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On parabolic induction associated with a p-adic symmetric space

Omer Offen[1]

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 3200003, Israel

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We provide some tools to study distinguished induced representations in the setting of a general p-adic symmetric space.

Contents

E-mail address: offen@tx.technion.ac.il.

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1. Introduction

Let G be a reductive p-adic group, θ an involution on G and $H = G^\theta$. Fix a character χ of H. A (smooth, complex valued) representation π of G is (H,χ)-distinguished if $\mathrm{Hom}_H(\pi,\chi) \neq 0$.

This note examines the relation between distinction of an induced representation and distinction of the inducing data (or, more precisely, of some Jacquet module of the inducing data) in the setting of any p-adic symmetric space. Our main result is a generalization to this setting of a necessary condition for an induced representation of G to be (H,χ)-distinguished in terms of distinction of some Jacquet module of the inducing data.

The result we have in mind proved to be a useful tool in the study of distinction problems in the special cases where it was already proved (see [Off06,FLO12,Gur15]). It further led to applications in the study of period integrals of automorphic forms. The relevant results were proved separately in each case.

For future reference, it will be of use to have the results available in a general setting. In particular, the results of this paper will be applied to the study of distinguished representations of classical groups. A particular case of study, also related to the descent construction of Ginzburg, Rallis and Soudry (see e.g. [GRS11]), is the case where $G = U_{2n}(E/F)$ is a unitary group associated with the quadratic extension E/F and $H = \mathrm{Sp}_{2n}(F)$. In a work in progress joint with Arnab Mitra we study distinguished representations in this setting. Combining ideas of Ash, Ginzburg and Rallis ([AGR93]) with the results of this note we can already show the following result. If an irreducible representation of G is H-distinguished, then its cuspidal support is a representation of a Levi subgroup that is contained in the Siegel Levi. We expect the tools developed here to have many further application for this and other symmetric spaces.

We consider the restriction to H of a parabolically induced representation of G. Our main tool is a combination of the geometric lemma of Bernstein and Zelevinsky [BZ77] and a careful study of parabolic orbits on the symmetric space G/H, where we generalize results of Lapid and Rogawski [LR03].

Our main result can be formulated as follows (see Theorem 4.2 for a more notationally involved formulation). Fix a minimal parabolic subgroup P_0 of G and let δ_A be the modulus function of a locally compact group A.

Theorem 1.1. *Let $P \supseteq P_0$ be a parabolic subgroup of G with standard Levi subgroup M, σ a representation of M, $\pi = \mathrm{Ind}_P^G(\sigma)$ the normalized parabolic induction and χ a character of H. If π is (H,χ)-distinguished then there exists $\eta \in G$ so that the following conditions are satisfied:*

- *The group $L = M \cap \eta\theta(\eta^{-1}M\eta)\eta^{-1}$ is the standard Levi subgroup of a parabolic $Q \supseteq P_0$ of G contained in P.*

- *The normalized Jacquet module $r_{L,M}(\sigma)$ is $(L\cap\eta^{-1}H\eta,\Delta)$-distinguished where $\Delta = \delta_{Q\cap\eta^{-1}H\eta}\delta_Q^{-1/2}\chi^{\eta^{-1}}$ and $\chi^{\eta^{-1}}(h) = \chi(\eta h\eta^{-1})$.*

In Section 6 we provide tools to compute the character $\delta_{Q\cap\eta^{-1}H\eta}\delta_Q^{-1/2}$, in order to facilitate the application of our main theorem to special cases. In particular, if θ is a Galois involution, we deduce that this character is trivial. This is a generalization of (the local analogue of) [LR03, Proposition 4.3.2].

In section 7 we provide sufficient conditions for distinction of an induced representation in terms of distinction of its inducing data. The first condition, Proposition 7.1, is the contribution of a closed parabolic orbit. Its proof is straightforward. The second, Proposition 7.2, is the contribution of an open parabolic orbit. Its proof is an application of the main result of [BD08].

2. Notation and preliminaries

Let F be a non-archimedean local field, $\mathbb{G}$ a reductive group defined over F and $G = \mathbb{G}(F)$. We denote by e the identity element of any group and let δ_Q be the modulus function of a locally compact group Q.

2.1. The symmetric space

Let θ be an involution on $\mathbb{G}$ defined over F and $H = G^\theta = \mathbb{G}^\theta(F)$. Let

$$X = \{g \in G : g\theta(g) = e\}$$

be the G-space with G-action $g \cdot x = gx\theta(g)^{-1}$. For any subgroup Q of G and $x \in X$ let $Q_x = \mathrm{Stab}_Q(x)$. Thus $Q_e = Q \cap H$ and in particular, $H = G_e$. Note that $g \mapsto g \cdot e$ defines an imbedding of G/H in X.

We further observe that for $x \in X$ the automorphism

$$\theta_x(g) = x\theta(g)x^{-1}, \quad g \in G$$

is an involution on G and

$$Q_x = (Q \cap \theta_x(Q))^{\theta_x} \tag{1}$$

for any subgroup Q of G.

2.2. Parabolic subgroups and Weyl groups

Fix a minimal parabolic subgroup P_0 of G and a θ-stable maximal split torus T of G contained in P_0 (see [HW93, Lemma 2.4]). Let $M_0 = C_G(T)$ and U_0 the unipotent radical of P_0. Then $P_0 = M_0 \ltimes U_0$ is a θ-stable Levi decomposition.

We call a parabolic subgroup P of G standard if it contains P_0 and semi-standard if it contains T (or equivalently M_0). If P is a standard parabolic subgroup of G then it contains a unique Levi subgroup M containing M_0. The group M is then called a standard Levi subgroup. If U is the unipotent radical of P we say that $P = M \ltimes U$ is a standard Levi decomposition.

We adopt the following convention. By saying, $P = M \ltimes U$ is a standard parabolic subgroup of G, we further mean that $M \ltimes U$ is the standard Levi decomposition of P.

Let $W = N_G(T)/M_0$ be the Weyl group of G with respect to T. More generally, for a standard Levi subgroup M of G let $W_M = N_M(T)/M_0$ be the Weyl group of M with respect to T.

