \in Q$ then ${}^t(hq)J_{2m}hq = {}^t q J_{2m} q$. Since q preserves the space $\langle e_1, \dots, e_\ell\rangle$ and since J_{2m} is totally isotropic on $\langle e_1, \dots, e_\ell\rangle$ we get that ${}^t q J_{2m} q$ is also totally isotropic on $\langle e_1, \dots, e_\ell\rangle$. To prove the other direction let $g \in G_{2m}$ be such that ${}^t g J_{2m} g$ is totally isotropic on $\langle e_1, \dots, e_\ell\rangle$. Then

$$x = {}^t g J_{2m} g = \begin{pmatrix} 0_\ell & A \\ -{}^t A & D \end{pmatrix} \in G_{2m}$$

for some $D = -{}^t D \in M_{2m-\ell\times 2m-\ell}(F)$. We must show that there exists $q \in Q$ such that ${}^t qxq = J_{2m}$. Since x is invertible and $\ell \leqslant 2m - \ell$ the matrix A is of rank ℓ. Performing elementary operations, there exist $\alpha \in G_\ell$ and $\gamma \in G_{2m-\ell}$ such that ${}^t\alpha A\gamma = (0_{\ell\times 2(m-\ell)}, w_\ell)$. It follows that for $q = \mathrm{diag}(\alpha, \gamma) \in Q$, ${}^tqxq$ has the form

$$\begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b \\ -w_\ell & -{}^tb & d \end{pmatrix}$$

where $a = -{}^ta \in G_{2(m-\ell)}$ and $d = -{}^td \in M_{\ell\times\ell}(F)$. Write

$$\beta = (\beta_1, \beta_2) \quad \text{with } \beta_1 \in M_{\ell\times 2(m-\ell)}(F) \text{ and } \beta_2 \in M_{\ell\times\ell}(F).$$

Note that

$$\begin{pmatrix} I_\ell & 0 & 0 \\ {}^t\beta_1 & I_{2(m-\ell)} & 0 \\ {}^t\beta_2 & 0 & I_\ell \end{pmatrix}\begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b \\ -w_\ell & -{}^tb & d \end{pmatrix}\begin{pmatrix} I_\ell & \beta_1 & \beta_2 \\ 0 & I_{2(m-\ell)} & 0 \\ 0 & 0 & I_\ell \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b + {}^t\beta_1 w_\ell \\ -w_\ell & -{}^tb - w_\ell\beta_1 & d + {}^t\beta_2 w_\ell - w_\ell\beta_2 \end{pmatrix}.$$

We may now take $\beta_1 = -w_\ell\, {}^tb$. Any skew symmetric matrix in $M_{\ell\times\ell}(F)$ can be written as a difference $X - {}^tX$ for some $X \in M_{\ell\times\ell}(F)$. Thus, there also exists β_2 such that

$${}^t\beta_2 w_\ell - w_\ell\beta_2 = -d.$$

We get that there exists $q \in Q$ such that

$${}^tqxq = \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & 0 \\ -w_\ell & 0 & 0 \end{pmatrix}.$$

Let $y \in G_{2(m-\ell)}$ be such that ${}^tyay = J_{2(m-\ell)}$. Thus $q' = q\,\mathrm{diag}(I_\ell, y, I_\ell) \in Q$ and ${}^tq'xq' = J_{2m}$. $\square$

For $x \in G_\ell$ let

$$\tilde{x} = w_\ell\, x^\tau\, w_\ell.$$

The following property of the group $Sp(J_{2m})$ will be used several times in the proof of $\mathcal{P}(g, r, r')$. Assume that $\ell \leqslant m$.

$$\begin{gathered}\text{For all } x \in G_\ell,\ s \in Sp(J_{2(m-\ell)}) \text{ and } y \text{ there exists } y^* \text{ uniquely determined} \\ \text{by } x,\ s \text{ and } y \text{ and dependent linearly on } y \text{ and there exists } z \text{ such that} \\ \begin{pmatrix} x & y^* & z \\ 0 & s & y \\ 0 & 0 & \tilde{x} \end{pmatrix} \left(\text{respectively } \begin{pmatrix} x & 0 & 0 \\ y^* & s & 0 \\ z & y & \tilde{x} \end{pmatrix}\right) \text{ lies in } Sp(J_{2m}).\end{gathered} \tag{24}$$

We now choose a convenient representative for g.

