ON LOCAL ROOT NUMBERS AND DISTINCTION

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ABSTRACT. In the context of GL_n over a quadratic extension of p-adic fields (p odd) we show that the Rankin-Selberg epsilon factor at $\frac{1}{2}$ of a pair of distinguished irreducible representations equals 1.

In his dissertation [Ok97], under the supervision of Hervé Jacquet, Youngbin Ok indicated an interesting relation between distinction of representations and special values of local Rankin-Selberg gamma factors. In a special case, Ok characterizes distinction in terms of the values of gamma factors. The results in Ok's dissertation were never published and Jacquet kindly suggested to extend his results. In this work we generalize one direction in Ok's characterization. We hope to address the other direction in the future.

Let E be a local non-archimedean field and ψ a non-trivial character of E. For positive integers r and t and for smooth irreducible representations π of $GL_r(E)$ and π' of $GL_t(E)$ Jacquet, Piatetskii-Shapiro and Shalika attached in [JPSS83] a local Rankin-Selberg L-factor $L(s,\pi\times\pi')$ and ϵ-factor $\epsilon(s,\pi\times\pi';\psi)$. We also set

$$\gamma(s,\pi\times\pi';\psi)=\frac{L(1-s,\tilde{\pi}\times\tilde{\pi}')\,\epsilon(s,\pi\times\pi';\psi)}{L(s,\pi\times\pi')} \tag{0.1}$$

where $\tilde{\pi}$ is the contragredient of π. Let E/F be a quadratic extension of non-archimedean local fields. Assume that the characteristic of the residual field of F is odd. A representation (π,V) of $GL_r(E)$ is called $GL_r(F)$-*distinguished* if there exists a non-zero linear form $\mu:V\to\mathbb{C}$ such that

$$\mu(\pi(h)v)=\mu(v),\ v\in V,\ h\in GL_r(F).$$

We may now state the main result of this work (which is [Ana08, Conjecture 5.1]).

Theorem 0.1. *Let π (resp. π') be a smooth, irreducible and $GL_r(F)$-distinguished (resp. $GL_t(F)$-distinguished) representation of $GL_r(E)$ (resp. $GL_t(E)$). If ψ is a non-trivial character of E with a trivial restriction to F then*

$$\epsilon(\frac{1}{2},\pi\times\pi';\psi)=\gamma(\frac{1}{2},\pi\times\pi';\psi)=1.$$

Let us make a few straightforward observations first. The ϵ-factor satisfies the identity

$$\epsilon(s,\pi\times\pi';\psi)\,\epsilon(1-s,\tilde{\pi}\times\tilde{\pi}';\psi^{-1})=1. \tag{0.2}$$

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Denote by $x \mapsto \bar{x}$ the Galois action associated to E/F and let $\bar{f}(g) = f(\bar{g})$, $g \in GL_r(E)$ for any function f with domain $GL_r(E)$. We then have

$$L(s, \bar{\pi} \times \bar{\pi}') = L(s, \pi \times \pi') \text{ and } \epsilon(s, \bar{\pi} \times \bar{\pi}'; \bar{\psi}) = \epsilon(s, \pi \times \pi'; \psi). \tag{0.3}$$

Flicker proved in [Fli91, Proposition 12] that any smooth, irreducible and $GL_r(F)$-distinguished representation π of $GL_r(E)$ satisfies

$$\bar{\pi} \simeq \tilde{\pi}. \tag{0.4}$$

Let π and π' be as in the assumption of Theorem 0.1. It follows from (0.3) and (0.4) that

$$L(s, \pi \times \pi') = L(s, \tilde{\pi} \times \tilde{\pi}') \quad \text{and} \quad \epsilon(s, \tilde{\pi} \times \tilde{\pi}'; \bar{\psi}) = \epsilon(s, \pi \times \pi'; \psi).$$

Furthermore, if ψ is trivial on F then $\bar{\psi} = \psi^{-1}$ and it therefore follows from (0.2) that

$$\epsilon(\frac{1}{2}, \pi \times \pi'; \psi)^2 = 1.$$

Thus,

$$\epsilon(\frac{1}{2}, \pi \times \pi'; \psi) = \gamma(\frac{1}{2}, \pi \times \pi'; \psi) \in \{1, -1\}.$$

The difficulty in Theorem 0.1 is therefore to determine the sign of the local Rankin-Selberg root number $\epsilon(\frac{1}{2}, \pi \times \pi'; \psi)$. Theorem 0.1 was proved by Ok in his thesis under the further assumptions that $t \le r$, that π is supercuspidal and that π' is unitary and non-degenerate if $t < r$ and supercuspidal if $t = r$. For a supercuspidal representation, Ok further showed that the γ-factor of enough distinguished twists being 1 at $s = \frac{1}{2}$ characterizes distinction. More precisely, he showed that if π is a smooth, irreducible, supercuspidal representation of $GL_r(E)$ with a central character trivial on $F^\times$ and such that $\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = 1$ for every smooth, irreducible, unitarizable, non-degenerate and $GL_{r-1}(F)$-distinguished representation of $GL_{r-1}(E)$ then π is $GL_r(F)$-distinguished. It will be interesting to examine to what extent this characterization of distinction can be generalized for other π. At this point, however, we do not have a strategy to generalizing this 'converse theorem'. The case $r = 2$ (for *any* irreducible π) of this characterization of distinction was proved by Hakim [Hak91, Theorem 4.1].

Distinction in our context is further related to the existence of poles of the local Asai (twisted tensor) L-function as observed in [AKT04]. The Asai root number of a distinguished representation is computed in [Ana08] in the square-integrable case. The study of the ϵ-factor in [loc. cit.] applies an idea of Lapid-Rallis [LR03] further pursued in [Lap04] to study root numbers. In the context of the triple product for GL_2 Prasad relates between distinction and root values in [Pra07]. We emphasize that the study of the local ϵ-factor in all the above mentioned papers is by global means. In contrast, the methods used in this paper are purely local.

We now discuss in more detail the content of this work beginning with the very last section. Recently, a new tool was introduced in the study of local harmonic analysis on p-adic symmetric spaces. Kato and Takano introduced in [KT08] the concept of relative cuspidality (see also [Lag08] for an independent treatment) and proved a relative version of Jacquet's sub-representation theorem. In Section 7 we explicate the Kato-Takano sub-representation Theorem in our special setting and apply it to reduce Theorem 0.1 to the case where both π and π' are relatively cuspidal. This special case is formulated as Proposition 6.1 and the rest of the paper focuses on its proof. We remark that a distinguished supercuspidal representation is also relatively cuspidal. We adjust Ok's method of proof (for supercuspidal representations) to the relatively cuspidal case. After setting the notation in Section 1 we prove Proposition 6.1 in Section 2 in the special case $t = 1$, i.e. for Godement-Jacquet γ-factors, using the Godement-Jacquet theory [GJ72]. In Section 3 we show that on a $GL_r(F)$-distinguished, irreducible, unitarizable and non-degenerate representation π every linear form invariant under the mirabolic subgroup $P_r(F)$ of $GL_r(F)$ is automatically $GL_r(F)$-invariant (see [AKT04, Theorem 1.1] for a stronger statement for tempered representations). This is a relative analogue of [Ber84, Theorem A]. It allows us to deduce in Section 4 that the two linear forms on the Whittaker model of π defined by

$$W \mapsto \int_{U_r(F)\backslash P_r(F)} W(p)\ dp \quad \text{and} \quad W \mapsto \int_{U_r(F)\backslash P_r(F)} W(\begin{pmatrix} 0 & 1 \\ I_{r-1} & 0 \end{pmatrix} p)\ dp$$

(here $U_r(F)$ is the unipotent radical of $P_r(F)$) are $GL_r(F)$-invariant. Flicker's multiplicity one ([Fli91, Proposition 11]) then implies that the two forms are proportional and we denote by $c(\pi)$ the proportionality constant. In fact, π is $GL_r(F)$-distinguished if and only if the two forms are proportional (Proposition 4.1, see also [AKT04, Theorem 1.3] for π tempered).

Ok's origional proof of our main Theorem in the supercuspidal case relies on the local Rankin-Selberg integrals of [JPSS83]. In order to apply the Rankin-Selberg construction to relatively cuspidal representations we need to know that they admit a Whittaker model. This is carried out in Section 5 by adjusting to the relative setting arguments from [BZ76]. We emphasize that this step is not vacuous. If σ is an irreducible, supercuspidal, $GL_n(F)$-distinguished representation of $GL_n(E)$ such that $\tilde{\sigma} \not\simeq \sigma$ then, using the results of Kato-Takano, it can be shown that the (non-supercuspidal) representation of $GL_{2n}(E)$ parabolically induced from $\sigma \otimes \tilde{\sigma}$ is $GL_{2n}(F)$-relatively supercuspidal. For the case $n = 1$ these representations were considered by Hakim. Curiously, they are the only irreducible representations of $GL_2(E)$ that are distinguished by $GL_2(F)$ but not by its non-trivial inner forms ([Hak91, Theorem 6.1]). In Section 6 we apply the Rankin-Selberg integrals of Jacquet-P. Shapiro-Shalika to prove Proposition 6.1. Once Theorem 0.1 is proved we deduce in Section 7 that for distinguished, unitarizable, non-degenerate representations π we have $c(\pi) = 1$ (Corollary 7.2).

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1. Notation

Let F be a non-archimedean local field and let ψ_F be a non-trivial character of F. Denote respectively by $\mathcal{O}_F$, $\mathfrak{p}_F$, ϖ_F and q_F the ring of integers of F, its maximal ideal, a uniformizer in $\mathfrak{p}_F$ and the size of the residual field $\mathcal{O}_F/\mathfrak{p}_F$. Denote by $|\cdot|_F$ the standard absolute value on F so that $|\varpi_F|_F = q_F^{-1}$. We assume throughout this work that q_F is odd. Let E be a quadratic extension of F and let $x \mapsto \bar{x}$ denote the associated Galois action. Similar notation applies to E. Thus, for example, $|x|_E = |x|_F^2$, $x \in F$. The symbol $|\cdot|$ without a subscript will be used solely for the standard absolute value on $\mathbb{C}$. Fix once and for all $\iota \in E^\times$ such that $\bar{\iota} = -\iota$ and a non-trivial additive character ψ_F of F. Let

$$\psi(x) = \psi_F\left(\frac{x-\bar{x}}{2\iota}\right),\ x \in E.$$

Thus ψ is a non-trivial character of E with trivial restriction to F and any non-trivial character of E with trivial restriction to F is obtained in this way from some ψ_F.

Denote by $M_{r\times t}(F)$ the vector space of all $r \times t$ matrices with entries in F and set $M_r(F) = M_{r\times r}(F)$. We often denote both $M_{r\times 1}(F)$ and $M_{1\times r}(F)$ by F^r and hope the context is clear enough to distinguish between the two vector spaces. Let $G_r = GL_r$ be considered as an algebraic group over F and let Z_r denote the center of G_r. Denote by U_r the subgroup of G_r consisting of upper triangular unipotent matrices and denote by ψ_r the character of $U_r(E)$ defined by

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}),\ u \in U_r(E).$$

Let P_r be the *mirabolic* subgroup of G_r consisting of matrices with last row equal to

$$\eta_r = (0, \dots, 0, 1) \in M_{1\times r}(F) \tag{1.1}$$

and let V_r be the unipotent radical of P_r. Thus,

$$V_r(E) = \left\{ \begin{pmatrix} I_{r-1} & x \\ 0 & 1 \end{pmatrix} : x \in E^{r-1} \right\}.$$

We denote by $w_r \in G_r(F)$ the permutation matrix with $(i,j)^{th}$ entry equal to $\delta_{i,r+1-j}$. We will sometimes also consider w_r as the associated permutation of the set $\{1, 2, \dots, r\}$, i.e. $w_r(i) = r + 1 - i$.

By a representation, we always mean a smooth representation. Let (π, V) be a representation of $G_r(E)$. If π has a central character then it is denoted by ω_π. We denote by $(\tilde{\pi}, \tilde{V})$ the *contragredient* or smooth dual of π. The representation π is called *non-degenerate* if it has a non-trivial Whittaker functional, i.e. if there exists a non-zero linear form λ on V such that

$$\lambda(\pi(u)v) = \psi_r(u)\lambda(v),\ v \in V,\ u \in U_r(E).$$

The representation π is said to be of *Whittaker type* if it is admissible, has a central character and has precisely a one dimensional space of Whittaker functionals. If π is irreducible and non-degenerate then it is of Whittaker type. If π is of Whittaker type, we denote its Whittaker model by $\mathcal{W}(\pi,\psi)$. It is the space of functions W on $G_r(E)$ of the form

$$W(g)=\lambda(\pi(g)v),\ g\in G_r(E)$$

for $v\in V$ where λ is a non-zero Whittaker functional of π. If π is irreducible and non-degenerate then so is $\tilde{\pi}$ and its Whittaker model $\mathcal{W}(\tilde{\pi},\psi^{-1})$ is given by

$$\mathcal{W}(\tilde{\pi},\psi^{-1})=\{\tilde{W}:W\in\mathcal{W}(\pi,\psi)\}$$

where

$$\tilde{W}(g)=W(w_r\,{}^tg^{-1}).$$

Denote by $R(g)$ the action of $G_r(E)$ by right translations on any space of functions on $G_r(E)$. Note that if π is irreducible and non-degenerate then

$$\widetilde{R(g)W}=R({}^tg^{-1})\tilde{W},\ g\in G_r(E),\ W\in\mathcal{W}(\pi,\psi). \tag{1.2}$$

We say that π is $G_r(F)$-*distinguished* or simply *distinguished* if it has a non-trivial $G_r(F)$-invariant linear form, i.e. if there exists a non-zero linear form μ on V such that

$$\mu(\pi(h)v)=\mu(v),\ v\in V,\ h\in G_r(F). \tag{1.3}$$

Furthermore, π is called $G_r(F)$-*relatively cuspidal* or simply *relatively cuspidal* if π is distinguished and if for any $G_r(F)$-invariant linear form μ on V and any $v\in V$ the generalized matrix coefficient

$$f_{v,\mu}(g)=\mu(\pi(g)v)$$

lies in $C_c^\infty(G_r(F)\backslash G_r(E))$.