2.3. The involution θ'

A careful analysis of the double coset space $P\backslash G/H$ where P is a standard parabolic subgroup of G plays the key technical role of this work. Such an analysis was carried out by Lapid and Rogawski in [LR03, §4] under the assumption that θ is a Galois involution that stabilizes P_0. This analysis is based on the notion of twisted involutions associated to an involution σ on $\mathfrak{a}_0^* = X^*(M_0) \otimes_{\mathbb{Z}} \mathbb{R}$ where $X^*(M_0)$ is the lattice of F-rational characters on M_0. The necessary results are obtained in [LR03, §3] under the assumption that the basis of simple roots of G on T with respect to P_0 is σ-stable. Many of the results of [LR03, §4] relevant to us carry over verbatim when we remove the assumption that θ is Galois. However, in order to remove the assumption that P_0 is θ-stable, we need to carry out the following modification.

The set of minimal semi-standard parabolic subgroups of G forms a W-torsor. In particular, (since T is θ-stable) there exists a unique Weyl element $\tau \in W$ such that

$$\theta(P_0) = \tau P_0 \tau^{-1}.$$

Applying θ to this identity we also have $P_0 = \theta(\tau)\theta(P_0)\theta(\tau)^{-1} = \theta(\tau)\tau P_0 \tau^{-1}\theta(\tau)^{-1}$. Hence,

$$\theta(\tau)\tau = e.$$

Fix once and for all an element $n \in \tau$. Then $\theta(n)n \in M_0$. Define the automorphism $\theta' : G \to G$ by

$$\theta'(g) = n^{-1}\theta(g)n.$$

Then θ' preserves T (and M_0). It therefore induces the involution

$$\theta'(w) = \tau^{-1}\theta(w)\tau$$

on W and acts as an involution on $\mathfrak{a}_0^*$. Furthermore, $\theta'(P_0) = P_0$. Thus, the results and constructions in [LR03, §3] apply with $\sigma = \theta'$. This allows us to generalize the results we need from [LR03, §4] to our general setting of a p-adic reductive symmetric space.

2.4. Bruhat decomposition

Let $P = M \ltimes U$ and $P' = M' \ltimes U'$ be standard parabolic subgroups of G with their standard Levi decompositions. Let ${}_MW_{M'}$ be the set of all $w \in W$ that are left W_M-reduced and right $W_{M'}$-reduced. It is a complete set of representatives for the set of double cosets $W_M\backslash W/W_{M'}$ consisting of the unique elements of minimal length in their double coset. The Bruhat decomposition provides a bijection $P\backslash G/P' \to {}_MW_{M'}$ so that $PgP' \mapsto w$ whenever $PgP' = PwP'$.

Recall that if $Q = L \ltimes V$ and $Q' = L' \ltimes V'$ are any two parabolic subgroups of G and $\mathrm{pr}_L : Q \to L$ is the projection to the Levi subgroup then $\mathrm{pr}_L(Q \cap Q')$ is a parabolic subgroup of L. For $w \in {}_MW_{M'}$ we have

$$P \cap wP'w^{-1} = (M \cap wP'w^{-1})(U \cap wP'w^{-1}).$$

Let

$$P(w) := \mathrm{pr}_M(P \cap wP'w^{-1}) = M \cap wP'w^{-1}.$$

Then $P(w)$ is a standard parabolic subgroup of M (with respect to $M\cap P_0$) with standard Levi decomposition

$$P(w) = M(w) \ltimes U(w)$$

where $M(w) = M \cap wM'w^{-1}$ and $U(w) = M \cap wU'w^{-1}$.

For the rest of the paper, for a standard parabolic subgroup $P = M \ltimes U$ of G we let

$$P' = \theta'(P).$$

Then $P' = M' \ltimes U'$ is also a standard parabolic subgroup of G, $M' = \theta'(M)$ and $U' = \theta'(U)$.

Note that $Pg\theta(P)n = PgnP'$, $g \in G$ and therefore

$$(Pg\theta(P) \mapsto PgnP') : P\backslash G/\theta(P) \to P\backslash G/P'$$

is a bijection. In the sequel, we apply this bijection in order to translate the above results on $P\backslash G/P'$ to $P\backslash G/\theta(P)$.

2.5. Twisted involutions

By the θ-invariance of T, θ also acts as an involution on W. Let

$$\mathfrak{J}_0(\theta) = \{w \in W : w\theta(w) = e\}$$

be the set of θ-twisted involutions in W.

For future reference, note that

$$w \mapsto w\tau : \mathfrak{J}_0(\theta) \to \mathfrak{J}_0(\theta') \text{ is a bijection.} \tag{2}$$

3. Parabolic orbits

It is well known that $P_0\backslash G/H$ is a finite set. In fact, the set $P_0\backslash X$ of P_0 orbits in X is finite ([HW93, Proposition 6.15]). The following result follows from [HW93, Propositions 6.6 and 6.8] (see also [LR03, Proposition 4.1.1]) and is based on results of Springer [Spr86].

Lemma 3.1. *For every $x \in X$, the set $P_0 \cdot x \cap N_G(T)$ is not empty. Furthermore, the map $(P_0 \cdot x \mapsto P_0 \cdot x \cap N_G(T)) : P_0\backslash X \to M_0\backslash(X \cap N_G(T))$ is a bijection.* □

The lemma implies that for $x \in X$ there is a unique $w \in W$ such that $P_0 \cdot x \cap w \neq \emptyset$. Since $P_0 \cdot x \subseteq P_0 x\theta(P_0)$ we observe that $P_0 x\theta(P_0) = P_0 w\theta(P_0)$ or equivalently that

$$P_0 xnP_0 = P_0 w\tau P_0.$$

Note that $\theta'(xn)^{-1} = n^{-1}\theta(n)^{-1}\theta(x)^{-1}n \in M_0 xn$ and therefore, by applying $\theta'(\cdot)^{-1}$ to the above identity, we get that $w\tau \in \mathfrak{J}_0(\theta')$ or equivalently (by (2)), that $w \in \mathfrak{J}_0(\theta)$.

Let $\imath_0 : P_0\backslash X \to \mathfrak{J}_0(\theta)$ be defined by $w = \imath_0(P_0 \cdot x)$ if the equivalent conditions:

- $P_0 \cdot x \cap w \neq \emptyset$,
- $P_0 x\theta(P_0) = P_0 w\theta(P_0)$,
- $P_0 xnP_0 = P_0 w\tau P_0$

are satisfied.

Next, we consider a standard parabolic subgroup $P = M \ltimes U$ of G. Then $P' = \theta'(P) = M' \ltimes U'$ is a standard parabolic subgroup of G.