Lemma 10. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

belong to the totally isotropic case. Then there exists $\gamma \in G_{n-(r+r')}$ such that

$$\begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g.$$

Proof. Since $-\mathcal{I}_1^{-1} = g_1^{-1} J_{2k} g_1^{\tau}$ is totally isotropic on $\langle e_1, \ldots, e_{r'} \rangle$ and $\mathcal{I}_2 = g_2^{\tau} J_{2k} g_2^{-1}$ is totally isotropic on $\langle e_1, \ldots, e_r \rangle$, it follows from Lemma 9 that

$$g_1 \in Sp(J_{2k}) \begin{pmatrix} \alpha_1 & 0 \\ \beta_1' & \gamma_1 \end{pmatrix} \quad \text{and} \quad g_2 \in \begin{pmatrix} \alpha_2 & \beta_2' \\ 0 & \gamma_2 \end{pmatrix} Sp(J_{2k'})$$

for some $\alpha_1 \in G_{r'}$, $\gamma_1 \in G_{2k-r'}$, $\alpha_2 \in G_r$, $\gamma_2 \in G_{2k'-r}$ and β_1' and β_2' of the appropriate size. Therefore,

$$\begin{pmatrix} 0 & \alpha_2 & \beta_2' \\ \alpha_1 & 0 & 0 \\ \beta_1' & 0 & \gamma_1\gamma_2 \end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & \beta_1' & \gamma_1 \end{pmatrix} \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} I_{r'} & 0 & 0 \\ 0 & \alpha_2 & \beta_2' \\ 0 & 0 & \gamma_2 \end{pmatrix} \in \mathcal{H} \cdot g.$$

Note that $\mathrm{diag}(\alpha_1, I_{2(k-r')}, \tilde{\alpha}_1) \in Sp(J_{2k})$ and $\mathrm{diag}(\alpha_2, I_{2(k'-r)}, \tilde{\alpha}_2) \in Sp(J_{2k'})$ and therefore that

$$h = \mathrm{diag}\big(I_r, \alpha_1^{-1}, I_{2(k-r')}, \tilde{\alpha}_1^{-1}\big) \in H \quad \text{and} \quad h' = \mathrm{diag}\big(I_{r'}, \alpha_2^{-1}, I_{2(k'-r)}, \tilde{\alpha}_2^{-1}\big) \in \overline{H'}.$$

Thus,

$$h \begin{pmatrix} 0 & \alpha_2 & \beta_2' \\ \alpha_1 & 0 & 0 \\ \beta_1' & 0 & \gamma_1\gamma_2 \end{pmatrix} h' = \begin{pmatrix} 0 & I_r & \beta_2 \\ I_{r'} & 0 & 0 \\ \beta_1 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g$$

for some $\gamma \in G_{n-(r+r')}$, β_1 and β_2. Now note that

$$\begin{pmatrix} I_r & \beta_2\gamma^{-1}\beta_1 & -\beta_2\gamma^{-1} \\ 0 & I_{r'} & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} 0 & I_r & \beta_2 \\ I_{r'} & 0 & 0 \\ \beta_1 & 0 & \gamma \end{pmatrix} \begin{pmatrix} I_{r'} & 0 & 0 \\ 0 & I_r & 0 \\ -\gamma^{-1}\beta_1 & 0 & I_{n-(r+r')} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g. \qquad \square$$

Lemma 11. *Let $\gamma \in G_{n-(r+r')}$ and let*

$$g = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Then $\mathcal{P}(g, r, r')$ holds.

Proof. Recall that $r+r'>0$. Let

$$\sigma_1=\begin{pmatrix} & I_{2(k-r')}\\ w_{r'} & \end{pmatrix}\quad\text{and}\quad \sigma_2=\begin{pmatrix} & I_{2(k'-r)}\\ w_r & \end{pmatrix}.$$

For $x=\sigma_1^{-1}\gamma\sigma_2$ we have by the induction hypothesis $\mathcal{P}_{n-(r+r')}(x,r,r')$. Fix $y\in H_{r,2(k'-r)}$ such that either

$$x^{-1}yx\in\overline{H}_{r',2(k-r')}\quad\text{and}\quad\theta\big(y,x^{-1}yx\big)\neq 1 \tag{25}$$

or

$$r=r',\qquad x^{-1}y\,{}^tx\in\overline{H}_{r',2(k-r')}\quad\text{and}\quad\theta\big(y,x^{-1}y\,{}^tx\big)=1. \tag{26}$$

For every invertible matrix z denote by $z^\star$ the matrix z if y satisfies (25) and the matrix ${}^tz$ otherwise. Note that if (26) holds then $\sigma_1=\sigma_2$ and therefore in either case we have

$$x^\star=\sigma_1^{-1}\gamma^\star\sigma_2.$$

There exist $s'\in Sp(J_{2(k-r')})$, $u'\in U_{r'}$ and $\varrho'\in M_{r'\times 2(k-r')}(F)$ such that

$$\sigma_1 y\sigma_1^{-1}=\begin{pmatrix} s' & \\ \varrho' & {}^t(\tilde{u'})\end{pmatrix}$$

and there exist $s\in Sp(J_{2(k'-r)})$, $u\in U_r$ and $\varrho\in M_{2(k'-r)\times r}(F)$ such that

$$\gamma^{-1}\sigma_1 y\sigma_1^{-1}\gamma^\star=\sigma_2x^{-1}yx^\star\sigma_2^{-1}=\begin{pmatrix} s & \varrho\\ & \tilde{u}\end{pmatrix}.$$