Remark 1. We recall that for the symmetric space at hand we have multiplicity one ([Fli91, Proposition 11]). Since furthermore $G_r(F)\backslash Z_r(E)G_r(F)\simeq F^\times\backslash E^\times$ is compact our definition is compatible with that of Kato and Takano in [KT08]. (In particular, relative cuspidality in the sense of Kato-Takano with respect to some non-zero invariant linear form is equivalent to our definition of relative cuspidality.)

For a parabolic subgroup $Q=LV$ of G_r with Levi subgroup L and unipotent radical V and for a representation τ of $L(E)$ we denote by $I_{Q(E)}^{G_r(E)}(\tau)$ the representation of $G_r(E)$ obtained from τ by parabolic induction. Denote by $\mathbf{1}_\Gamma$ the characteristic function of a set Γ.

2. Distinction for GL_n and γ-factors at $\frac{1}{2}$.

Fix a positive integer n and let $G=G_n(E)$ and $H=G_n(F)$. For an irreducible representation π of G, Godement and Jacquet attached in [GJ72] a local L-factor $L(s,\pi)$ and ϵ-factor $\epsilon(s,\pi;\psi)$. We also set

$$\gamma(s,\pi;\psi)=\frac{\epsilon(s,\pi;\psi)\,L(1-s,\tilde{\pi})}{L(s,\pi)}.$$

The Godement-Jacquet L and ϵ-factors are a special case of the Rankin-Selberg L and ϵ-factors. That is

$$L(s,\pi)=L(s,\pi\times \mathbf{1}_{E^\times}) \text{ and } \epsilon(s,\pi;\psi)=\epsilon(s,\pi\times\mathbf{1}_{E^\times};\psi).$$

The purpose of this section is to prove Theorem 0.1 in the special case where $t=1$ and π is relatively cuspidal.

Proposition 2.1. *Let π be an irreducible H-relatively cuspidal representation of G. Then $\gamma(\frac{1}{2},\pi;\psi)=1$.*

The Godement-Jacquet construction of the local L and ϵ-factors is essential for the proof and we therefore begin in Section 2.1 with a review of the local non archemedean Godement-Jacquet theory. In Section 2.2 we adjust the theory to the relative setting by allowing generalized matrix coefficients. Based on the ideas of Ok, we prove Proposition 2.1 in Section 2.3. We remark that already at this stage we could, with little effort, apply the subrepresentation theorem of Kato-Takano (see Section 7.1) to remove the relative cuspidality assumption on π. However, since only the relatively cuspidal case is used in our proof of Theorem 0.1 we find it superfluous to carry this out here.

2.1. **Godement-Jacquet Zeta integrals.** We review the Godement-Jacquet construction of the local L and ϵ-factors for representations of GL_n over a non-acrchimedean local field [GJ72]. We begin by introducing some notation.

Let (π,V) be an irreducible representation of G, $(\tilde{\pi},\tilde{V})$ its contragredient representation and $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{\pi\times\tilde{\pi}}$ a non-degenerate G-invariant bi-linear form on $V\times\tilde{V}$. Let $\mathcal{C}(\pi)$ denote the space of matrix coefficients of π, i.e. the space of complex valued functions on G spanned by functions of the form

$$f_{v,\tilde{v}}(g)=\langle\pi(g)v,\tilde{v}\rangle\,,\; g\in G$$

where $v\in V$, $\tilde{v}\in\tilde{V}$. For every function f on G we denote by $f^\vee$ the function $f^\vee(g)=f(g^{-1})$. Note that $f\mapsto f^\vee$ is a bijection between $\mathcal{C}(\pi)$ and $\mathcal{C}(\tilde{\pi})$. Let $\mathcal{H}=C_c^\infty(G)$ be the Hecke algebra of G with multiplication given by convolution. The space $C^\infty(G)$ is an $\mathcal{H}$-module, the action given again by the convolution

$$f*\Phi(x)=\int_G f(g)\Phi(g^{-1}x)\;dg \tag{2.1}$$

where $x\in G$, $f\in\mathcal{H}$, $\Phi\in C^\infty(G)$. A function $\Phi\in C^\infty(G)$ is called left (resp. right) uniformly smooth if there is an open compact subgroup K of G such that

$$f(kg)=f(g) \text{ (resp. } f(gk)=f(g)),\; k\in K,\; g\in G.$$

Following J. Tate's thesis for the case $n=1$, Godement and Jacquet considered an extremely useful $\mathcal{H}$-submodule of $C^\infty(G)$. The space $C_c^\infty(M_n(E))$ is an $\mathcal{H}$-module under the convolution given by (2.1) where we allow x to be in $M_n(E)$ and let Φ be in $C_c^\infty(M_n(E))$. Since G is open and dense in $M_n(E)$ the restriction map $\Phi\mapsto\Phi_{|G}$ is an $\mathcal{H}$-module embedding of $C_c^\infty(M_n(E))$ in $C^\infty(G)$. We will often not distinguish between a function Φ in $C_c^\infty(M_n(E))$ and its restriction to G and we view $C_c^\infty(M_n(E))$ as an $\mathcal{H}$-submodule of $C^\infty(G)$. Thus, $\mathcal{H} \subset C_c^\infty(M_n(E)) \subset C^\infty(G)$ is a sequence of $\mathcal{H}$-modules. Note that $C_c^\infty(M_n(E))$ consists of left and right uniformly smooth functions (in fact, as a function on $M_n(E)$, every $\Phi \in C_c^\infty(M_n(E))$ is left and right invariant by some open compact subgroup K of G). Since $M_n(E)$ is an E-vector space, there is a Fourier transform on $C_c^\infty(M_n(E))$ defined by

$$\mathcal{F}_\psi(\Phi)(x) = \int_{M_n(E)} \Phi(y)\psi(\mathrm{Tr}(xy))\, dy,\ x \in M_n(E)$$

where the Haar measure dy on $M_n(E)$ is self dual with respect to the pairing $(x,y) \mapsto \psi(\mathrm{Tr}(xy))$, i.e. it satisfies $\mathcal{F}_{\psi^{-1}}(\mathcal{F}_\psi(\Phi)) = \Phi$. The fact that this Fourier transform maps the space $C_c^\infty(M_n(E))$ to itself is a key to the definition, that we now recall, of the L and ϵ-factors.

Let π be an irreducible representation of G. There exists $s_0 \in \mathbb{R}$ such that for every $\Phi \in C_c^\infty(M_n(E))$ and every $f \in \mathcal{C}(\pi)$ the integral

$$Z(\Phi, f, s) = \int_G \Phi(g) f(g)\, |\det g|_E^{s+\frac{n-1}{2}}\ dg \tag{2.2}$$

is absolutely convergent when $\mathrm{Re}\, s > s_0$ and it extends to a rational function in q^s that we denote again by $Z(\Phi, f, s)$. Here the Haar measure dg on G is given by restriction to G of the measure $\mathbf{1}_G(y)\, |\det y|_E^{-n}\ dy$ on $M_n(E)$. Let $X = q^{-s}$ and denote by $\mathcal{Z}(\pi)$ the vector sub-space of $\mathbb{C}(X)$ consisting of all the Godement-Jacquet Zeta integrals $Z(\Phi, f, s)$. The space $\mathcal{Z}(\pi)$ is a fractional ideal for the ring $\mathbb{C}[X, X^{-1}]$ that contains $\mathbb{C}[X, X^{-1}]$. Therefore, there is a unique polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$ and such that

$$\mathcal{Z}(\pi) = P(X)^{-1}\, \mathbb{C}[X, X^{-1}].$$

The L-factor $L(s, \pi)$ attached to the representation π is defined by

$$L(s, \pi) = P(X)^{-1}.$$

Furthermore, there is a rational function $\gamma(s, \pi; \psi) \in \mathbb{C}(X)$ such that for every $\Phi \in C_c^\infty(M_n(E))$ and for every $f \in \mathcal{C}(\pi)$ we have

$$Z(\mathcal{F}_\psi \Phi, f^\vee, 1-s) = \gamma(s, \pi; \psi)\ Z(\Phi, f, s). \tag{2.3}$$

The function $L(1-s, \tilde{\pi})^{-1} L(s, \pi)\gamma(s, \pi; \psi)$ is of the form cX^m for some $c \in \mathbb{C}^\times$ and $m \in \mathbb{Z}$. The local ϵ-factor attached to π is the monomial defined by

$$\epsilon(s, \pi; \psi) = \frac{L(s, \pi)}{L(1-s, \tilde{\pi})}\gamma(s, \pi; \psi).$$

Let $Q = LV$ be a standard parabolic subgroup of G of type $(n_1, \dots, n_r)$ with a standard Levi decomposition so that V is the unipotent radical of Q and $L \simeq G_{n_1}(E) \times \cdots \times G_{n_r}(E)$. If $\tau = \tau_1 \otimes \cdots \otimes \tau_r$ is an irreducible representation of L then for any irreducible sub-quotient π of $I_Q^G(\tau)$ we have

$$\gamma(s, \pi; \psi) = \prod_{i=1}^r \gamma(s, \tau_i : \psi). \tag{2.4}$$

We recall further that for any irreducible representation π of G we have

$$L(s,\bar{\pi})=L(s,\pi),\ \ \epsilon(s,\bar{\pi};\bar{\psi})=\epsilon(s,\pi;\psi) \tag{2.5}$$

and

$$\epsilon(s,\pi;\psi)\epsilon(1-s,\tilde{\pi};\psi^{-1})=1. \tag{2.6}$$

2.2. **Generalized coefficients and regularized Zeta integrals.** Let (π,V) be a representation of G. By a generalized matrix coefficient of π we mean a function $f_{v,\alpha}\in C^\infty(G)$ of the form

$$f_{v,\alpha}(g)=\alpha(\pi(g)v)$$

where $\alpha:V\to\mathbb{C}$ is any (not necessarily smooth) linear form on V. Denote by $\mathcal{D}(\pi)$ the space spanned by all generalized matrix coefficients. Of course $\mathcal{C}(\pi)\subseteq\mathcal{D}(\pi)$. We can extend the domain of the Godement-Jacquet integrals $Z(\Phi,f,s)$ by allowing any $f\in\mathcal{D}(\pi)$. This can be done formally in the sense of a regularized integral. Let $e_K=\mathrm{vol}(K)^{-1}\mathbf{1}_K\in\mathcal{H}$ be the idempotent associated to an open compact subgroup K of G. The linear form $\alpha\circ\pi(e_K)$ on V is smooth, i.e. it lies in $\tilde{V}$ and therefore $e_K*f_{v,\alpha}=f_{v,\alpha\circ\pi(e_K)}\in\mathcal{C}(\pi)$. Thus every function $f\in\mathcal{D}(\pi)$ is right uniformly smooth on G and $e_K*f\in\mathcal{C}(\pi)$ for every open compact K. Let K be such that $\Phi(kx)=\Phi(x)$ for all $k\in K$ and $x\in M_n(E)$. Then clearly $e_K*\Phi=\Phi$. Thus, for every $f\in\mathcal{D}(\pi)$, $Z(\Phi,e_K*f,v,s)\in\mathcal{Z}(\pi)$ is a rational function of q^s that is defined as the meromorphic continuation of a Godement-Jacquet integral for $\mathrm{Re}\,s>s_0$. Note that this integral is independent of the choice of the open compact K as long as it is small enough so that Φ is left K-invariant. Indeed, if $K'\subset K$ is an open compact subgroup of K then for $\mathrm{Re}\,s>s_0$ and $k\in K$ the change of variables $g\mapsto kg$ gives

$$\int_G\Phi(g)(e_{K'}*f)(k^{-1}g)\left|\det g\right|^s\ dg=\int_G\Phi(g)(e_{K'}*f)(g)\left|\det g\right|^s\ dg.$$

Averaging over K we get that

$$Z(\Phi,e_K*(e_{K'}*f),s)=Z(\Phi,e_{K'}*f,s).$$

Since $e_K*e_{K'}=e_K$ we get that

$$Z(\Phi,e_K*f,s)=Z(\Phi,e_{K'}*f,s).$$

By meromorphic continuation this holds as an identity of rational functions in q^s. Thus, for $f\in\mathcal{D}(\pi)$ and $\Phi\in C_c^\infty(M_n(E))$ we set

$$Z(\Phi,f,s)=Z(\Phi,e_K*f,s)$$

for any open compact subgroup K of G such that Φ is left K-invariant. Note that if it so happens that for some $s_1\in\mathbb{R}$ and for all $\mathrm{Re}\,s>s_1$ and $\Phi\in C_c^\infty(M_n(F))$ the integral

$$\int_G\Phi(g)f(g)\left|\det g\right|^{s+\frac{n-1}{2}}\ dg \tag{2.7}$$

is absolutely convergent, then it extends to a rational function of q^s that agrees with the regularized zeta integral $Z(\Phi,f,s)$. We would like to be able to write the functional equations (2.3) for generalized matrix coefficients. For that purpose, we need to overcome the technical obstacle that the map $f \mapsto f^\vee$ does not take $\mathcal{D}(\pi)$ to $\mathcal{D}(\tilde{\pi})$. However, it is not the only natural map between $\mathcal{C}(\pi)$ and $\mathcal{C}(\tilde{\pi})$. For every function f with domain G let ${}^t f(g) = f({}^t g)$ and let $f^* = {}^t f^\vee$. Since $\tilde{\pi} \simeq \pi^*$ (cf. [GK75]), $f \mapsto f^*$ is also a map between $\mathcal{C}(\pi)$ and $\mathcal{C}(\tilde{\pi})$, (thus, $f \mapsto {}^t f$ is a map from $\mathcal{C}(\pi)$ to itself). For $\Phi \in C_c^\infty(M_n(E))$ we define the Fourier transform

$$\hat{\Phi} = {}^t\mathcal{F}_\psi(\Phi) = \mathcal{F}_\psi({}^t\Phi).$$

The change of variables $g \mapsto {}^t g$ in the Godement-Jacquet integral shows that $Z(\mathcal{F}_\psi\Phi, f^\vee, s) = Z(\hat{\Phi}, f^*, s)$. Thus, the functional equations (2.2) for $f \in \mathcal{C}(\pi)$ can be rewritten as

$$Z(\Phi, f, s) = \gamma(s, \pi; \psi)\ Z(\hat{\Phi}, f^*, 1 - s). \tag{2.8}$$

Note further that if Φ is left K-invariant then so is $\hat{\Phi}$ and that for $f \in C^\infty(G)$ and an open compact subgroup K of G we have

$$(e_K * f)^* = e_K * f^*.$$

Furthermore, $f \mapsto f^*$ maps $\mathcal{D}(\pi)$ to $\mathcal{D}(\tilde{\pi})$. Thus, the identity (2.8) holds for $f \in \mathcal{D}(\pi)$.