For $x \in X$ we have $P \cdot x \subseteq Px\theta(P) = PxnP'n^{-1}$. By the Bruhat decomposition, there exists a unique $w \in {}_M W_{M'}\tau^{-1}$ such that $PxnP' = Pw\tau P'$. Arguing as in the minimal parabolic case, we observe that $Pw\tau P' = P\theta'(w\tau)^{-1}P'$. By the Bruhat decomposition (§2.4) we have $W_M w\tau W_{M'} = W_M\theta'(w\tau)^{-1}W_{M'}$. Since, furthermore, $\theta'(P_0) = P_0$ it follows that $\theta'(w\tau)^{-1}$ is also of minimal length in $W_M w\tau W_{M'}$ and therefore $w\tau \in \mathfrak{J}_0(\theta')$. By (2), the assignment $\imath_M(P \cdot x) = w$ defines a map

$$\imath_M : P\backslash X \to {}_M W_{M'} \tau^{-1} \cap \mathfrak{J}_0(\theta).$$

It is characterized by the identity

$$Px\theta(P) = Pw\theta(P).$$

Fix $x \in X$ and let

$$w = \imath_M(P \cdot x) \quad \text{and} \quad L = M(w\tau) = M \cap w\theta(M)w^{-1}.$$

Note that by §2.4, L is a standard Levi subgroup of M satisfying $L = w\theta(L)w^{-1}$. Furthermore,

$$P \cap w\theta(P)w^{-1} = P \cap w\tau P'(w\tau)^{-1} = L \ltimes Z$$

where the unipotent radical Z of $P \cap w\theta(P)w^{-1}$ satisfies

$$\begin{aligned} Z = U(w\tau)(U \cap w\tau M'(w\tau)^{-1})(U \cap w\tau U'(w\tau)^{-1}) = \\ (M \cap w\theta(U)w^{-1})(U \cap w\theta(M)w^{-1})(U \cap w\theta(U)w^{-1}). \end{aligned} \tag{3}$$

We therefore also have $Z = w\theta(Z)w^{-1}$. The following is a generalization of [LR03, Proposition 4.2.1] and we adapt its proof.

Lemma 3.2. *With the above notation $P \cdot x \cap Lw$ is not empty, in fact, it is a unique L-orbit in X. Furthermore, for $y \in P \cdot x \cap Lw$ we have $P_y = L_y \ltimes Z_y$.*

Proof. By Lemma 3.1, the set $P \cdot x \cap N_G(T) \supseteq P_0 \cdot x \cap N_G(T)$ is non-empty. Let $w' \in W$ be such that $w'\tau$ is of minimal length in the set

$$\{w\tau : w \in W,\ P \cdot x \cap w \neq \emptyset\} = \{\imath_0(P_0 \cdot y) : y \in P \cdot x\}\tau.$$

Then $w' \in \mathfrak{J}_0(\theta)$ is such that $Pw'\tau P' = Pw\tau P'$. Hence, there exists a reduced expression $w'\tau = w_1 w''(w\tau)w_2$ (i.e., such that the length of $w'\tau$ is the sum of length of w_1, w'', $w\tau$ and w_2) with $w'' \in W_L$, $w_1 \in W_M$ right W_L-reduced and $w_2 \in W_{M'}$ left $W_{L'}$-reduced. Such a decomposition is unique.

Since both $w\tau$, $w'\tau \in \mathfrak{J}_0(\theta')$, we also have $w'\tau = \theta'(w_2)^{-1}w\tau\theta'(w'')^{-1}\theta'(w_1)^{-1}$. Note that $w\tau\theta'(W_L)(w\tau)^{-1} = W_L$. It follows from the uniqueness of the decomposition that $w_2 = \theta'(w_1)^{-1}$.

Let $n_1 \in w_1 \subseteq M$ and $y \in P \cdot x \cap w'$. Then $n_1^{-1} \cdot y \in P \cdot x \cap w''w$. Now the minimality of the length of $w'\tau$ implies that $w_1 = e$ and therefore $w' = w''w \in W_L w$. This shows that $y \in P \cdot x \cap Lw$ as required.

If $y \in P \cdot x \cap Lw$ then, by the paragraph preceding the lemma, θ_y restricts to an involution on $L \ltimes Z$ stabilizing L and Z. The decomposition $(L \ltimes Z)_y = L_y \ltimes Z_y$ therefore follows from (1).

Assume now that $y', y \in P \cdot x \cap Lw$ and let $p \in P$ be such that $p \cdot y = y'$. Since $Lw = w\theta(L)$ we have that $p = y'\theta(p)y^{-1} \in P \cap w\theta(P)w^{-1} = L \ltimes Z$ and θ_y is an involution on $L \ltimes Z$ stabilizing L and Z.

Note that $p = (y'y^{-1})\theta_y(p)$ and $y'y^{-1} \in L$. Decomposing $p = mz$ with $m \in L$ and $z \in Z$ and projecting to L we get that $m = y'y^{-1}\theta_y(m)$, i.e., that $m \cdot y = y'$. The lemma follows. $\square$

Next, by adapting the proof of [LR03, Proposition 4.2.2 (1)], we obtain its following generalization. Let $\mathcal{O} \in P\backslash X$, $w = \imath_M(\mathcal{O})$, $L = M(w\tau)$ and $x \in \mathcal{O} \cap Lw$ be given by Lemma 3.2. By the same Lemma and in its notation $P_x = L_x \ltimes Z_x$ and in particular Z_x is the unipotent radical of P_x.

Lemma 3.3. *With the above notation, $\mathrm{pr}_M(Z_x) = U(w\tau) = M \cap w\theta(U)w^{-1}$ is a normal subgroup of $\mathrm{pr}_M(P_x)$.*

Proof. Note that $P_x \subseteq P \cap w\theta(P)w^{-1} = P \cap w\tau P'(w\tau)^{-1}$. Therefore $\mathrm{pr}_M(P_x) \subseteq P(w\tau)$ (see §2.4). Since $U(w\tau)$ is normal in $P(w\tau)$ it is enough to show that $U(w\tau) = \mathrm{pr}_M(Z_x)$. Clearly, $\mathrm{pr}_M(Z_x) \subseteq \mathrm{pr}_M(Z) = U(w\tau)$ (see (3)). We now show the other inclusion.