Note then that

$$\theta\big(y,x^{-1}yx^\star\big)=\psi_r(u)\psi_{r'}(u')^{-1}. \tag{27}$$

By (24) there exist $(\varrho')^*\in M_{2(k-r')\times r'}(F)$, $\varrho^*\in M_{r\times 2(k'-r)}(F)$, z' and z such that

$$h=\begin{pmatrix} {}^tu' & 0 & 0\\ (\varrho')^* & s' & 0\\ z' & \varrho' & {}^t\tilde{u}'\end{pmatrix}\in Sp(J_{2k})\quad\text{and}\quad h'=\begin{pmatrix} u & \varrho^* & z\\ 0 & s & \varrho\\ 0 & 0 & \tilde{u}\end{pmatrix}\in Sp(J_{2k'}).$$

Note that

$$g^\star=\begin{pmatrix} 0 & I_r & 0\\ I_{r'} & 0 & 0\\ 0 & 0 & \gamma^\star\end{pmatrix}.$$

Let

$$\zeta_1=(\varrho^*,z)(\gamma^\star)^{-1}\quad\text{and}\quad\zeta=(0_{r\times r'},\zeta_1).$$

Then

$$Y = \begin{pmatrix} u & \zeta \\ 0 & h \end{pmatrix} \in H, \qquad g^{-1} Y g^{\star} = \begin{pmatrix} {}^t u' & 0 \\ \zeta' & h' \end{pmatrix} \in \overline{H'}$$

where

$$\zeta' = \begin{pmatrix} 0_{r \times r'} \\ \zeta_1' \end{pmatrix}, \qquad \zeta_1' = \gamma^{-1} \begin{pmatrix} (\varrho')^* \\ z' \end{pmatrix} \quad \text{and} \quad \theta\big(Y, g^{-1} Y g^{\star}\big) = \psi_r(u) \psi_{r'}(u')^{-1}.$$

The property $\mathcal{P}_n(g, r, r')$ therefore follows from (27) and the fact that either (25) holds or (26) holds. □

6.3. Conclusion

For $g \in G$, by (7) there exists $w \in {}_M W_{M'}$ such that $g \in P w \overline{P'}$. If I_w is not empty then $\mathcal{P}(g, r, r')$ is proved in Lemma 5. If I_w is empty then we separated in Section 6.2 the statement $\mathcal{P}(g, r, r')$ into two cases. If g belongs to the totally isotropic case then $\mathcal{P}(g, r, r')$ follows from Lemmas 2, 10 and 11. Otherwise $\mathcal{P}(g, r, r')$ follows from Lemmas 6 and 8. It follows that for every $g \in G$ we have $\mathcal{P}(g, r, r')$. Proposition 2 now follows from Lemma 1. Therefore, Proposition 1 follows from Section 4.1 and Theorem 1 follows from Section 3.1.

Acknowledgments

We wish to thank the Weizmann Institute of Science where this work was initiated and the Hausdorff Research Institute for Mathematics (HIM) in Bonn where this work was completed. We further thank Prof. Joseph Bernstein for his insight regarding uniqueness theorems and Prof. Erez Lapid for his insistence that an elementary proof is within reach. During the preparation of this work the second named author was supported by the Ann Stone Postdoctoral Fellowship, the Clore Postdoctoral Fellowship at the Weizmann Institute and BSF grant # 2004200.

References

[1] I.N. Bernšteĭn, A.V. Zelevinskiĭ, Representations of the group $GL(n, F)$, where F is a local non-Archimedean field, Uspekhi Mat. Nauk 31 (3(189)) (1976) 5–70.

[2] L. Clozel, Combinatorial consequences of Arthur's conjectures and the Burger–Sarnak method, Int. Math. Res. Not. 2004 (11) (2004) 511–523.

[3] I.M. Gelfand, D.A. Kajdan, Representations of the group $GL(n, K)$ where K is a local field, in: Lie Groups and Their Representations, Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971, Halsted, New York, 1975, pp. 95–118.

[4] D. Goldstein, R.M. Guralnick, Alternating forms and self-adjoint operators, J. Algebra 308 (1) (2007) 330–349.

[5] M.J. Heumos, S. Rallis, Symplectic–Whittaker models for Gl_n, Pacific J. Math. 146 (2) (1990) 247–279.

[6] N.F.J. Inglis, J. Saxl, An explicit model for the complex representations of the finite general linear groups, Arch. Math. (Basel) 57 (5) (1991) 424–431.

[7] A.A. Klyachko, Models for complex representations of the groups $GL(n, q)$, Mat. Sb. (N.S.) 48 (2) (1984) 365–378.

[8] O. Offen, E. Sayag, Global mixed periods and local Klyachko models for the general linear group, Int. Math. Res. Not. (2007), article ID rnm136.

[9] O. Offen, E. Sayag, On unitary representations of GL_{2n} distinguished by the symplectic group, J. Number Theory 125 (2007) 344–355.