2.3. **Proof of Proposition 2.1.** Let (π, V) be an irreducible H-relatively cuspidal representation of G. Let $\mathcal{C}_H(\pi)$ be the space of generalized matrix coefficients of π associated with the symmetric space $H\backslash G$. That is, $\mathcal{C}_H(\pi)$ is the vector space spanned by all functions of the form $f_{v,\mu}$ where $v \in V$ and μ is an H-invariant linear form on V. Recall that $\mathcal{C}_H(\pi) \subseteq C_c^\infty(H\backslash G)$. Thus π can be realized in $C_c^\infty(H\backslash G)$ and is therefore unitarizable.

Lemma 2.1. *For every $f \in \mathcal{C}_H(\pi)$, $\Phi \in C_c^\infty(M_n(E))$ and $s \in \mathbb{C}$ such that* $\operatorname{Re} s > 0$ *the integral*

$$\int_G \Phi(g) f(g)\, |\det g|_E^{s+\frac{n-1}{2}}\ dg \tag{2.9}$$

is absolutely convergent.

Proof. In order to prove the absolute convergence we may assume that $\Phi \geq 0$ and that $s > 0$ is real. For every $g \in G$ the function $x \mapsto \Phi(xg)$, $x \in M_n(F)$ is in $C_c^\infty(M_n(F))$ and therefore by [GJ72, Proposition 1.1] the integral

$$\int_H \Phi(hg)\, |\det h|_E^{s+\frac{n-1}{2}}\ dh = \int_H \Phi(hg)\, |\det h|_F^{2s+n-1}\ dh$$

converges. Let $f \in \mathcal{C}_H(\pi)$ and let

$$F(g) = |f(g)|\, |\det g|_E^{s+\frac{n-1}{2}} \int_H \Phi(hg)\, |\det h|_E^{s+\frac{n-1}{2}}\ dh.$$

Then $F \in C_c^\infty(H\backslash G)$ and therefore the integral

$$\int_G \Phi(g)\, |f(g)|\, |\det g|_E^{s+\frac{n-1}{2}}\ dg = \int_{H\backslash G} F(g)\ dg$$

converges. $\square$

It follows from Lemma 2.1 and the discussion in Section 2.2 that for $f \in \mathcal{C}_H(\pi)$ and $\Phi \in C_c^\infty(M_n(E))$ the regularized integral $Z(\Phi, f, s)$ is given by the meromorphic continuation of the integral (2.9).

For $\Phi \in C_c^\infty(M_{n\times m}(E))$ we define the Fourier transform

$$\hat{\Phi}(x) = \int_{M_{n\times m}(E)} \Phi(y)\psi(\mathrm{Tr}({}^t yx))\, dy$$

where dy is the self dual Haar measure with respect to ψ, i.e. with respect to the pairing $(x, y) \mapsto \psi(\mathrm{Tr}({}^t yx))$. The Fourier transform satisfies a relative Poisson integral formula.

Lemma 2.2. *For every $\Phi \in C_c^\infty(M_{n\times m}(E))$ we have*

$$\int_{M_{n\times m}(F)} \Phi(x)\, dx = \int_{M_{n\times m}(F)} \hat{\Phi}(x)\, dx.$$

Proof. For any finite dimensional vector space W over F fix a symmetric, non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $W \times W$ identifying W with its dual. Let $\mathcal{F}$ denote the Fourier transform on $C_c^\infty(W)$ defined by

$$\mathcal{F}(\Phi)(w') = \int_W \Phi(w)\psi_F(\langle w, w'\rangle)\, dw$$

where the Haar measure dw is self dual with respect to $\psi_F(\langle\cdot,\cdot\rangle)$. Let $W = W_1 \oplus W_2$ be a direct sum decomposition and let $V_i = W_i^\perp$ be the orthogonal complement of W_i with respect to $\langle\cdot,\cdot\rangle$. Then $W = V_1 \oplus V_2$. The Haar measure dw can be decomposed as $dw = dw_1\, dw_2 = dv_1\, dv_2$ where the Haar measures dw_i on W_i and dv_i on V_i are such that

$$\int_{V_1}\int_{W_2} f(w_2)\,\psi_F(\langle w_2, v_1\rangle)\, dw_2\, dv_1 = f(0)$$

for $f \in C_c^\infty(W_2)$. Thus from this Partial Fourier inversion formula we have

$$\int_{W_1} \Phi(w_1)\, dw_1 = \int_{V_1} \mathcal{F}(\Phi)(v_1)\, dv_1. \tag{2.10}$$

Define the F bi-linear, non-degenerate form

$$\langle x, y\rangle = \frac{1}{2\iota}(\mathrm{Tr}({}^t xy) - \overline{\mathrm{Tr}({}^t xy)}),\ x,\ y \in M_{n\times m}(E).$$

Note that for $\Phi \in C_c^\infty(M_{n\times m}(E))$ we have

$$\hat{\Phi}(x) = \int_{M_{n\times m}(E)} \Phi(y)\,\psi_F(\langle x, y\rangle)\, dy.$$

We may decompose $M_{n\times m}(E) = M_{n\times m}(F) \oplus \iota M_{n\times m}(F)$. Note that with respect to $\langle\cdot,\cdot\rangle$ we have

$$M_{n\times m}(F)^\perp = M_{n\times m}(F) \text{ and } (\iota M_{n\times m}(F))^\perp = \iota M_{n\times m}(F).$$

The lemma therefore follows from (2.10). $\square$

Lemma 2.3. *Let π be an irreducible, H-relatively cuspidal representation of G. For every $\Phi \in C_c^\infty(M_n(E))$ and $f \in \mathcal{C}_H(\pi)$ we have*

$$Z(\Phi, f, \frac{1}{2}) = Z(\hat{\Phi}, f^*, \frac{1}{2}). \tag{2.11}$$

Proof. Since $H = {}^tH$ any H-invariant linear form on π is also an H-invariant linear form on π^*. Thus, $\tilde{\pi}$ is also H-relatively cuspidal and $f^* \in \mathcal{C}_H(\tilde{\pi})$. Lemma 2.1 justifies the convergence in the following computation.

$$\begin{aligned} Z(\Phi, f, \frac{1}{2}) &= \int_G \Phi(g) f(g) \, |\det g|_E^{\frac{n}{2}} \; dg \\ &= \int_{H\backslash G} f(g) \, |\det g|_E^{\frac{n}{2}} \int_H \Phi(hg) \, |\det h|_F^n \; dh \; dg \\ &= \int_{H\backslash G} f(g) \, |\det g|_E^{\frac{n}{2}} \int_{M_n(F)} \Phi(xg) \; dx \; dg. \end{aligned}$$

Note that if $\Psi(x) = \Phi(xg)$, $x \in M_n(E)$ then $\hat{\Psi}(x) = |\det g|_E^{-n} \, \hat{\Phi}(x \; {}^tg^{-1})$. It now follows from Lemma 2.2 applied to Ψ that

$$Z(\Phi, f, \frac{1}{2}) = \int_{H\backslash G} f(g) \, |\det g|_E^{-\frac{n}{2}} \int_{M_n(F)} \hat{\Phi}(x \; {}^tg^{-1}) \; dx \; dg$$

and after the change of variables $g \mapsto {}^tg^{-1}$ we get that

$$Z(\Phi, f, \frac{1}{2}) = \int_{H\backslash G} f^*(g) \, |\det g|_E^{\frac{n}{2}} \int_{M_n(F)} \hat{\Phi}(xg) \; dx \; dg = Z(\hat{\Phi}, f^*, \frac{1}{2}).$$

□

Lemma 2.4. *Let (π, V) be an irreducible and H-distinguished representation of G. Then there exists $f \in \mathcal{C}_H(\pi)$ and $\Phi \in C_c^\infty(G)$ such that $Z(\Phi, f, s) = 1$ for all $s \in \mathbb{C}$.*

Proof. Let μ be a non-zero H-invariant linear form on V and let $v \in V$ be such that $\alpha(v) = 1$. Let K be an open compact subgroup of G such that $\pi(k)v = v$ for all $k \in K$. Set $\Phi = e_K$ and $f = f_{v,\alpha}$. Since $\Phi \in \mathcal{H}$ the integral defining $Z(\Phi, f, s)$ is absolutely convergent for all $s \in \mathbb{C}$ and we have

$$Z(\Phi, f, s) = \mathrm{vol}(K)^{-1} \int_K f(k) \; dk = 1.$$

□

Proposition 2.1 now follows from the functional equation (2.8) together with Lemmas 2.3 and 2.4.

3. A relative variant of a property of Bernstein

We recall from [BZ76] that a topological space X is called an l-space if it is Hausdorff, locally compact and such that every point has a fundamental system of open compact neighborhoods. For an l-space X we denote by $C_c^\infty(X)$ the space of locally constant, complex valued functions on X of compact support and let $\mathcal{D}(X) = C_c^\infty(X)^*$ be the dual space of distributions on X. If Q is an l-group acting on X then the action induces an action of Q on $C_c^\infty(X)$ and on $\mathcal{D}(X)$. We denote by $\mathcal{D}(X)^Q$ the space of Q-invariant distributions on X.

As in Section 2 we set $G = G_n(E)$ and $H = G_n(F)$. We also denote by $P = P_n(F)$ the mirabolic subgroup of H. The main goal of this section is to show the following.

Theorem 3.1. *Let (π, V) be an irreducible, H-distinguished representation of G then*

$$(V^*)^P = (V^*)^H.$$

Remark 2. For π tempered this Theorem follows from [AKT04, Theorem 1.1].

Theorem 3.1 can be viewed as a relative variant of [Ber84, Theorem A]. In his thesis, Ok observed that as in Bernstein's proof one can reduce Theorem 3.1 to a statement about invariant distributions on G (a relative variant of [Ber84, Theorem B]) and that in fact the required statement about invariant distributions can be derived, not quite from [Ber84, Theorem B], but rather from [Ber84, Section 2.2, Statement $X(n)$] that we now recall. Consider $M_n(F)$ as an l-space on which H acts by conjugation. Then Bernstein showed that

$$\mathcal{D}(M_n(F))^P = \mathcal{D}(M_n(F))^H. \tag{3.1}$$

We will prove Theorem 3.1 by several reductions that we now begin.

3.1. Reduction to invariant distributions on G.

Here we consider G as an l-space with $G \times G$ acting on it by

$$x \cdot (g_1, g_2) = g_1^{-1} x g_2,\ x,\ g_1,\ g_2 \in G.$$

Proposition 3.1. *We have*

$$\mathcal{D}(G)^{P\times H} = \mathcal{D}(G)^{H\times H}.$$

We prove that Proposition 3.1 implies Theorem 3.1. Let (π, V) be an irreducible, H-distinguished representation of G and let $\mu \in (V^*)^P$. We need to show that μ is H-invariant. As already observed, the contragredient $(\tilde{\pi}, \tilde{V})$ of π is then also H-distinguished. Fix a non zero linear form $\lambda \in (\tilde{V}^*)^H$. According to the Lemma in [Ber84, Section 5.1] there is a surjective morphism of $G \times G$-modules

$$A_\pi : C_c^\infty(G) \to V \otimes \tilde{V}.$$

Thus, the dual map is an injective morphism of $G \times G$-modules

$$A_\pi^* : V^* \otimes \tilde{V}^* \hookrightarrow \mathcal{D}(G).$$

It follows that $A_\pi^*(\mu\otimes\lambda)\in\mathcal{D}(G)^{P\times H}$ and applying Proposition 3.1 we get that $A_\pi^*(\mu\otimes\lambda)\in\mathcal{D}(G)^{H\times H}$. Since A_π^* is an injective morphism of $G\times G$-modules it follows that $\mu\otimes\lambda\in(V^*\otimes\tilde{V}^*)^{H\times H}$ and in particular that $\mu\in(V^*)^H$. This shows that Proposition 3.1 implies Theorem 3.1.

In order to prove Proposition 3.1 we reduce it to a statement about distributions on the symmetric space $H\backslash G$. This reduction requires a property of invariant distribution on l-groups that we prove first.

3.2. **A lemma on invariant distributions on an l-group.** Let Q be an l-group and let R be a closed subgroup of Q. Assume for convenience that both Q and R are unimodular (this suffices for our needs). Consider Q as a $Q\times Q$-space with the action given by

$$(q_1,q_2)\cdot q\mapsto q_1^{-1}qq_2,\ q_1,\ q_2,\ q\in Q.$$

For $f\in C_c^\infty(Q)$ we denote by $\Phi_f\in C_c^\infty(R\backslash Q)$ the function given by

$$\Phi_f(q)=\int_R f(rq)\ dr.$$

The map $f\mapsto\Phi_f$ from $C_c^\infty(Q)$ to $C_c^\infty(R\backslash Q)$ is surjective and equivariant (in the obvious sense) with respect to the action of $\{e\}\times Q$ on $C_c^\infty(Q)$ and the natural action of Q on $C_c^\infty(R\backslash Q)$. Here e denotes the identity element in Q. The dual of the surjective morphism $f\mapsto\Phi_f$ is an embedding $D\mapsto D'$ of $\mathcal{D}(R\backslash Q)$ into $\mathcal{D}(Q)$ given by $D'(f)=D(\Phi_f)$ that satisfies similar equivariance properties. For every $f\in C_c^\infty(Q)$ and $q\in Q$ let $f^q\in C_c^\infty(Q)$ be given by

$$f^q(x)=f(q^{-1}x),\ x\in Q.$$

The function f^q is the result of letting $(q,e)\in Q\times\{e\}$ act on f. Since

$$\Phi_f=\Phi_{f^r},\ r\in R$$

we see that $D'(f)=D'(f^r)$ for every $r\in R$. Thus

$$D\mapsto D':\mathcal{D}(R\backslash Q)\hookrightarrow\mathcal{D}(Q)^{R\times\{e\}} \tag{3.2}$$

is an embedding that is equivariant with respect to the action of Q on $\mathcal{D}(R\backslash Q)$ and $\{e\}\times Q$ on $\mathcal{D}(Q)^{R\times\{e\}}$.