Let $u \in U(w\tau)$ and let $v = \theta_x(u) \in U \cap w\theta(M)w^{-1}$. Since $U(w\tau) \subseteq M$ it follows that $u^{-1}vu \in U$. Therefore the commutator $z := [v^{-1}, u^{-1}] \in U$. Also, since $v \in w\theta(M)w^{-1}$ and $u \in w\theta(U)w^{-1}$ it follows that $v^{-1}u^{-1}v \in w\theta(U)w^{-1}$ and therefore $z \in U'' := U \cap w\theta(U)w^{-1}$.

Note that, $\theta_x(z) = [u^{-1}, v^{-1}] = z^{-1} \in U''$. Thus, z satisfies the cocycle condition $z\theta_x(z) = 1$ with respect to the involution θ_x on U''. Since U'' is a unipotent group we have $H^1(\langle\theta_x\rangle, U'') = 1$, ([HW93, Lemma 0.1]), i.e., z must be a co-boundary.

There exists therefore $u' \in U''$ such that $z = u'\theta_x(u')^{-1}$. Note that this means that $v^{-1}u^{-1}vu = u'x\theta(u')^{-1}x^{-1}$ or $v^{-1}u^{-1}vux = u' \cdot x$. Since $v = \theta_x(u)$ and $u = \theta_x(v)$ we get that $(uv)^{-1} \cdot x = u' \cdot x$, i.e., that $(uvu') \cdot x = x$. Note that u, v, $u' \in Z$ and therefore $uvu' \in Z_x$. But v, $u' \in U$ and therefore $\mathrm{pr}_M(uvu') = u$. The Lemma follows. $\square$

4. The geometric lemma

Let $P = M \ltimes U$ be a standard parabolic subgroup of G and σ a smooth, complex valued representation of M. Let $\mathrm{Ind}_P^G(\sigma)$ be its normalized parabolic induction and for any standard Levi subgroup $L \subseteq M$ let $r_{L,M}(\sigma)$ be the normalized Jacquet module as defined in [BZ77, §2.3]. We recall a consequence of the geometric lemma of Bernstein and Zelevinsky [BZ77, Theorem 5.2] (see also [BD08, Proposition 1.17]).

It follows from [BZ76, §1.5] (see also [BD08, Lemma 3.1]), that we can order the double cosets in $P\backslash G/H$ as $\{P\eta_i H\}_{i=1}^N$ in such a way that

$$Y_i = \cup_{j=1}^i P\eta_j H$$

is open in G for all $i = 1, \dots, N$. Let

$$V_i = \{\varphi \in \mathrm{Ind}_P^G(\sigma) : \mathrm{Supp}(\varphi) \subseteq Y_i\}.$$

Then $V_0 := \{0\} \subseteq V_1 \subseteq \cdots \subseteq V_N = \mathrm{Ind}_P^G(\sigma)$ is a filtration of the restriction to H of $\mathrm{Ind}_P^G(\sigma)$. The factors of this filtration can be described as follows:

$$V_i/V_{i-1} \simeq \mathrm{ind}_{P_i}^H((\sigma\delta_P^{1/2}|_{P_{x_i}})^{\eta_i}) \tag{4}$$

where $x_i = \eta_i \cdot e$ and

$$P_i = \eta_i^{-1} P \eta_i \cap H = \eta_i^{-1} P_{x_i} \eta_i.$$

Here, $(\sigma\delta_P^{1/2}|_{P_{x_i}})^{\eta_i}$ is the representation of P_i obtained from $\sigma\delta_P^{1/2}|_{P_{x_i}}$ by η_i-conjugation and $\mathrm{ind}_{P_i}^H$ is non-normalized induction with compact support.

For the rest of this section fix a character χ of H. Fix i and let $\eta = \eta_i$, $x = x_i \in X$, $w = \iota_M(P \cdot x)$ and $L = M(w\tau)$. It follows from Lemma 3.2 that the representative η of the double coset in $P\backslash G/H$ can be chosen so that $x \in Lw$. Assume this is the case and let $Q = L \ltimes V$ be the standard parabolic subgroup of G with standard Levi subgroup L and unipotent radical V.

Proposition 4.1. *With the above notation,*

$$\mathrm{Hom}_H(\mathrm{ind}_{\eta^{-1}P_x\eta}^H((\sigma\delta_P^{1/2}|_{P_x})^\eta), \chi) = \mathrm{Hom}_{L_x}(r_{L,M}(\sigma), \delta_{Q_x}\delta_Q^{-1/2}\chi^{\eta^{-1}}).$$

Proof. By Frobenious reciprocity [BZ76, Proposition 2.29], for a character χ of H we have

$$\mathrm{Hom}_H(\mathrm{ind}_{\eta^{-1}P_x\eta}^H((\sigma\delta_P^{1/2}|_{P_x})^\eta), \chi) = \mathrm{Hom}_{\eta^{-1}P_x\eta}((\sigma\delta_P^{1/2}\delta_{P_x}^{-1})^\eta, \chi)$$

and conjugation by η identifies it with

$$\mathrm{Hom}_{P_x}(\sigma\delta_P^{1/2}\delta_{P_x}^{-1}, \chi^{\eta^{-1}}) = \mathrm{Hom}_{P_x}(\sigma, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}}).$$

Recall that, by Lemma 3.2 (and in its notation), $P_x = L_x \ltimes Z_x$ and, by Lemma 3.3, $\mathrm{pr}_M(Z_x) = U(w\tau)$.

Clearly, δ_{P_x} is trivial on the unipotent radical Z_x of P_x and for the same reason $\delta_P^{1/2}$ is trivial on $U \cap w\theta(P)w^{-1}$. Any smooth character of a reductive group is trivial on unipotent elements, since any unipotent element can be conjugated into an arbitrarily small neighborhood of the identity (and in particular into the kernel of the character). Therefore, $\delta_P^{1/2}|_M$ is trivial on $U(w\tau)$ and similarly, $\chi^{\eta^{-1}}|_{Z_x} = 1$. By (3) we now have $\delta_P^{1/2}|_Z = 1$. Thus, all together, $\delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}}$ is trivial on Z_x.

Note further, that $M \cap Q = L \ltimes (M \cap V)$ is the standard parabolic subgroup of M with standard Levi subgroup L, i.e., that $M \cap Q = P(w\tau)$ and therefore also $U(w\tau) = M \cap V \subseteq Q$. Since $U \subseteq V \subseteq Q$, it follows form (3) that $Z \subseteq Q$ and therefore that $P_x = L_x \ltimes Z_x = Q_x$.