Lemma 3.1. *The embedding* (3.2) *is in fact an isomorphism. Thus, for any subgroup Q' of Q we have*

$$\mathcal{D}(R\backslash Q)^{Q'}\simeq\mathcal{D}(Q)^{R\times Q'}. \tag{3.3}$$

Proof. We only need to show that the embedding (3.2) is also surjective. The proof is, in fact, given in [Bou63, chap. VII, Section 2, Proposition 4] and we present it here for the convenience of the reader. Note that $(f,g)\mapsto f\Phi_g$ is a bilinear map from $C_c^\infty(Q)\times C_c^\infty(Q)$ to $C_c^\infty(Q)$. We will show that for any $D'\in\mathcal{D}(Q)^{R\times\{e\}}$ we have

$$D'(f\Phi_g)=D'(\Phi_f g). \tag{3.4}$$

But first let us show that (3.4) implies the required surjectivity. We need to show that for any $D'\in\mathcal{D}(Q)^{R\times\{e\}}$ and f such that $\Phi_f=0$ we also have $D'(f)=0$. When this is the case then the distribution $D \in \mathcal{D}(R\backslash Q)$ where $D(\Phi) = D'(g)$ for any g such that $\Phi_g = \Phi$ is a well defined pre-image of D'. But if $\Phi_f = 0$ then from (3.4) we get that $D'(f\Phi_g) = 0$ for any $g \in C_c^\infty(Q)$ and in particular, for $g = \mathrm{vol}(R \cap \mathrm{supp}(f))^{-1}\mathbf{1}_{\mathrm{supp}(f)}$. But for such g we also have $f\Phi_g = f$ and therefore $D'(f) = 0$. It is now only left to prove the identity (3.4). Let K be an open compact subgroup of Q such that both f and g are $\{e\} \times K$-invariant. Then both f and g can be written as finite linear combinations of functions of the form $\mathbf{1}_{xK}$, $x \in Q$. It is therefore enough to prove (3.4) when $f = \mathbf{1}_{xK}$ and $g = \mathbf{1}_{yK}$ for some x, $y \in K$. Note then that

$$\Phi_f = \mathrm{vol}(R \cap xKx^{-1})\mathbf{1}_{RxK}$$

and therefore that

$$\Phi_f g = \mathrm{vol}(R \cap xKx^{-1})\mathbf{1}_{RxK \cap yK} \text{ and } \Phi_g f = \mathrm{vol}(R \cap yKy^{-1})\mathbf{1}_{RyK \cap xK}.$$

Thus if $RxK \neq RyK$ then $\Phi_f g = \Phi_g f = 0$. Assume now that $RxK = RyK$ then there exists $r \in R$ such that $g = \mathbf{1}_{rxK}$. We then have

$$\Phi_f g = (\Phi_g f)^r.$$

Since D' is $R \times \{e\}$-invariant the identity (3.4) holds. □

3.3. **Reduction to invariant distributions on the symmetric space $H\backslash G$.** Recall that

$$X = \{g \in G : g\bar{g} = I_n\}$$

is an l-space with a G-action given by the twisted conjugation $x \cdot g = \bar{g}^{-1}xg$. Note that the restricted action of H on X is by conjugation. The map $Hg \mapsto I_n \cdot g$, $g \in G$ defines an isomorphism $H\backslash G \simeq X$ as G-spaces and therefore we have

$$\mathcal{D}(H\backslash G)^Q \simeq \mathcal{D}(X)^Q \tag{3.5}$$

for any subgroup Q of G. Our next reduction is the following.

Proposition 3.2. *We have*

$$\mathcal{D}(X)^P = \mathcal{D}(X)^H.$$

We now show that Proposition 3.2 implies Proposition 3.1. From Proposition 3.2 and (3.5) we get that

$$\mathcal{D}(H\backslash G)^P = \mathcal{D}(H\backslash G)^H. \tag{3.6}$$

Now apply Lemma 3.1 with $Q = G$ and $R = H$ twice. Once with $Q' = P$ and once with $Q' = H$. We get that the map (3.2) induces the isomorphisms

$$\mathcal{D}(H\backslash G)^P \simeq \mathcal{D}(G)^{H \times P} \tag{3.7}$$

and

$$\mathcal{D}(H\backslash G)^H \simeq \mathcal{D}(G)^{H \times H}. \tag{3.8}$$

It therefore follows from (3.7), (3.8) and (3.6) that

$$\mathcal{D}(G)^{H \times P} = \mathcal{D}(G)^{H \times H},$$

i.e. that Proposition 3.1 holds.

3.4. **Proof of Proposition 3.2.** We begin with the following observations.

Lemma 3.2. *Let Y be an l-space on which H acts.*

(1) *Let Z be a closed H-subspace of Y.*

$$\text{If } \mathcal{D}(Y)^P = \mathcal{D}(Y)^H \text{ then } \mathcal{D}(Z)^P = \mathcal{D}(Z)^H.$$

(2) *Let $\mathcal{U} = \{U\}$ be a collection of open H-subspaces of Y such that $Y = \cup_{U\in\mathcal{U}} U$.*

$$\text{If } \mathcal{D}(U)^P = \mathcal{D}(U)^H,\ U \in \mathcal{U} \text{ then } \mathcal{D}(Y)^P = \mathcal{D}(Y)^H.$$

Proof. For the first part it is enough to recall that the restriction map $f \mapsto f_{|Z}$ from $C_c^\infty(Y)$ to $C_c^\infty(Z)$ is surjective and H-equivariant and therefore its dual map is an H-equivariant embedding of $\mathcal{D}(Z)$ into $\mathcal{D}(Y)$. Thus

$$\mathcal{D}(Z)^H = \mathcal{D}(Z) \cap \mathcal{D}(Y)^H = \mathcal{D}(Z) \cap \mathcal{D}(Y)^P = \mathcal{D}(Z)^P.$$

For the second part let $f \in C_c^\infty(Y)$ and $D \in \mathcal{D}(Y)^P$. Denote by f^h the result of the action of an element $h \in H$ on f. We need to show that

$$D(f^h) = D(f),\ h \in H. \tag{3.9}$$

Since f has compact support, there are $U_1, \dots, U_r \in \mathcal{U}$ such that $\mathrm{supp}(f) \subseteq \cup_{i=1}^r U_i$. For every $y \in \mathrm{supp}(f)$ let $1 \le i_y \le r$ and let O_y be an open compact neighborhood of y that is contained in U_{i_y} and on which f is constant. Let $y_1, \dots, y_t$ be such that $\mathrm{supp}(f) = \cup_{j=1}^t O_{y_j}$. Then $C_i = O_{y_i} \setminus (\cup_{j=1}^{i-1} O_{y_j})$ is open and compact and $\mathrm{supp}(f)$ is the disjoint union of the open compact subsets C_i, $i = 1, \dots, t$. We can therefore express f as a sum $f = \sum_{i=1}^r f_i$ where $\mathrm{supp}(f_i) \subseteq U_i$. It is therefore enough to prove that for every $U \in \mathcal{U}$ we have (3.9) for $f \in C_c^\infty(U)$. But this is satisfied since

$$D_{|C_c^\infty(U)} \in \mathcal{D}(U)^P = \mathcal{D}(U)^H.$$

□

Let

$$\mathfrak{X} = \{x \in M_n(E) : x + \bar{x} = 0\} = \iota M_n(F).$$

Thus $x \mapsto \iota x$ is an isomorphism $M_n(F) \simeq \mathfrak{X}$ of H-spaces, where H acts on both of them by conjugation. It therefore follows from (3.1) that

$$\mathcal{D}(\mathfrak{X})^P = \mathcal{D}(\mathfrak{X})^H. \tag{3.10}$$

Let $\Xi : \mathfrak{X} \to F$ be the continuous map defined by

$$\Xi(x) = \det(I_n + x)\det(I_n - x) = \det(I_n + x)\overline{\det(I_n + x)}.$$

For $m \in \mathbb{Z}$ let $F(m)$ denote the open compact subset of elements of F of valuation m. Let

$$\mathfrak{X}_0 = \Xi^{-1}(F^\times) \text{ and } \mathfrak{X}(m) = \Xi^{-1}(F(m)),\ m \in \mathbb{Z}.$$

Thus $\mathfrak{X}_0$ is an open H-subspace of $\mathfrak{X}$ and $\{\mathfrak{X}(m)\}_{m\in\mathbb{Z}}$ is a covering of $\mathfrak{X}_0$ consisting of open and closed H-subspaces of $\mathfrak{X}$. It follows from part (1) of Lemma 3.2 that

$$\mathcal{D}(\mathfrak{X}(m))^P = \mathcal{D}(\mathfrak{X}(m))^H$$

for all m and therefore from part (2) of Lemma 3.2 that

$$\mathcal{D}(\mathfrak{X}_0)^P = \mathcal{D}(\mathfrak{X}_0)^H. \tag{3.11}$$

Next, we define a covering of X by open H-subspaces as follows. Note that if $\lambda \in E^\times$ is an eigenvalue of an element of $x \in X$ then $\lambda\bar{\lambda} = 1$, i.e. λ is in the subgroup E^1 of $E^\times$ consisting of elements of norm 1. For every $\lambda \in E^1$ let

$$X_\lambda = \{x \in X : \det(x - \lambda I_n) \neq 0\}.$$

Thus X_λ is an open H-subspace of X and the collection $\{X_\lambda : \lambda \in E^1\}$ is an open covering of X. Next we define a Cayley transform that provides an isomorphism between the H-spaces $\mathfrak{X}_0$ and X_λ. For $x \in X_\lambda$ let

$$\xi_\lambda(x) = (x + \lambda I_n)(x - \lambda I_n)^{-1}$$

and for $x \in \mathfrak{X}_0$ let

$$\eta_\lambda(x) = -\lambda(I_n + x)(I_n - x)^{-1}.$$

Let $\lambda \in E^1$ and $x \in X_\lambda$. Recall that $\bar{x} = x^{-1}$ and $\bar{\lambda} = \lambda^{-1}$ thus

$$\begin{aligned}
&\xi_\lambda(x) + \overline{\xi_\lambda(x)} \\
&= (x + \lambda I_n)(x - \lambda I_n)^{-1} + (x^{-1} + \lambda^{-1} I_n)(x^{-1} - \lambda^{-1} I_n)^{-1} \\
&= (x - \lambda I_n)^{-1}(x^{-1} - \lambda^{-1} I_n)^{-1} \times \\
&\quad \big[(x + \lambda I_n)(x^{-1} - \lambda^{-1} I_n) + (x^{-1} + \lambda^{-1} I_n)(x - \lambda I_n)\big] = 0
\end{aligned}$$

since the term in the brackets on the right hand side vanishes. Thus, $\xi_\lambda(x) \in \mathfrak{X}$. Furthermore

$$I_n + \xi_\lambda(x) = [(x - \lambda I_n) + (x + \lambda I_n)](x - \lambda I_n)^{-1} = 2x(x - \lambda I_n)^{-1} \in G$$

and

$$I_n - \xi_\lambda(x) = [(x - \lambda I_n) - (x + \lambda I_n)](x - \lambda I_n)^{-1} = -2\lambda(x - \lambda I_n)^{-1} \in G$$

and therefore $x \in \mathfrak{X}_0$. If $y \in \mathfrak{X}_0$ then recall that $\bar{y} = -y$. We then have

$$\eta_\lambda(y)\overline{\eta_\lambda(y)} = \lambda(I_n + y)(I_n - y)^{-1}\bar{\lambda}(I_n - y)(I_n + y)^{-1} = I_n$$

and therefore $\eta_\lambda(y) \in X$. Furthermore

$$\begin{aligned}
\eta_\lambda(y) - \lambda I_n &= -\lambda[(I_n + y)(I_n - y)^{-1} + I_n] \\
&= -\lambda[(I_n + y) + (I_n - y)](I_n - y)^{-1} = -2\lambda(I_n - y)^{-1} \in G
\end{aligned}$$

and therefore $\eta_\lambda(y) \in X_\lambda$. It is now easy to verify that $\eta_\lambda \circ \xi_\lambda = \mathbf{1}_{X_\lambda}$, that $\xi_\lambda \circ \eta_\lambda = \mathbf{1}_{\mathfrak{X}_0}$, and that for $x \in X_\lambda$, $y \in \mathfrak{X}_0$ and $h \in H$ we have

$$\xi_\lambda(h^{-1}xh) = h^{-1}\xi_\lambda(x)h \text{ and } \eta_\lambda(h^{-1}yh) = h^{-1}\eta_\lambda(y)h.$$

It follows that $\xi_\lambda : X_\lambda \to \mathfrak{X}_0$ and $\eta_\lambda : \mathfrak{X}_0 \to X_\lambda$ are isomorphisms of H-spaces and they induce an isomorphism of H-modules

$$\mathcal{D}(X_\lambda) \simeq \mathcal{D}(\mathfrak{X}_0).$$

Applying (3.11) we therefore get that

$$\mathcal{D}(X_\lambda)^P = \mathcal{D}(X_\lambda)^H,\ \lambda \in E^1$$

and applying part (2) of Lemma 3.2 we finally obtain

$$\mathcal{D}(X)^P = \mathcal{D}(X)^H.$$

This completes the proof of Proposition 3.2 and therefore also of Proposition 3.1 and of Theorem 3.1.

4. An integral formula for the $G_r(F)$-period

Let π be an irreducible, unitarizable and non-degenerate representation of $G_r(E)$. Applying results of Flicker from [Fli88], we construct in this section a $P_r(F)$-invariant linear form on the Whittaker model $\mathcal{W}(\pi, \psi)$ of π and provide a criterion for π to be $G_r(F)$-distinguished.

For $W \in \mathcal{W}(\pi, \psi)$ let

$$\mu_\pi(W) = \int_{U_r(F)\backslash P_r(F)} W(p)\, dp = \int_{U_{r-1}(F)\backslash G_{r-1}(F)} W\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh. \tag{4.1}$$

In the Lemma in [Fli88, Section 4] Flicker proved that the integral defining $\mu_\pi(W)$ is absolutely convergent. Thus, μ_π is a $P_r(F)$-invariant linear form on $\mathcal{W}(\pi, \psi)$. It can also be read off the Proposition in p. 309 of [loc., cit.] that $\mu_\pi \neq 0$. For the convenience of the reader we include a proof.

Lemma 4.1. *Let π be an irreducible, unitarizable and non-degenerate representation of $G_r(F)$. Then μ_π is not identically zero.*

Proof. Let $\mathcal{K}(\pi, \psi) = \{W_{|P_r(E)} : W \in \mathcal{W}(\pi, \psi)\}$ and let

$$\mathcal{K}_0(\psi) = \mathrm{ind}_{U_r(E)}^{P_r(E)}(\psi_r)$$

where ind denotes smooth induction with compact support. It is proved in [GK75] that $\mathcal{K}_0(\psi) \subseteq \mathcal{K}(\pi, \psi)$. It is therefore enough to show that there exists $\phi \in \mathcal{K}_0(\psi)$ such that

$$\int_{U_r(F)\backslash P_r(F)} \phi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} dh \neq 0. \tag{4.2}$$

Let $K_r(m) = I_r + M_r(\mathfrak{p}_E^m)$ be the congruence subgroup of $G_r(E)$ associated to m. Let m be large enough so that ψ is trivial on $\mathfrak{p}_E^m$. The function

$$\phi(p) = \begin{cases} \psi_r(u) & p = uk,\ u \in U_r(E),\ k \in P_r(E) \cap K \\ 0 & p \notin U_r(E)(P_r(E) \cap K) \end{cases}$$

is then well defined and belongs to $\mathcal{K}_0(\psi)$. To complete the lemma we show that $\phi_{|P_r(F)} = \mathbf{1}_{U_r(F)(P_r(F)\cap K_r(m))}$, i.e. that

$$[U_r(E)(P_r(E) \cap K_r(m))] \cap P_r(F) = U_r(F)(P_r(F) \cap K_r(m)) \tag{4.3}$$

and therefore that (4.2) holds. The right hand side of (4.3) is clearly contained in the left hand side. To prove the other containment note first that it is enough to show that

$$U_t(E)K_t(m) \cap G_t(F) = U_t(F)(K_t(m) \cap G_t(F)). \tag{4.4}$$

Indeed if $u = \begin{pmatrix} u_{r-1} & y \\ 0 & 1 \end{pmatrix} \in U_r(E)$ and $k = \begin{pmatrix} k_{r-1} & z \\ 0 & 1 \end{pmatrix} \in P_r(E) \cap K_r(m)$ are such that

$$uk = \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} \in P_r(F)$$

for some $g = u_{r-1}k_{r-1} \in G_{r-1}(F)$ and $x \in F^{r-1}$ then for

$$u' = \begin{pmatrix} I_{r-1} & x \\ 0 & 1 \end{pmatrix} \in U_r(F)$$

we have $uk = u' \operatorname{diag}(g, 1)$ and from (4.4) for $t = r-1$ we get that $g = u'_{r-1}k'_{r-1}$ for some $u' \in U_{r-1}(F)$ and $k' \in K_{r-1}(m) \cap G_{r-1}(F)$. It follows that

$$uk = (u' \operatorname{diag}(u'_{r-1}, 1)) \operatorname{diag}(k'_{r-1}, 1) \in U_r(F)(P_r(F) \cap K_r(m)).$$

We now show (4.4) by induction. Assume that $u \in U_t(E)$ and $k \in K_t(m)$ are such that $uk \in G_t(F)$ and assume by induction that

$$uk \in U_t(F) \begin{pmatrix} u_1 & * & * \\ 0 & 1 & x \\ 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} * & * & * \\ * & a & b \\ * & c & d \end{pmatrix}$$

for some $u_1 \in U_{r-s-1}(E)$, $x \in E^s, a \in E$, $d \in G_s(E)$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{s+1}(m)$. We put $*$ for blocks that are irrelevant for our argument. Multiplying u and k in the above block form the $(2,3)$-block gives that $b + xd \in F^s$ while b has entries in $\mathfrak{p}_E^m$ and $d \in K_s(m)$. Write $x = x_1 + \iota x_2$ where x_1, $x_2 \in F^s$. It follows that x_2 has entries in $\mathfrak{p}_E^m \cap F$. Thus we can decompose

$$\begin{pmatrix} u_1 & * & * \\ 0 & 1 & x \\ 0 & 0 & I_s \end{pmatrix} = \begin{pmatrix} I_{t-s-1} & 0 & 0 \\ 0 & 1 & x_1 \\ 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} u_1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} I_{t-s-1} & 0 & 0 \\ 0 & 1 & \iota x_2 \\ 0 & 0 & I_s \end{pmatrix}$$

and

$$\begin{pmatrix} I_{r-s-1} & 0 & 0 \\ 0 & 1 & \iota x_2 \\ 0 & 0 & I_s \end{pmatrix} \in K_r(m).$$

It follows that

$$uk \in U_t(F) \begin{pmatrix} u_1 & * \\ 0 & I_{s+1} \end{pmatrix} k_1$$

for some $k_1 \in K_r(m)$. The identity (4.4) and therefore the lemma now follows by induction on s. $\square$

Corollary 4.1. *Let π be an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation of $G_r(E)$. Then μ_π is a non-zero $G_r(F)$-invariant linear form on $\mathcal{W}(\pi, \psi)$.*

Proof. This is immediate from Lemma 4.1 and Theorem 3.1. $\square$

In [Fli91, Proposition 11] Flicker proved the multiplicity one of $G_r(F)$-periods, i.e. that for any irreducible representation (π, V) of $G_r(E)$ we have

$$\dim_{\mathbb{C}}((V^*)^{G_r(F)}) \leq 1. \tag{4.5}$$

Let π be an irreducible, unitarizable and non-degenerate representation of $G_r(E)$. We define on $\mathcal{W}(\pi, \psi)$ the linear form $\tilde{\mu}_\pi$ by

$$\tilde{\mu}_\pi(W) = \mu_{\tilde{\pi}}(\tilde{W}),\ W \in \mathcal{W}(\pi, \psi).$$

If π is $G_r(F)$-distinguished then so is $\tilde{\pi}$. In this case, Corollary 4.1 asserts that $\mu_{\tilde{\pi}}$ is a non-zero $G_r(F)$-invariant linear form on $\mathcal{W}(\tilde{\pi}, \psi^{-1})$. Applying (1.2) and the $G_r(F)$-invariance of $\mu_{\tilde{\pi}}$ we get that for every $h \in G_r(F)$ and every $W \in W(\pi, \psi)$ we have

$$\tilde{\mu}_\pi(R(h)W) = \mu_{\tilde{\pi}}(R({}^t h^{-1})\tilde{W}) = \mu_{\tilde{\pi}}(\tilde{W}) = \tilde{\mu}_\pi(W).$$

Thus, μ_π and $\tilde{\mu}_\pi$ are both non-zero $G_r(F)$-invariant linear forms on π. It follows from (4.5) that for an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation π of $G_r(E)$ there exists a scalar $c(\pi) \in \mathbb{C}^\times$ such that

$$\tilde{\mu}_\pi = c(\pi)\, \mu_\pi. \tag{4.6}$$

Remark 3. As suggested by the notation, the scalar $c(\pi)$ is independent of the character ψ. Indeed, any other non-trivial character of E with trivial restriction to F is of the form $\psi_a(x) = \psi(ax)$, $x \in E$ for some $a \in F^\times$. We then have $\mathcal{W}(\pi, \psi_a) = \{W^a : W \in \mathcal{W}(\pi, \psi)\}$ where $W^a(g) = W(d_a\, g)$, $g \in G_r(E)$ and $d_a = \mathrm{diag}(a^{r-1}, \dots, a, 1) \in P_r(F)$. It follows from the definition of μ_π in (4.1) that

$$\mu_\pi(W^a) = \delta(d_a)\mu_\pi(W) \tag{4.7}$$

where $\delta(d_a)$ is the Jacobian of the homeomorphism of $U_r(F)\backslash P_r(F)$ given by $p \mapsto d_a^{-1}\, p\, d_a$. Note further that $\widetilde{W^a}(g) = W(d_a\, w_r\, {}^t g^{-1})$ and that $w_r\, d_a w_r = a^{r-1} d_a^{-1}$. Since π is distinguished $\omega_{\pi|F^\times} = \mathbf{1}_{F^\times}$ and therefore $\widetilde{W^a}(g) = W(w_r\, d_a^{-1}\, {}^t g^{-1})$, i.e. $\widetilde{W^a} = (\tilde{W})^a$. Applying (4.7) to $\tilde{\pi}$ we therefore also have

$$\mu_{\tilde{\pi}}(\widetilde{W^a}) = \delta(d_a)\mu_{\tilde{\pi}}(\tilde{W}). \tag{4.8}$$

The identities (4.7) and (4.8) show that $c(\pi)$ is independent of ψ.

The following proposition provides a criterion for distinction. For π tempered it was proved in [AKT04, Theorem 1.3].

Proposition 4.1. *Let π be an irreducible, unitarizable and non-degenerate representation of $G_r(E)$ such that $\omega_{\pi|F^\times} = \mathbf{1}_{F^\times}$. Then π is $G_r(F)$-distinguished if and only if there exists a scalar $c \in \mathbb{C}^\times$ such that $\tilde{\mu}_\pi = c\mu_\pi$.*

Proof. If π is $G_r(F)$-distinguished the definition of $c(\pi)$ asserts the existence of the required scalar c. Assume that there exists a scalar $c \in \mathbb{C}^\times$ such that $\tilde{\mu}_\pi = c\mu_\pi$. Then μ_π is invariant under $P_r(F)$, $Z_r(F)$ and ${}^tP_r(F)$. Since these 3 subgroups generate $G_r(F)$ we get that μ_π is $G_r(F)$-invariant. By Lemma 4.1 $\mu_\pi \neq 0$ and therefore π is $G_r(F)$-distinguished. $\square$

Let π be an irreducible, unitarizable and non-degenerate representation of $G_r(E)$. For $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in C_c^\infty(F^r)$ Flicker introduced the Asai integrals

$$Z(s, W; \Phi) = \int_{U_r(F)\backslash G_r(F)} W(h)\Phi(\eta_r h) \left|\det h\right|_F^s \, dh. \tag{4.9}$$

It is proved in [Fli88, Section 4, Proposition (i)] that the integral (4.9) is absolutely convergent whenever $\mathrm{Re}\, s \geq 1$. Flicker studied the Asai integrals $Z(s, W; \Phi)$ in [Fli88] and in [Fli93] in order to analyze the local (and global) Asai L and ϵ-factors. They will not play a role in this work except for an identity, observed by Ok, satisfied by the Asai integrals at $s = 1$.

Lemma 4.2. *Let π be an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation of $G_r(E)$. For every $\Phi \in C_c^\infty(E^r)$ and $W \in \mathcal{W}(\pi, \psi)$ we have*

$$Z(1, \tilde{W}, \hat{\Phi}_{|F^r}) = c(\pi)\, Z(1, W, \Phi_{|F^r}).$$

Remark 4. In the main Theorem of the Appendix of [Fli93] Flicker developed an analogue, for the Asai integrals, of the Godement-Jacquet theory (see also [AKT04, Theorem 2.1]). We emphasize that Lemma 4.2 is not a formal consequence of the functional equations obtained by Flicker. It is an extra symmetry satisfied by the Asai integrals at $s = 1$.

Proof. Note that the modulus function of $P_r(F)$ is given by $\delta_{P_r(F)}(p) = \left|\det p\right|_F$. Thus for any function f on $U_r(F)\backslash G_r(F)$ we may decompose the measure as

$$\int_{U_r(F)\backslash G_r(F)} f(h)\, dh = \int_{P_r(F)\backslash G_r(F)} \int_{U_r(F)\backslash P_r(F)} \left|\det p\right|_F^{-1} f(ph)\, dp\, dh.$$

Since the Asai integral is absolutely convergent for $s = 1$ we have

$$Z(1, W, \Phi_{|F^r}) = \int_{P_r(F)\backslash G_r(F)} \Phi(\eta_r h) \left|\det h\right|_F \int_{U_r(F)\backslash P_r(F)} W(ph)\, dp\, dh.$$

It follows from Corollary 4.1 that

$$\int_{U_r(F)\backslash P_r(F)} W(ph)\, dp = \mu_\pi(W)$$

is independent of h and therefore

$$Z(1, W, \Phi_{|F^r}) = \mu_\pi(W) \int_{P_r(F)\backslash G_r(F)} \Phi(\eta_r h) \left|\det h\right|_F \, dh.$$

Note further that $h \mapsto \eta_r h$ identifies $P_r(F)\backslash G_r(F)$ with $F^r \setminus \{0\}$ and that $\left|\det h\right|_F dh$ transforms to a Haar measure dx on the vector space F^r. We further assume that the

$G_r(F)$-invariant measure dh on $P_r(F)\backslash G_r(F)$ is so normalized that dx is self dual with respect to ψ. We get that

$$Z(1, W, \Phi_{|F^r}) = \mu_\pi(W) \int_{F^r} \Phi(x)\ dx.$$

The same line of argument applied to $\tilde{\pi}$ gives that

$$Z(1, \tilde{W}, \hat{\Phi}_{|F^r}) = \mu_{\tilde{\pi}}(\tilde{W}) \int_{F^r} \hat{\Phi}(x)\ dx.$$

Recall that by definition $\mu_{\tilde{\pi}}(\tilde{W}) = c(\pi)\mu_\pi(W)$. The lemma now follows from Lemma 2.2. $\square$

5. Relatively cuspidal representations are non-degenerate

As is the case with cuspidal representations, it turns out that $G_r(F)$-relatively cuspidal representations of $G_r(E)$ are non-degenerate. To show this we apply the Gelfand-Kazhdan theory and study the restriction to $P_r(E)$ of relatively cuspidal representations.