Since $\delta_P|_L = \delta_Q|_L\delta_{M\cap Q}|_L$, taking the normalization of the Jacquet module into consideration and factoring through Z_x we get that

$$\mathrm{Hom}_{P_x}(\sigma, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}}) = \mathrm{Hom}_{L_x}(r_{L,M}(\sigma), \delta_{Q_x}\delta_Q^{-1/2}\chi^{\eta^{-1}}).$$

The lemma follows. □

We now reformulate and prove Theorem 1.1 in the notation of this section.

Theorem 4.2. *If the representation* $\mathrm{Ind}_P^G(\sigma)$ *is* (H,χ)*-distinguished then there exist a* P*-orbit* $\mathcal{O}$ *in* $P\backslash(G\cdot e) \subseteq P\backslash X$ *and* $\eta \in G$ *satisfying* $x = \eta \cdot e \in \mathcal{O} \cap Lw$ *(where* $w = \iota_M(\mathcal{O})$ *and* $L = M(w\tau)$*) such that* $r_{L,M}(\sigma)$ *is* $(L_x, \delta_{Q_x}\delta_Q^{-1/2}\chi^{\eta^{-1}})$*-distinguished. Here* $Q = L \ltimes (U(w\tau)U)$ *is the standard parabolic subgroup of* G *with standard Levi subgroup* L*.*

Proof. Let ℓ be a non-zero (H,χ)-equivariant linear form on $\mathrm{Ind}_P^G(\sigma)$, i.e., on V_N. Then there exists a minimal $i \in \{1,\dots,N\}$ such that $\ell|_{V_i} \neq 0$ and therefore ℓ defines an (H,χ)-equivariant linear form on V_i/V_{i-1}. The theorem now follows from (4) and Proposition 4.1. □

In the other direction we have the following simple observation.

Lemma 4.3. *If there exists* $\eta \in G$ *so that* $P\eta H$ *is closed in* G *and* $r_{L,M}(\sigma)$ *is* $(L_x, \delta_{Q_x}\delta_Q^{-1/2}\chi^{\eta^{-1}})$*-distinguished then* $\mathrm{Ind}_P^G(\sigma)$ *is* (H,χ)*-distinguished. Here,* $x = \eta\cdot e$*,* $w = \iota_M(P\cdot x)$*,* $L = M(w\tau)$ *and* $Q = L \ltimes (U(w\tau)U)$*.*

Proof. In the notation of this section, we may choose the order on $P\backslash G/H$ in such a way that $\eta = \eta_N$. It follows from (4) and Proposition 4.1 that there exists $0 \neq \ell \in \mathrm{Hom}_H(V_N/V_{N-1},\chi)$. The lemma follows by composing ℓ with the projection $V_N \to V_N/V_{N-1}$. □

5. Admissible orbits

Let $P = M \ltimes U$ be a standard parabolic subgroup of G.

Definition 5.1. We say that $x \in X$ (or $P \cdot x$) is M-admissible if $M = w\theta(M)w^{-1}$ where $w = \iota_M(P\cdot x)$.

As a simple consequence of Theorem 4.2 we observe that when the inducing data is cuspidal then only admissible orbits can contribute towards (H,χ)-distinction.

Corollary 5.2. *Let σ be a cuspidal representation of M. If $\mathrm{Ind}_P^G(\sigma)$ is (H,χ)-distinguished then there exists an M-admissible P-orbit $\mathcal{O}$ in $P\backslash(G\cdot e)\subseteq P\backslash X$, and $x=\eta\cdot e\in\mathcal{O}\cap Mw$ (where $w=\iota_M(\mathcal{O})$) such that σ is $(M_x,\delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$-distinguished.*

Proof. Let x be given by Theorem 4.2. Then, in particular, in its notation $r_{L,M}(\sigma)\neq 0$ and therefore, by cuspidality, $L=M$, i.e. x is M-admissible. $\square$

In order to apply Theorem 4.2 in particular cases, in its notation, it is helpful to have a more explicit description of the stabilizer L_x and the character $\delta_{Q_x}\delta_Q^{-1/2}$ on it. It is a simple observation that x is L-admissible. Therefore, Theorem 4.2 already reduces the description of the character $\delta_{Q_x}\delta_Q^{-1/2}$ to the case of admissible orbits.

If θ is a Galois involution so that $\theta(P_0)=P_0$ then it follows from the proof of [LR03, Proposition 4.3.2] that $\delta_{Q_x}\delta_Q^{-1/2}$ is always trivial. We can remove the assumption that $\theta(P_0)=P_0$. However, as already observed in [Off06] for the case that $G=\mathrm{GL}_{2n}(F)$ and $H=\mathrm{Sp}_{2n}(F)$, the character $\delta_{Q_x}\delta_Q^{-1/2}$ is not always trivial for a general θ. Nevertheless, its computation can be reduced further to the case of minimal orbits in a sense that will be defined in the sequel.

The next section contains further reductions that have proved useful in the special cases where they were already applied. Such reductions are also crucial for the study of period integrals of Eisenstein series as observed in [LO].

6. Some further reductions

Let $x\in X$ and $P=M\ltimes U$ be a standard parabolic subgroup of G. Let $w=\imath_M(P\cdot x)$, so that $PxnP'=Pw\tau P'$ and let $p\in P$ be such that $xn\in pw\tau P'$. Then

$$\begin{aligned}P\cap x\theta(P)x^{-1}&=P\cap xnP'(xn)^{-1}=p(P\cap w\tau P'(w\tau)^{-1})p^{-1}\\&=p(P\cap w\theta(P)w^{-1})p^{-1}.\end{aligned}$$

Since $P\cap w\tau P(w\tau)^{-1}=P(w\tau)(U\cap w\tau P(w\tau)^{-1})=(M\cap w\theta(P)w^{-1})(U\cap w\theta(P)w^{-1})$ and p normalizes U we have

$$\mathrm{pr}_M(P\cap x\theta(P)x^{-1})=\mathrm{pr}_M(p)(M\cap w\theta(P)w^{-1})\,\mathrm{pr}_M(p)^{-1}.\tag{5}$$

In particular, the following conditions are equivalent

(1) $\mathrm{pr}_M(P\cap x\theta(P)x^{-1})=M$,
(2) $(P\cap x\theta(P)x^{-1})U=P$,
(3) $M\cap w\theta(P)w^{-1}=M$,

(4) $M \cap w\theta(M)w^{-1} = M$,
(5) $M \cap w\theta(U)w^{-1} = 1$,
(6) $Mw\theta(M) \subseteq N_{G,\theta}(M) := \{g \in G : M = g\,\theta(M)g^{-1}\}$.