Let $\theta_r = \psi_{r|V_r}$. We recall from [BZ77, Section 3.2] the definition of the functor Φ^- from representations of $P_r(E)$ to representations of $P_{r-1}(E)$. Let (τ, W) be a representation of $P_r(E)$ and let $W(V_r, \theta_r)$ be the subspace of W spanned by vectors of the form

$$\theta_r(m)v - \tau(m)v,\ m \in V_r,\ v \in W.$$

Thus $\Phi^-(\tau)$ is the representation of $P_{r-1}(E)$, that we also denote by (τ_1, W_1), defined by $W_1 = W/W(V_r, \theta_r)$ and

$$\tau_1(p)\bar{v} = |\det p|_E^{-\frac{1}{2}}\ \overline{\tau(\mathrm{diag}(p, 1))v},\ p \in P_{r-1}(E),\ v \in W$$

where $\bar{v} = v + W(V_r, \theta_r)$ denotes the projection of v to W_1. If a linear form $\mu \in (W^*)^{P_r(F)}$ is not zero and the associated generalized matrix coefficients

$$f_{v,\mu}(p) = \mu(\tau(p)v),\ v \in W,\ p \in P_r(E)$$

all lie in $C_c^\infty(P_r(F)\backslash P_r(E))$ then we say that μ is a relatively cuspidal linear form of τ or that the triple (τ, W, μ) is $P_r(F)$-relatively cuspidal. Given a relatively cuspidal linear form μ of τ we construct a linear form $\mu_1 \in (W_1^*)^{P_{r-1}(F)}$ as follows. We set

$$\mu_1(\bar{v}) = \int_{V_r(F)\backslash V_r(E)} \mu(\tau(m)v)\theta_r^{-1}(m)\ dm.$$

Since μ is relatively cuspidal the integrand is of compact support and it is then easy to see that μ_1 is a well defined linear form on W_1. To see that it is $P_{r-1}(F)$ invariant we note that for $p \in P_{r-1}(F)$ we have

$$\mu_1(\tau_1(p)\bar{v}) = \int_{V_r(F)\backslash V_r(E)} |\det p|_E^{-\frac{1}{2}}\ \mu\left(\tau\left[m\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\right]v\right)\theta_r^{-1}(m)\ dm.$$

Note that $m \mapsto \mathrm{diag}(p,1)m\,\mathrm{diag}(p^{-1},1)$ is a homeomorphism of $V_r(F)\backslash V_r(E)$ with Jacobian equal to $|\det p|_F = |\det p|_E^{\frac{1}{2}}$ and that

$$\theta_r(\mathrm{diag}(p,1)m\,\mathrm{diag}(p^{-1},1)) = \theta_r(m),\ m \in V_r(E).$$

Thus, the change of variables

$$m \mapsto \mathrm{diag}(p,1)m\,\mathrm{diag}(p^{-1},1)$$

and the fact that μ is $P_r(F)$-invariant imply that

$$\mu_1(\tau_1(p)\bar{v}) = \mu_1(\bar{v}).$$

Thus for any relatively cuspidal triple (τ, W, μ) where (τ, W) is a representation of $P_r(E)$ we assigned a triple (τ_1, W_1, μ_1) where (τ_1, W_1) is a representation of $P_{r-1}(E)$ and $\mu_1 \in (W_1^*)^{P_{r-1}(F)}$.

Proposition 5.1. *For $r > 1$ and a $P_r(F)$-relatively cuspidal triple (τ, W, μ) the triple (τ_1, W_1, μ_1) is $P_{r-1}(F)$-relatively cuspidal.*

Proof. We only need to show that μ_1 is not zero (the compact support condition is straightforward from that for μ). Assume by contradiction that $\mu_1 = 0$. Note that for $x \in F^{r-1}$ we have

$$\theta_r\left[\begin{pmatrix} I_{r-1} & \iota\, x \\ 0 & 1 \end{pmatrix}\right] = \psi_F(\eta_{r-1}x).$$

Let dx be the Haar measure on F^{r-1} such that for any function f on $V_r(F)\backslash V_r(E)$ we have the integration formula

$$\int_{V_r(F)\backslash V_r(E)} f(m)\ dm = \int_{F^{r-1}} f(\begin{pmatrix} I_{r-1} & \iota\, x \\ 0 & 1 \end{pmatrix})\ dx.$$

Let $v \in W$ and let $f = f_{v,\mu}$. Our assumption implies that

$$\int_{F^{r-1}} f\left[\begin{pmatrix} I_{r-1} & \iota\, x \\ 0 & 1 \end{pmatrix} p\right] \psi_F(\eta_{r-1}x)\ dx = 0,\ p \in P_r(E)$$

and since f is left $P_r(F)$-invariant and

$$\begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I_{r-1} & \iota\, x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{r-1} & \iota\, h^{-1}x \\ 0 & 1 \end{pmatrix}$$

for $h \in G_{r-1}(F)$, $x \in F^{r-1}$ that

$$\int_{F^{r-1}} f\left[\begin{pmatrix} I_{r-1} & \iota\, h^{-1}x \\ 0 & 1 \end{pmatrix} p\right] \psi_F(\eta_{r-1}x)\ dx = 0,\ p \in P_r(E),\ h \in G_{r-1}(F).$$

The change of variables $x \mapsto hx$ now shows that

$$\int_{F^{r-1}} f\left[\begin{pmatrix} I_{r-1} & \iota\, x \\ 0 & 1 \end{pmatrix} p\right] \psi_F(\eta_{r-1}hx)\ dx = 0,\ p \in P_r(E),\ h \in G_{r-1}(F).$$

Since $\{{}^t(\eta_{r-1}h) : h \in G_{r-1}(F)\} = F^{r-1} \setminus \{0\}$ it follows that the function

$$\phi(x) = f\left[\begin{pmatrix} I_{r-1} & \iota\, x \\ 0 & 1 \end{pmatrix}\right] \in C_c^\infty(F^{r-1})$$

is such that

$$\int_{F^{r-1}} \phi(x)\psi_F({}^t yx)\, dx = 0,\ y \in F^{r-1} \setminus \{0\}.$$

Since the Fourier transform of ϕ is also smooth and of compact support on F^{r-1} it follows that it is identically zero and therefore also that $\phi = 0$. In particular, $\mu(v) = f(I_r) = \phi(0) = 0$. Since $v \in W$ was arbitrary this shows that $\mu = 0$, a contradiction. □

As a consequence of Proposition 5.1 we get that a relatively cuspidal representation of $G_r(E)$ is non-degenerate and its Whittaker functional can be expressed in terms of the $G_r(F)$-invariant linear form.

Corollary 5.1. *Any representation of $P_r(E)$ that has a relatively cuspidal linear form is also non-degenerate. In particular, any relatively cuspidal representation of $G_r(E)$ is non-degenerate. Furthermore, if (π, V) is a relatively cuspidal representation of $G_r(E)$ and $\mu \in (V^*)^{G_r(F)}$ is non-zero then*

$$\lambda(v) = \int_{U_r(F)\backslash U_r(E)} \mu(\pi(u)v)\psi_r^{-1}(u)\, du \tag{5.1}$$

is a non-zero Whittaker functional on π.

Proof. As observed in [BZ76, Section 5.15] a representation τ of $P_r(E)$ is non-degenerate if and only if $(\Phi^-)^{r-1}(\tau) \neq 0$. Here $(\Phi^-)^m$ is the functor from representations of $P_r(E)$ to representations of $P_{r-m}(E)$ obtained by applying Φ^- repeatedly m times. In fact, in [BZ76] the functor Φ^- was the non-normalized θ_r-twisted Jacquet functor (whereas here as in [BZ77] we use the normalized θ_r-twisted Jacquet functor) but the underlying vector space for $(\Phi^-)^{r-1}(\tau)$ is the same whether or not Φ^- is normalized. Now if μ is a relatively cuspidal linear form on a representation τ of $P_r(E)$ then applying Proposition 5.1 repeatedly, and setting $\mu_m = (\mu_{m-1})_1$ we obtain that μ_m is a relatively cuspidal linear form on $(\Phi^-)^m(\tau)$ for all $m = 1, \ldots, r-1$. In particular $(\Phi^-)^{r-1}(\tau)$ admits a non-zero linear form and is therefore non-zero. Thus, τ is non-degenerate. Let (π, V) be a relatively cuspidal representation of $G_r(E)$ and let $\mu \in (V^*)^{G_r(F)}$ be non-zero. Thus μ is a relatively cuspidal linear form on the restriction (τ, V) of π to $P_r(E)$. Based on [BZ76, Section 2.32], it is observed in [BZ76, Section 5.15] that $(\Phi^-)^{r-1}(\tau)$ can be realized as (the representation of $P_1(E) = \{1\}$ on) the vector space $V/V(U_r, \psi_r)$ where $V(U_r, \psi_r)$ is spanned by vectors of the form

$$\psi_r(u)v - \tau(u)v,\ u \in U_r(E),\ v \in V.$$

With this realization, and with the $U_r(E)$-invariant measure on $U_r(F)\backslash U_r(E)$ normalized appropriately, following our construction above we obtain that

$$\mu_{r-1}(v + V(U_r, \psi_r)) = \lambda(v)$$

where λ is defined by (5.1). Since we have seen that $\mu_{r-1} \neq 0$ it follows indeed that λ is a non-zero Whittaker functional on π. □

To conclude this section, we note that if π is an irreducible and relatively cuspidal representation of $G_r(E)$ then Corollary 5.1 implies that π is non-degenerate and for every $W \in \mathcal{W}(\pi, \psi)$ there exists a generalized matrix coefficient $f \in \mathcal{C}_{G_r(F)}(\pi)$ such that

$$W(g) = \int_{U_r(F)\backslash U_r(E)} f(ug)\psi_r^{-1}(u)\, du,\ g \in G_r(E). \tag{5.2}$$

6. Relatively cuspidal Rankin-Selberg γ-factors at $\frac{1}{2}$

In this section we prove Theorem 0.1 in the special case that both π and π' are relatively cuspidal.

Proposition 6.1. *Let π (resp. π') be an irreducible and $G_r(F)$-relatively cuspidal (resp. $G_t(F)$-relatively cuspidal) representation of $G_r(E)$ (resp. $G_t(E)$). Then*

$$\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = 1.$$

The explicit construction of the Rankin-Selberg L and ϵ-factors for $G_r(E) \times G_t(E)$ by Jacquet-P. Shapiro-Shalika is quintessential to our proof. We begin by recalling it.

6.1. **Rankin-Selberg integrals.** Let $r \geq t > 0$ be integers. Let π (resp. π') be a representation of $G_r(E)$ (resp. $G_t(E)$) of Whittaker type. If $r = t$ for every $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ and $\Phi \in C_c^\infty(E^n)$ we define the Rankin-Selberg integral

$$\Psi(s, W, W'; \Phi) = \int_{G_r(E)} W(g)W'(g)\Phi(\eta_r g)\, |\det g|_E^s\ dg. \tag{6.1}$$

If $r > t$ for every $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ and an integer j such that $0 \leq j \leq r-t-1$ we define the Rankin-Selberg integral

$$\Psi(s, W, W'; j) = \int_{M_{j\times t}(E)} \int_{G_t(E)} W\left[\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{k+1} \end{pmatrix}\right] W'(g)\, |\det g|_E^{s-\frac{r-t}{2}}\ dg\ dx \tag{6.2}$$

where $k = r - t - 1 - j$. Note that if $t = r - 1$ then we must have $j = k = 0$. In this case we set

$$\Psi(s, W, W') = \Psi(s, W, W'; 0).$$

The content of [JPSS83, Theorem 2.7] is as follows. There exists $s_0 \in \mathbb{R}$ such that $\Psi(s, W, W'; \Phi)$ when $r = t$ (resp. $\Psi(s, W, W'; j)$ when $r > t$) is defined by an absolutely convergent integral whenever $\operatorname{Re} s > s_0$. Furthermore, $\Psi(s, W, W'; \Phi)$ (resp. $\Psi(s, W, W'; j)$) extends to a rational function of $X = q^{-s}$. Denote by $\mathcal{I}(\pi, \pi')$ the subspace of $\mathbb{C}(X)$ spanned by $\Psi(s, W, W'; \Phi)$, $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi, \psi^{-1})$, $\Phi \in C_c^\infty(E^r)$ (resp. $\Psi(s, W, W'; j)$, $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi, \psi^{-1})$). Then $\mathcal{I}(\pi, \pi') \subseteq \mathbb{C}(X)$ is a fractional ideal over $\mathbb{C}[X, X^{-1}]$ containing $\mathbb{C}[X, X^{-1}]$ and independent of j if $r > t$. It is therefore of the form $\mathcal{I}(\pi, \pi') = P(X)^{-1}\mathbb{C}[X, X^{-1}]$ for a unique $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$. The local Rankin-Selberg L-factor associated to π and π' is defined by

$$L(s, \pi \times \pi') = P(q^{-s})^{-1}.$$

There exists a rational function $\gamma(s,\pi\times\pi';\psi)\in\mathbb{C}(X)$ such that if $r=t$ then for all $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$ and $\Phi\in C_c^\infty(E^n)$ we have

$$\Psi(1-s,\tilde{W},\tilde{W}';\hat{\Phi})=\omega_{\pi'}(-1)^{r-1}\gamma(s,\pi\times\pi';\psi)\Psi(s,W,W';\Phi) \tag{6.3}$$

and if $r>t$ then for all $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$ and $0\le j\le r-t-1$ we have

$$\Psi(1-s,R(w_{r,t})\tilde{W},\tilde{W}';k)=\omega_{\pi'}(-1)^{r-1}\gamma(s,\pi\times\pi';\psi)\Psi(s,W,W';j) \tag{6.4}$$

where

$$w_{r,t}=\begin{pmatrix} I_{r-t} & 0\\ 0 & w_t\end{pmatrix}.$$

The local Rankin-Selberg ϵ-factor is then defined to be

$$\epsilon(s,\pi\times\pi';\psi)=\frac{L(1-s,\tilde{\pi}\times\tilde{\pi}')\gamma(s,\pi\times\pi';\psi)}{L(s,\pi\times\pi')}$$

and $\epsilon(s,\pi\times\pi';\psi)$ is a monomial, i.e. of the form cX^m for some $c\in\mathbb{C}^\times$ and $m\in\mathbb{Z}$. We will apply the Rankin-Selberg integrals explicitly, only in the cases where $t=r-1$ or $t=r$.

Let now π (resp. π') be any irreducible representation of $G_r(E)$ (resp. $G_t(E)$). Let (Q,τ,λ) (resp. (Q',τ',λ')) be the Langlands data for π (resp. π'). That is, $Q=LV$ is a standard parabolic subgroup of $G_r(E)$ (L is its standard Levi subgroup and V the unipotent radical), say of type $(r_1,\dots,r_k)$, $\tau=\tau_1\otimes\cdots\otimes\tau_k$ is an irreducible, tempered representation of L, $\lambda=(\lambda_1,\dots,\lambda_k)\in\mathbb{R}^k$ is such that $\lambda_1>\lambda_2>\cdots>\lambda_k$ and π is the unique irreducible quotient of $\xi=I_Q^{G_r(E)}(\tau[\lambda])$ where $\tau[\lambda]=|\det\cdot|_E^{\lambda_1}\tau_1\otimes\cdots\otimes|\det\cdot|_E^{\lambda_k}\tau_k$. The triple (Q',τ',λ') is the unique such data for π' and we set $\xi'=I_{Q'}^{G_r(E)}(\tau'[\lambda'])$. The representations ξ and ξ' are then of Whittaker type and we set

$$L(s,\pi\times\pi')=L(s,\xi\times\xi')\text{ and }\epsilon(s,\pi\times\pi';\psi)=\epsilon(s,\xi\times\xi';\psi).$$

Furthermore, we set (recall that we assumed $t\le r$)

$$L(s,\pi'\times\pi)=L(s,\pi\times\pi')\text{ and }\epsilon(s,\pi'\times\pi;\psi)=\epsilon(s,\pi\times\pi';\psi).$$

This is well defined when $t=r$. Recall that $\gamma(s,\pi\times\pi';\psi)$ is defined by (0.1). Assume now that r and t are any positive integers. Let $Q=LV$ (resp. $Q'=L'V'$) be a standard parabolic subgroup of $G_r(E)$ (resp. $G_t(E)$) of type $(r_1,\dots,r_k)$ (resp. $(t_1,\dots,t_{k'})$) and let $\tau=\tau_1\otimes\cdots\otimes\tau_k$ (resp. $\tau'=\tau_1'\otimes\cdots\otimes\tau_{k'}'$) be an irreducible representation of L (resp. L'). If π (resp. π') is an irreducible sub-representation of $I_Q^{G_r(E)}(\tau)$ (resp. $I_{Q'}^{G_t(E)}(\tau')$) then

$$\gamma(s,\pi\times\pi';\psi)=\prod_{\substack{1\le i\le k\\ 1\le j\le k'}}\gamma(s,\tau_i\times\tau_j';\psi). \tag{6.5}$$

We recall further the following properties of the Rankin-Selberg L and ϵ-factors. We have

$$L(s,\bar{\pi}\times\bar{\pi}')=L(s,\pi\times\pi'),\ \epsilon(s,\bar{\pi}\times\bar{\pi}';\bar{\psi})=\epsilon(s,\pi\times\pi';\psi) \tag{6.6}$$

and

$$\epsilon(s,\pi\times\pi';\psi)\,\epsilon(1-s,\tilde{\pi}\times\tilde{\pi}';\psi^{-1})=1. \tag{6.7}$$

6.2. **Proof of Proposition 6.1.** We begin this section with a series of 3 lemmas that, in certain cases, relate $\gamma(\frac{1}{2}, \pi \times \pi'; \psi)$ to the scalar $c(\pi)$ defined in (4.6). These observations will be the key to our proof of Proposition 6.1. The structure of proof of the 3 lemmas is rather similar. We apply directly the definition of $\gamma(s, \pi \times \pi'; \psi)$ as the quotient of two Rankin-Selberg integrals. Thus we first show that the Rankin-Selberg integrals converge absolutely at $s = \frac{1}{2}$ in all cases we consider. We then prove each lemma with a rather long, yet elementary, manipulation of the integrals. It will be used in all 3 lemmas without further mention that if $r = t$ there exists $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ and $\Phi \in C_c^\infty(E^r)$ such that $\Psi(s, W, W'; \Phi) = 1$ and if $t = r - 1$ then there exists $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$ such that $\Psi(s, W, W') = 1$. It will also be used without further mention that the central character of an irreducible and distinguished representation π' of $G_r(E)$ is trivial on $F^\times$ and in particular that $\omega_{\pi'}(-1) = 1$.

Lemma 6.1. *Let π be an irreducible and relatively cuspidal representation of $G_r(E)$ and let π' be an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation of $G_r(E)$. Then*

$$\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = c(\pi').$$

Proof. Let $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ and $\Phi \in C_c^\infty(E^r)$. Let $f \in \mathcal{C}_{G_r(F)}(\pi)$ satisfy (5.2). Thus,

$$\begin{aligned}
&\int_{U_r(E)\backslash G_r(E)} |W(g)W'(g)\Phi(\eta_r g)|\, |\det g|_E^s \ dg \\
&\leq \int_{U_r(E)\backslash G_r(E)} \int_{U_r(F)\backslash U_r(E)} |f(ug)| \ du\, |W'(g)\Phi(\eta_r g)|\, |\det g|_E^s \ dg \\
&= \int_{U_r(F)\backslash G_r(E)} |f(g)W'(g)\Phi(\eta_r g)|\, |\det g|_E^s \ dg \\
&= \int_{G_r(F)\backslash G_r(E)} |f(g)|\, |\det g|_E^s \int_{U_r(F)\backslash G_r(F)} |W'(hg)\Phi(\eta_r hg)|\, |\det h|_F^{2s} \ dh\ dg.
\end{aligned}$$

The inner integral is convergent for all $\mathrm{Re}\, s \geq \frac{1}{2}$ by the absolute convergence of the Asai integrals proved by Flicker [Fli88, Section 4, Proposition (i)] and already discussed in Section 5. Since π is relatively cuspidal $f \in C_c^\infty(G_r(F)\backslash G_r(E))$ and therefore the outer integral is over a compact set. It follows that $\Psi(\frac{1}{2}, W, W'; \Phi)$ is defined by an absolutely convergent integral. This justifies our further computation. We have

$$\begin{aligned}
(6.8)\qquad \Psi(\frac{1}{2},W,W';\Phi) &= \int_{U_r(E)\backslash G_r(E)} W(g)W'(g)\Phi(\eta_r g)\,|\det g|_E^{\frac{1}{2}}\ dg \\
&= \int_{U_r(E)\backslash G_r(E)} \int_{U_r(F)\backslash U_r(E)} f(ug)\psi_r^{-1}(u)\ du\, W'(g)\Phi(\eta_r g)\,|\det g|_E^{\frac{1}{2}}\ dg \\
&= \int_{U_r(F)\backslash G_r(E)} f(g)W'(g)\Phi(\eta_r g)\,|\det g|_E^{\frac{1}{2}}\ dg \\
&= \int_{G_r(F)\backslash G_r(E)} f(g)\,|\det g|_E^{\frac{1}{2}} \int_{U_r(F)\backslash G_r(F)} W'(hg)\Phi(\eta_r hg)\,|\det h|_F\ dh\ dg \\
&= \int_{G_r(F)\backslash G_r(E)} f(g)\,|\det g|_E^{\frac{1}{2}}\, Z(1,R(g)W',\Phi(\cdot\, g)_{|F^r})\ dg.
\end{aligned}$$

Recall that $f^* \in \mathcal{C}_{G_r(F)}(\tilde\pi)$. Applying the change of variables $u \mapsto w_r{}^t u^{-1} w_r^{-1}$, and the fact that $f(w_r g) = f(g)$ it follows from the definitions that

$$(6.9)\qquad \tilde W(g) = \int_{U_r(F)\backslash U_r(E)} f^*(ug)\psi_r(u)\ du.$$

Our computation applied to $\tilde\pi$ and $\tilde\pi'$ therefore yields

$$\Psi(\frac{1}{2},\tilde W,\tilde W';\hat\Phi) = \int_{G_r(F)\backslash G_r(E)} f^*(g)\,|\det g|_E^{\frac{1}{2}}\, Z(1,R(g)\tilde W',\hat\Phi(\cdot\, g)_{|F^r})\ dg$$

and the change of variables $g \mapsto {}^t g^{-1}$ gives

$$(6.10)\qquad \Psi(\frac{1}{2},\tilde W,\tilde W';\hat\Phi) = \int_{G_r(F)\backslash G_r(E)} f(g)\,|\det g|_E^{-\frac{1}{2}}\, Z(1,R({}^t g^{-1})\tilde W',\hat\Phi(\cdot\, {}^t g^{-1})_{|F^r})\ dg.$$

Note that the Fourier transform of $\Phi(\cdot\, g)$ equals $|\det g|_E^{-1}\,\hat\Phi(\cdot\, {}^t g^{-1})$. It follows from (1.2) and Lemma 4.2 that

$$Z(1,R({}^t g^{-1})\tilde W',\hat\Phi(\cdot {}^t g^{-1})_{|F^r}) = c(\pi')\,|\det g|_E\, Z(1,R(g)W',\Phi(\cdot\, g)_{|F^r}).$$

Plugging this into (6.10), by (6.8) we obtain that

$$\Psi(\frac{1}{2},\tilde W,\tilde W';\hat\Phi) = c(\pi')\Psi(\frac{1}{2},W,W';\Phi)$$

for all $W \in \mathcal{W}(\pi,\psi)$, $W' \in \mathcal{W}(\pi',\psi^{-1})$ and $\Phi \in C_c^\infty(E^r)$. The lemma follows. □

Lemma 6.2. *Let $r > 1$ be an integer. Let π be an irreducible and relatively cuspidal representation of $G_r(E)$ and let π' be an irreducible, unitarizable, non-degenerate and $G_{r-1}(F)$-distinguished representation of $G_{r-1}(E)$. Then*

$$\gamma(\frac{1}{2},\pi\times\pi';\psi) = c(\pi').$$

Proof. Let $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$. Let $f \in \mathcal{C}_{G_r(F)}(\pi)$ satisfy (5.2). We then have

$$\begin{aligned}
&\int_{U_{r-1}(E)\backslash G_{r-1}(E)} \left| W\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right] W'(g)\right| \, dg \\
&= \int_{U_{r-1}(E)\backslash G_{r-1}(E)} \left|\int_{U_r(F)\backslash U_r(E)} f\left[u\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right] \psi_r^{-1}(u)\, du\, W'(g)\right| \, dg \\
&\le \int_{U_{r-1}(E)\backslash G_{r-1}(E)} |W'(g)| \cdot \\
&\qquad \int_{U_{r-1}(F)\backslash U_{r-1}(E)} \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} I_{r-1} & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} ug & 0 \\ 0 & 1\end{pmatrix}\right] \psi_r^{-1}(\eta_{r-1}x)\, dx\right| \, du\, dg \\
&= \int_{U_{r-1}(F)\backslash G_{r-1}(E)} |W'(g)| \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} I_{r-1} & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right] \psi_r^{-1}(\eta_{r-1}x)\, dx\right| \, dg \\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)} \int_{U_{r-1}(F)\backslash G_{r-1}(F)} |W'(hg)| \\
&\qquad \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} I_{r-1} & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} hg & 0 \\ 0 & 1\end{pmatrix}\right] \psi_r^{-1}(\eta_{r-1}x)\, dx\right| \, dh\, dg \\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)} \int_{U_{r-1}(F)\backslash G_{r-1}(F)} |W'(hg)| \\
&\qquad \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} I_{r-1} & h^{-1}x \\ 0 & 1\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right] \psi_r^{-1}(\eta_{r-1}x)\, dx\right| \, dh\, dg.
\end{aligned}$$

The last equality is obtained since f is left $G_r(F)$-invariant and for $h \in G_{r-1}(F)$ we have

$$\begin{pmatrix} h^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} I_{r-1} & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} I_{r-1} & h^{-1}x \\ 0 & 1\end{pmatrix}.$$

After a change of variables $x \mapsto hx$ the last expression we obtained equals

$$\begin{aligned}
(6.11)\quad &\int_{G_{r-1}(F)\backslash G_{r-1}(E)} \int_{U_{r-1}(F)\backslash G_{r-1}(F)} |W'(hg)|\, |\det h|_F \\
&\qquad \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} g & x \\ 0 & 1\end{pmatrix}\right] \psi_r^{-1}(\eta_{r-1}hx)\, dx\right| \, dh\, dg.
\end{aligned}$$

Decomposing the measure on $U_{r-1}(F)\backslash G_{r-1}(F)$ by first integrating over $U_{r-1}(F)\backslash P_{r-1}(F)$ and then over $P_{r-1}(F)\backslash G_{r-1}(F)$ (mind the modulus function) and recalling that for $p \in P_{r-1}(F)$ we have $\eta_{r-1}p = \eta_{r-1}$ the expression (6.11) equals

$$\begin{aligned}
(6.12)\quad &\int_{G_{r-1}(F)\backslash G_{r-1}(E)} \int_{P_{r-1}(F)\backslash G_{r-1}(F)} \left[\int_{U_{r-1}(F)\backslash P_{r-1}(F)} |W'(phg)|\, dp\right] \\
&\qquad |\det h|_F \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} g & x \\ 0 & 1\end{pmatrix}\right] \psi^{-1}(\eta_{r-1}hx)\, dx\right| \, dh\, dg.
\end{aligned}$$

Note that we may identify $P_{r-1}(F)\backslash G_{r-1}(F)$ with $F^{r-1}\setminus\{0\}$ by $h\mapsto \eta_{r-1}h$ and that the measure $|\det h|_F\ dh$ transforms to a Haar measure dx on F^{r-1}. By re-normalizing the measure dp we may assume that dx is self dual with respect to ψ_F. For $y\in F^{r-1}\setminus\{0\}$ let $h(y)\in G_{r-1}(F)$ be an element with last row y. The expression (6.12) then equals

$$\int_{G_{r-1}(F)\backslash G_{r-1}(E)}\int_{F^{r-1}}\left[\int_{U_{r-1}(F)\backslash P_{r-1}(F)}|W'(ph(y)g)|\ dp\right] \left|\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} g & x\\ 0 & 1\end{pmatrix}\right]\psi^{-1}(yx)\ dx\right|\ dy\ dg. \tag{6.13}$$