Each of them characterizes the condition that x is M-admissible. If P and P' are associate parabolic subgroups, (i.e., if $N_{G,\theta}(M)$ is not empty) then $N_{G,\theta}(M)$ is a left $N_G(M)$ (alternatively a right $N_G(\theta(M))$) coset.

Lemma 6.1. *An element $x \in X$ is M-admissible if and only if $x \in UN_{G,\theta}(M)\theta(U)$.*

Proof. If $x \in UN_{G,\theta}(M)\theta(U)$ then after acting on x by some $u \in U$ we may assume that $x \in N_{G,\theta}(M)\theta(U)$. When this is the case, $x\theta(P)x^{-1} \supseteq M$ and therefore $\mathrm{pr}_M(P \cap x\theta(P)x^{-1}) = M$, i.e. x is M-admissible.

Conversely, suppose that $x \in X$ is M-admissible and let $w = \imath_M(P \cdot x)$. Then $Mw\theta(M) \subseteq N_{G,\theta}(M)$ and $Pw\theta(P) = Px\theta(P)$. Let u, $u' \in U$ m, $m' \in M$, and $n_w \in w$ be such that $x = umn_w\theta(m')\theta(u')$. Then $x \in uMn_w\theta(M)u' \subseteq UN_{G,\theta}(M)U'$. □

Combined with Lemma 3.2 we can now summarize as follows.

Corollary 6.2. *The map $\mathcal{O} \mapsto \mathcal{O} \cap N_{G,\theta}(M)$ defines a bijection from the M-admissible orbits in $P\backslash X$ to $M\backslash(X \cap N_{G,\theta}(M))$.* □

We now make the following observation.

Lemma 6.3. *Let $x \in X \cap N_{G,\theta}(M)$. Then $P_x = M_x \ltimes U_x$.*

Proof. By Lemma 3.2 and in its notation we have $P_x = L_x \ltimes Z_x$. Since x is M-admissible $L = M$ and $Z = U$. □

For what follows, we introduce some more notation. Let $T_M = T \cap Z_M$ be the maximal split torus in the center Z_M of M. For a standard parabolic subgroup $Q = L \ltimes V \supseteq P$ let $\Sigma_M^L = R(T_M, L)$ be the set of roots of T_M in L and $\Sigma_M^{L,+} = R(T_M, L \cap P)$ the subset of positive roots with respect to $L \cap P$.

Recall that $\Sigma_{M_0}^G$ forms a root system and let Δ_0 be its basis of simple roots with respect to P_0. Let Δ_P be the set of non-zero restrictions to T_M of the elements of Δ_0. More generally, $\Delta_0^Q = \Delta_0 \cap \Sigma_{M_0}^L$ is a basis of simple roots for the root system $\Sigma_{M_0}^L$. Let $\Delta_P^Q = \Delta_P \cap \Sigma_M^L$ be the set of non-zero restrictions to T_M of the elements of Δ_0^Q.

Let $W^L(M)$ be the set of $w \in W_L$ so that w is of minimal length in wW_M and wMw^{-1} is a standard Levi subgroup of L. Let w_M^L denote the element of maximal length in $W^L(M)$.

For $\alpha \in \Delta_P$ let $s_\alpha \in W^G(M)$ be the elementary symmetry associated to α as in [MW95, §I.1.7]. There is a unique standard parabolic subgroup $Q = L \ltimes V \supseteq P$ such that $\Delta_P^Q = \{\alpha\}$ and a standard parabolic subgroup $P_1 = M_1 \ltimes U_1 \subseteq Q$ such that $s_\alpha \in W$ is of minimal length in wW_M, $M_1 = s_\alpha M s_\alpha^{-1}$ and $\Delta_{P_1}^Q = \{-s_\alpha \alpha\}$.

We remark that if σ is an involution on G so that $\sigma(M) = M$ then σ also stabilizes T_M and therefore acts on Σ_M^G.

In what follows, we verify that the proofs in [LO, §3.4] generalize to our setting. We define a directed edge-labeled graph $\mathfrak{G}$ in the spirit of [LR03, §3.3] as follows. The vertices of $\mathfrak{G}$ are pairs (M, x) where M is a standard Levi subgroup of G and $x \in X \cap N_{G,\theta}(M)$. Note that for a vertex (M, x) we have $\theta_x(M) = M$ (since x is M-admissible) and therefore θ_x acts on Σ_M^G. The (labeled) edges of $\mathfrak{G}$ are given by $(M, x) \xrightarrow{n_\alpha} (M_1, x_1)$ provided that:

(1) $\alpha \in \Delta_P$,
(2) $n_\alpha \in s_\alpha M$,
(3) $\theta_x(\alpha) \neq \pm\alpha$,
(4) $M_1 = s_\alpha M s_\alpha^{-1} = n_\alpha M n_\alpha^{-1}$,
(5) $x_1 = n_\alpha \cdot x$.

We will write $(M, x) \overset{n_\alpha}{\searrow} (M_1, x_1)$ if $(M, x) \xrightarrow{n_\alpha} (M_1, x_1)$ and $\theta_x(\alpha) < 0$ (i.e., $\theta_x(\alpha) \notin \Sigma_M^{G,+}$). Note that if $(M, x) \xrightarrow{n_\alpha} (M_1, x_1)$ then also $(M_1, x_1) \xrightarrow{n_\alpha^{-1}} (M, x)$. Moreover, either $(M, x) \overset{n_\alpha}{\searrow} (M_1, x_1)$ or $(M_1, x_1) \overset{n_\alpha^{-1}}{\searrow} (M, x)$ but not both. For a finite sequence of edges

$$(M, x) = (M_1, x_1) \xrightarrow{n_{\alpha_1}} (M_2, x_2) \xrightarrow{n_{\alpha_2}} \cdots \xrightarrow{n_{\alpha_k}} (M_{k+1}, x_{k+1}) = (M^*, x^*)$$

in $\mathfrak{G}$ we will write $(M, x) \overset{g}{\curvearrowright} (M^*, x^*)$ where $g = n_{\alpha_k} \dots n_{\alpha_1} \in G$. Note that $gMg^{-1} = M^*$ and $g \cdot x = x^*$. Similarly, we write $(M, x) \overset{g}{\downarrow} (M^*, x^*)$ if there exists a finite sequence

$$(M, x) = (M_1, x_1) \overset{n_{\alpha_1}}{\searrow} (M_2, x_2) \overset{n_{\alpha_2}}{\searrow} \cdots \overset{n_{\alpha_k}}{\searrow} (M_{k+1}, x_{k+1}) = (M^*, x^*).$$

Lemma 6.4. *Suppose that (M, x) and (M_1, x_1) are vertices in $\mathfrak{G}$ and $(M, x) \overset{n_\alpha}{\searrow} (M_1, x_1)$ for some $\alpha \in \Delta_P$. Let $Q = L \ltimes V$ be the parabolic subgroup of G containing P such that $\Delta_P^Q = \{\alpha\}$ and let $P_1 = M_1 \ltimes U_1$ be the parabolic subgroup of Q such that $\Delta_{P_1}^Q = \{-s_\alpha \alpha\}$. Then*

(1) *$V_{x_1} = n_\alpha U_x n_\alpha^{-1}$ and in particular $n_\alpha U_x n_\alpha^{-1} \subseteq (U_1)_{x_1}$.*
(2) *We have the following short exact sequence of subgroups normalized by $(M_1)_{x_1}$:*

$$1 \longrightarrow n_\alpha U_x n_\alpha^{-1} \longrightarrow (U_1)_{x_1} \xrightarrow{\mathrm{pr}_L} L \cap U_1 \longrightarrow 1.$$

(3) *We have*

$$(\delta_{P_x} \delta_P^{-\frac{1}{2}})(m) = (\delta_{(P_1)_{x_1}} \delta_{P_1}^{-\frac{1}{2}})(n_\alpha m n_\alpha^{-1}), \quad m \in M_x. \tag{6}$$

Proof. The first two parts are proved exactly as in [LR03, Lemma 4.3.1 (1) and (2)]. We omit the details.

Note that $P_1 \cap L = M_1 \ltimes (U_1 \cap L)$ and by Lemma 6.3 we also have $P_x = M_x \ltimes U_x$ and $(P_1)_{x_1} = (M_1)_{x_1} \ltimes (U_1)_{x_1}$. Therefore, as in the proof of [LR03, Proposition 4.3.2] the relation

$$\delta_{P_x}(m) = (\delta_{(P_1)_{x_1}} \delta_{P_1\cap L}^{-1})(n_\alpha m n_\alpha^{-1}), \quad m \in M_x$$

follows from part (2). It also follows from the proof of [LR03, Proposition 4.3.2] that

$$\delta_P^{-\frac{1}{2}}(m) = (\delta_{P_1}^{-\frac{1}{2}} \delta_{P_1\cap L})(n_\alpha m n_\alpha^{-1}).$$

The identity (6) follows. $\square$

A straightforward consequence of the lemma is

Corollary 6.5. *Suppose that $(M,x) \overset{g}{\curvearrowright} (M^*,x^*)$ in $\mathfrak{G}$ and let P^* be the standard parabolic subgroup with Levi subgroup M^*. Then*

$$(\delta_{P_x}\delta_P^{-\frac{1}{2}})(m) = (\delta_{P^*_{x^*}}\delta_{P^*}^{-\frac{1}{2}})(gmg^{-1}), \quad m \in M_x. \qquad \square$$

A graph of a similar nature, was defined in [LR03, §3.3] for an involution on $\mathfrak{a}_0^*$ that stabilizes Δ_0. In order to apply the results of Lapid and Rogawsky to $\mathfrak{G}$ we relate it to the graph they associate to θ'.

Observe that if $(M,x) \xrightarrow{n_\alpha} (M_1,x_1)$ is an edge in $\mathfrak{G}$ then $x_1 n = (n_\alpha \cdot x)n = n_\alpha x n\theta'(n_\alpha)^{-1}$, $w_1 = \iota_{M_1}(P_1 \cdot x_1) = s_\alpha w\theta(s_\alpha)^{-1}$ and $w_1\tau = s_\alpha w\tau\theta'(s_\alpha)^{-1}$.

Let $\mathfrak{G}'$ be the graph with vertices

$$\{(M, w\tau) : (M,x) \text{ a vertext in } \mathfrak{G} \text{ and } w = \iota_M(P\cdot x)\}$$

and edges

$$(M, w\tau) \xrightarrow{\alpha} (M_1, w_1\tau)$$

whenever $\alpha \in \Delta_P$ and there exists $n_\alpha \in s_\alpha M$ such that $(M,x) \xrightarrow{n_\alpha} (M_1,x_1)$ is an edge in $\mathfrak{G}$ and $\iota_{M_1}(P_1 \cdot x_1) = w_1$. The connected components of $\mathfrak{G}'$ and the graphs $\mathfrak{G}^0$ associated in [LR03, §3.3] to $\sigma = \theta'$ are the same.

We recall the following terminology from [LR03]. A twisted involution $w \in {}_MW_{M'} \cap \mathfrak{J}_0(\theta')$ is (M,θ')-admissible if $M = wM'w^{-1}$. An (M,θ')-admissible w is (M,θ')-minimal if there exists a θ'-stable standard Levi subgroup $L \supseteq M$ such that $w = w_{M'}^L$ and $w\theta'(\alpha) = -\alpha$ for all $\alpha \in \Delta_P^Q$.

It is a straight forward observation that $x \in X$ is M-admissible if and only if $\iota_M(P\cdot x)\tau$ is (M,θ')-admissible.

Definition 6.6. We say that $x \in X \cap N_{G,\theta}(M)$ is M-minimal if $\iota_M(P \cdot x)\tau$ is (M, θ')-minimal.

Combining Corollary 6.5 with [LR03, Lemma 3.2.1 and Proposition 3.3.1] we therefore get:

Corollary 6.7. *Let M be a standard Levi subgroup of G and $x \in X \cap N_{G,\theta}(M)$. Then there exists $g \in G$ such that $M^* = gMg^{-1}$ is standard, $x^* = g \cdot x$ is M^*-minimal and $(M, x) \overset{g}{\downarrow} (M^*, x^*)$. Therefore,*

$$(\delta_{P_x} \delta_P^{-\frac{1}{2}})(m) = (\delta_{P^*_{x^*}} \delta_{P^*}^{-\frac{1}{2}})(gmg^{-1}), \quad m \in M_x. \qquad \square$$

Assume now that $x \in X \cap N_{G,\theta}(M)$ is M-minimal. Let $w = \iota_M(P \cdot x)$ and let $Q = L \ltimes V$ be the standard parabolic subgroup with standard Levi subgroup L so that $w\tau = w^L_{M'}$ as in the definition of (M, θ')-minimality.

Lemma 6.8. *In the above notation Q is θ_x-stable, $\delta_{P_x} = \delta_{Q_x}|_{P_x}$ and $\delta_P|_{M_x} = \delta_Q|_{M_x}$, in particular,*

$$\delta_{P_x} \delta_P^{-1/2}|_{M_x} = \delta_{Q_x} \delta_Q^{-1/2}|_{M_x}.$$

Proof. Note that since L is θ'-stable and $Q' = \theta'(Q)$ is a standard parabolic subgroup we must also have $Q' = Q$. Furthermore, M-minimality implies that $xn \in L$. Since $\theta_x(g) = xn\theta'(g)(xn)^{-1}$, $Q = L \ltimes V$ is a θ_x-stable parabolic subgroup with a θ_x-stable decomposition and therefore

$$Q_x = L_x \ltimes V_x.$$

Furthermore, since $U \cap w^L_{M'}(L \cap U')(w^L_{M'})^{-1} = 1$ and $U' = (L \cap U')V$ we also have that $U_x \subseteq U \cap \theta_x(U) = U \cap w^L_{M'} U' (w^L_{M'})^{-1} \subseteq w^L_{M'} V (w^L_{M'})^{-1} = V$. It therefore follows from Lemma 6.3 that

$$P_x = M_x \ltimes V_x.$$

We therefore have that $\delta_{Q_x}|_{P_x} = \delta_{P_x}$. As in the proof of [LR03, Proposition 4.3.2], it follows from [LR03, Proposition 3.2.1 (2)] that $\delta_{L \cap P}|_{M_x} = 1$. Since $\delta_P|_M = \delta_Q|_M \cdot \delta_{L \cap P}|_M$, the lemma follows. $\square$

Combining Corollary 6.7, Lemma 6.8 and the proof of [LR03, Lemma 2.5.1], in the Galois case, we remove the assumption $\theta(P_0) = P_0$ in [LR03, Proposition 4.3.2].

Corollary 6.9. *Let $P = M \ltimes U$ be a standard Levi subgroup of G and $x \in X \cap N_{G,\theta}(M)$. If θ is a Galois involution then $\delta_{P_x} \delta_P^{-1/2}|_{M_x} = 1$.* $\square$

7. Sufficient conditions for distinction

In this section, we deduce distinction of an induced representation from distinction of the inducing data in two ways. The first applies Lemma 4.3 in the context of an admissible closed orbit. The second is related to the open orbit.

Proposition 7.1. *Let $P = M \ltimes U$ be a standard parabolic subgroup of G and σ a smooth representation of G. Suppose that $\eta \in G$ is such that $x = \eta \cdot e \in N_{G,\theta}(M)$ and $\theta_x(P) = P$. If σ is $(M_x, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$-distinguished then $\mathrm{Ind}_P^G(\sigma)$ is (H,χ)-distinguished. In particular, if θ is a Galois involution and σ is $(M_x, \chi^{\eta^{-1}})$-distinguished then $\mathrm{Ind}_P^G(\sigma)$ is (H,χ)-distinguished.*

Proof. Note that x is M-admissible and that $P\eta H$ is closed in G. Indeed, $P_x \cap G_x^\circ$ is a parabolic subgroup of the connected component G_x° of G_x (see e.g. [GO, Lemma 3.1]) and since $G_x^\circ \backslash G_x$ is finite it follows that $P_x\backslash G_x$ is compact. Therefore, $PG_x = P\eta H\eta^{-1}$ and also $P\eta H$ are closed in G. The proposition now follows from Lemma 4.3 for a general θ. If θ is Galois, the proposition follows from the general case and Corollary 6.9. $\square$

Based on the work of Blanc and Delorme [BD08], we end this work with another sufficient condition for distinction of an induced representation. We say that a representation of G is H-distinguished if it is $(H,1)$-distinguished.

Proposition 7.2. *Let $P = M \ltimes U$ be a standard parabolic subgroup of G and σ a smooth representation of G of finite length. Assume that $x \in (G \cdot e) \cap N_{G,\theta}(M)$ is such that $P \cap \theta_x(P) = M$. If σ is M_x-distinguished, then $\mathrm{Ind}_P^G(\sigma)$ is H-distinguished.*

Proof. Note that $\theta_x(P)$ is a parabolic subgroup of G opposite to P. Let $\mathcal{X}$ be the connected component of the identity in the complex space of unramified characters χ of M satisfying $\theta_x(\chi) = \chi^{-1}$. If $\mathfrak{a}$ is the -1-eigenspace of the involution θ_x on the complex vector space $X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}$, then there is a natural surjective map $\lambda \mapsto \chi_\lambda : \mathfrak{a} \to \mathcal{X}$. The induced representations $\mathrm{Ind}_P^G(\sigma \otimes \chi_\lambda)$, $\lambda \in \mathfrak{a}$ can all be realized in the same vector space V. It follows from [BD08, Theorem 2.8] that there is a non-zero meromorphic function $(\lambda \mapsto \ell_\lambda) : \mathfrak{a} \to V^*$ of linear forms so that $\ell_\lambda \in \mathrm{Hom}_{G_x}(\mathrm{Ind}_P^G(\sigma \otimes \chi_\lambda), 1)$ whenever holomorphic at λ. Taking a leading term at $\lambda = 0$ along a complex line through zero in a generic direction we obtain a non-zero element of $\mathrm{Hom}_{G_x}(\mathrm{Ind}_P^G(\sigma), 1)$.

If $\eta \in G$ is such that $x = \eta \cdot e$ then $G_x = \eta H \eta^{-1}$ and therefore $\mathrm{Ind}_P^G(\sigma)$ is also H-distinguished. $\square$

Remark 7.3. This argument was already applied in the case that E/F is a quadratic extension, $G = \mathrm{GL}_n(E)$ and $H = U_n(E/F)$ is a unitary group in [FLO12] and in the case that $G = \mathrm{GL}_{2n}(F)$ and $H = \mathrm{Sp}_{2n}(F)$ in [MOS], in order to show that distinction is preserved under parabolic induction. The result in the general framework of a symmetric space will be of use, in particular, for the study of distinguished representations of classical groups.

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