Note that the integral in the absolute value equals

$$\int_{F^{r-1}} f\left[\begin{pmatrix} g & \iota x\\ 0 & 1\end{pmatrix}\right]\psi_F^{-1}(yx)\ dx$$

and is the Fourier transform at y of a function in $C_c^\infty(F^{r-1})$. It is therefore also of compact support as a function of y. The integral over p is convergent by the result of Flicker. It follows that the support of the integration over y and over g in (6.13) is compact and therefore the integral converges. This shows that $\Psi(\frac{1}{2},W,W')$ is defined by an absolutely convergent integral and justifies our further computation. Since many of the integral manipulations will now be similar we will perform them in steps but with no further explanation. We compute as follows

$$\begin{aligned}
&\Psi(\frac{1}{2},W,W')\\
&= \int_{U_{r-1}(E)\backslash G_{r-1}(E)} W'(g)\int_{U_r(F)\backslash U_r(E)} f\left[u\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\right]\psi_r^{-1}(u)\ du\ dg\\
&= \int_{U_{r-1}(E)\backslash G_{r-1}(E)} W'(g)\cdot\\
&\qquad \int_{U_{r-1}(F)\backslash U_{r-1}(E)}\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} I_{r-1} & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} ug & 0\\ 0 & 1\end{pmatrix}\right]\psi^{-1}(\eta_{r-1}x)\ dx\,\psi_{r-1}^{-1}(u)\ du\ dg\\
&= \int_{U_{r-1}(F)\backslash G_{r-1}(E)} W'(g)\int_{F^{r-1}\backslash E^{r-1}} f\left[\begin{pmatrix} g & x\\ 0 & 1\end{pmatrix}\right]\psi^{-1}(\eta_{r-1}x)\ dx\ dg\\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)}\int_{U_{r-1}(F)\backslash G_{r-1}(F)} W'(hg)\int_{F^{r-1}} f\left[\begin{pmatrix} hg & \iota x\\ 0 & 1\end{pmatrix}\right]\psi_F^{-1}(\eta_{r-1}x)\ dx\ dh\ dg\\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)}\int_{U_{r-1}(F)\backslash G_{r-1}(F)} W'(hg)\int_{F^{r-1}} f\left[\begin{pmatrix} g & \iota x\\ 0 & 1\end{pmatrix}\right]\psi^{-1}(\eta_{r-1}hx)\ dx\ |\det h|_F\ dh\ dg\\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)}\int_{P_{r-1}(F)\backslash G_{r-1}(F)}\int_{U_{r-1}(F)\backslash P_{r-1}(F)} W'(phg)\ dp\\
&\qquad \int_{F^{r-1}} f\left[\begin{pmatrix} g & \iota x\\ 0 & 1\end{pmatrix}\right]\psi_F^{-1}(\eta_{r-1}hx)\ dx\,|\det h|_F\ dh\ dg.
\end{aligned}$$

Since $\mu_{\pi'}$ is a $G_{r-1}(F)$-invariant linear form on $\mathcal{W}(\pi',\psi^{-1})$ we get that

$$\int_{U_{r-1}(F)\backslash P_{r-1}(F)} W'(phg)\ dp = \mu_{\pi'}(R(hg)W') = \mu_{\pi'}(R(g)W') = \int_{U_{r-1}(F)\backslash P_{r-1}(F)} W'(pg)\ dp.$$

Thus,

$$\begin{aligned}\Psi(\frac{1}{2},W,W') = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} & \mu_{\pi'}(R(g)W') \cdot \\ & \int_{P_{r-1}(F)\backslash G_{r-1}(F)} \int_{F^{r-1}} f\left[\begin{pmatrix} g & \iota x \\ 0 & 1\end{pmatrix}\right] \psi_F^{-1}(\eta_{r-1}hx)\ dx\,\lvert\det h\rvert_F\ dh\ dg \\ = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} & \mu_{\pi'}(R(g)W') \int_{F^{r-1}}\int_{F^{r-1}} f\left[\begin{pmatrix} g & \iota x \\ 0 & 1\end{pmatrix}\right] \psi_F^{-1}(yx)\ dx\ dy\ dg.\end{aligned}$$

Applying the Fourier inversion formula for F^{r-1} for the integration over x and over y we obtain that

$$\Psi(\frac{1}{2},W,W') = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} \mu_{\pi'}(R(g)W') f\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right]\ dg. \tag{6.14}$$

Applying (6.14) to $\tilde{\pi}$ and $\tilde{\pi}'$ and keeping (6.9) in mind we get that

$$\Psi(\frac{1}{2},\tilde{W},\tilde{W}') = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} \mu_{\tilde{\pi}'}(R(g)\tilde{W}') f^*\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right]\ dg$$

and applying the change of variable $g \mapsto {}^t g^{-1}$ we get that

$$\Psi(\frac{1}{2},\tilde{W},\tilde{W}') = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} \mu_{\tilde{\pi}'}(R({}^t g^{-1})\tilde{W}') f\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right]\ dg. \tag{6.15}$$

By (1.2) and (4.6) we have

$$\mu_{\tilde{\pi}'}(R({}^t g^{-1})\tilde{W}') = \mu_{\tilde{\pi}'}(\widetilde{R(g)W'}) = c(\pi')\,\mu_{\pi'}(R(g)W').$$

Plugging this into (6.15) and comparing with (6.14) we get that

$$\Psi(\frac{1}{2},\tilde{W},\tilde{W}') = c(\pi')\,\Psi(\frac{1}{2},W,W')$$

for every $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$. The lemma follows. □

Lemma 6.3. *Let π be an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation of $G_r(E)$ and let π' be an irreducible and relatively cuspidal representation of $G_{r-1}(E)$. Then*

$$\gamma(\frac{1}{2},\pi\times\pi';\psi) = c(\pi).$$

Proof. Let $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$. Let $f' \in \mathcal{C}_{G_{r-1}(F)}(\pi')$ satisfy (5.2) for $(r-1,\psi^{-1},\pi',W',f')$ in the role of (r,ψ,π,W,f). We then have

$$\begin{aligned}
&\int_{U_{r-1}(E)\backslash G_{r-1}(E)} \left| W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] W'(g)\right| \, dg \\
&\quad \le \int_{U_{r-1}(E)\backslash G_{r-1}(E)} \left| W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right]\right| \int_{U_{r-1}(F)\backslash U_{r-1}(E)} |f'(ug)| \, du \, dg \\
&\quad = \int_{U_{r-1}(F)\backslash G_{r-1}(E)} \left| W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] f'(g)\right| \, dg \\
&\quad = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} |f'(g)| \int_{U_{r-1}(F)\backslash G_{r-1}(F)} \left| W\left[\begin{pmatrix} hg & 0 \\ 0 & 1 \end{pmatrix}\right]\right| \, dh \, dg.
\end{aligned}$$

On the right hand side, the inner integral converges by the absolute convergence of the integral defining μ_π proved by Flicker. The outer integral converges since the integrand has compact support. Thus, $\Psi(\frac{1}{2},W,W')$ is given by an absolutely convergent integral and the following computation is justified.

$$\begin{aligned}
&\Psi(\frac{1}{2},W,W') \\
&\quad = \int_{U_{r-1}(E)\backslash G_{r-1}(E)} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] W'(g) \, dg \\
&\quad = \int_{U_{r-1}(E)\backslash G_{r-1}(E)} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \int_{U_{r-1}(F)\backslash U_{r-1}(E)} f'(ug)\psi_{r-1}(u) \, du \, dg \\
&\quad = \int_{U_{r-1}(F)\backslash G_{r-1}(E)} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] f'(g) \, dg \\
&\quad = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} f'(g) \int_{U_{r-1}(F)\backslash G_{r-1}(F)} W\left[\begin{pmatrix} hg & 0 \\ 0 & 1 \end{pmatrix}\right] \, dh \, dg \\
&\quad = \int_{G_{r-1}(F)\backslash G_{r-1}(E)} f'(g)\mu_\pi\left(R\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] W\right) \, dg.
\end{aligned}$$

Applying this formula to $\tilde{\pi}$ and $\tilde{\pi}'$ and performing manipulations similar to those used in both Lemmas 6.1 and 6.2 we obtain

$$\begin{aligned}
&\Psi(\frac{1}{2}, \tilde{W}, \tilde{W}') \\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)} (f')^*(g)\mu_{\tilde{\pi}}\left(R\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right]\tilde{W}\right)\, dg \\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)} f'(g)\mu_{\tilde{\pi}}\left(R\left[\begin{pmatrix} {}^tg^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right]\tilde{W}\right)\, dg \\
&= \int_{G_{r-1}(F)\backslash G_{r-1}(E)} f'(g)\mu_{\tilde{\pi}}\left(\left(R\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right]W\right)^{\tilde{}}\right)\, dg \\
&= c(\pi)\int_{G_{r-1}(F)\backslash G_{r-1}(E)} f'(g)\mu_{\pi}(R\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right]W)\, dg \\
&= c(\pi)\,\Psi(\frac{1}{2}, W, W')
\end{aligned}$$

for every $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$. The lemma follows. ☐

Corollary 6.1. *Let π be an irreducible and $G_r(F)$-relatively cuspidal representation of $G_r(E)$. Then*

$$c(\pi) = 1.$$

Proof. For $r = 1$ an irreducible and $G_1(F)$-relatively cuspidal representation of $G_1(E)$ is a character π of $F^\times\backslash E^\times$. It is then clear that $c(\pi) = 1$. Now let $r > 1$ and let π be an irreducible and $G_r(F)$-relatively cuspidal representation of $G_r(E)$. In [HM02], Hakim and Murnaghan construct enough irreducible and supercuspidal distinguished representations to guarantee that for every $t > 0$ there exists an irreducible, $G_t(F)$-relatively cuspidal representation of $G_t(E)$. Let π' be an irreducible and $G_{r-1}(F)$-relatively cuspidal representation of $G_{r-1}(E)$. It follows from Lemmas 6.2 and 6.3 that

$$c(\pi) = \gamma(\frac{1}{2}, \pi \times \pi'; \psi) = c(\pi'). \tag{6.16}$$

The lemma follows by induction on r. ☐

It is an easy observation that in our setting distinction is preserved under induction with respect to standard parabolic subgroups. This fact which we now prove will soon be useful.

Lemma 6.4. *Let $Q = LV$ be a standard parabolic subgroup of G_r and let (τ, W) be an $L(F)$-distinguished representation of $L(E)$ then $I_{Q(E)}^{G_r(E)}(\tau)$ is $G_r(F)$-distinguished.*

Proof. Let μ be a non-zero $L(F)$-invariant linear form on W. Note that the modulus functions $\delta_{Q(E)}$ of $Q(E)$ and $\delta_{Q(F)}$ of $Q(F)$ satisfy the relation

$$\delta_{Q(E)}^{\frac{1}{2}}(q) = \delta_{Q(F)}(q),\ q \in Q(F)$$

and therefore that the linear form

$$\mu^G(f) = \int_{G_r(\mathcal{O}_F)} \mu(f(k))\, dk,\ f \in I_{Q(E)}^{G_r(E)}(\tau)$$

is $G_r(F)$-invariant. Let $v \in W$ be such that $\mu(v) = 1$ and let K be a congruence subgroup of $G(\mathcal{O}_E)$, small enough such that $v \in W^{K \cap L(E)}$. Recall that $K = (K \cap Q(E))(K \cap {}^tV(E))$ and that $Q(E)(K \cap {}^tV(E))$ is open in $G(E)$. Let

$$f(g) = \begin{cases} \delta_{Q(E)}^{\frac{1}{2}}(q)\tau(q)v & g = qk,\ q \in Q(E),\ k \in (K \cap {}^tV(E)) \\ 0 & g \notin Q(E)(K \cap {}^tV(E)). \end{cases}$$

Then $f \in I_{Q(E)}^{G(E)}(\tau)$ and the support of f is $Q(E)(K \cap {}^tV(E))$. If

$$k \in G_r(\mathcal{O}_F) \cap Q(E)(K \cap {}^tV(E))$$

then let $k = qk_0$ with $q \in Q(E)$ and $k_0 \in K \cap {}^tV(E)$. Since $qk_0 = \bar{q}\bar{k}_0$ we get that $q^{-1}\bar{q} = k_0\bar{k}_0^{-1} \in Q(E) \cap K \cap {}^tV(E) = \{I_r\}$ and therefore $q \in Q(F)$, i.e.

$$\mu(f(k)) = \delta_{Q(E)}^{\frac{1}{2}}(q)\mu(\tau(q)v) = \delta_{Q(E)}^{\frac{1}{2}}(q)\mu(v) = \delta_{Q(E)}^{\frac{1}{2}}(q) > 0.$$

Furthermore, $\mu(f(k)) = 1$ whenever $k \in G(\mathcal{O}_F) \cap K$ and therefore f is a non-negative function on $G(\mathcal{O}_F)$ which is positive on an open set. It follows that $\mu^G(f) \neq 0$. □

For a representation π of $G_r(E)$ we denote by $\pi[1]$ the representation of $G_{r+1}(E)$ parabolically induced from $\pi \otimes \mathbf{1}_{E^\times}$. For any positive integer m we set $\pi[m+1] = \pi[m][1]$.

Corollary 6.2. *Let π' be an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation of $G_r(E)$. Then*

$$c(\pi') = c(\pi'[1]).$$

Proof. Note that $\pi'[1]$ is an irreducible, unitarizable and non-degenerate representation of $G_{r+1}(E)$. It follows from Lemma 6.4 that it is also $G_{r+1}(F)$-distinguished. Let π be an irreducible and $G_{r+1}(F)$-relatively cuspidal representation of $G_{r+1}(E)$. As already explained, the existence of such π follows from [HM02]. It then follows from Lemma 6.2 that

$$\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = c(\pi')$$

and from Lemma 6.1 that

$$\gamma(\frac{1}{2}, \pi \times \pi'[1]; \psi) = c(\pi'[1]).$$

But from (6.5) we get that

$$\gamma(\frac{1}{2}, \pi \times \pi'[1]; \psi) = \gamma(\frac{1}{2}, \pi \times \pi'; \psi)\, \gamma(\frac{1}{2}, \pi \times \mathbf{1}_{E^\times}; \psi)$$

and from Proposition 2.1 we get that

$$\gamma(\frac{1}{2}, \pi \times \mathbf{1}_{E^\times}; \psi) = \gamma(\frac{1}{2}, \pi; \psi) = 1.$$

The lemma follows. □

We are now ready to prove Proposition 6.1. Without loss of generality we may assume that $r \geq t$. If $t = r$ or $t = r - 1$ the proposition follows from Lemmas 6.1, 6.2, 6.3 and Corollary 6.1. Assume now that $t < r - 1$. As in the proof of Corollary 6.2, (6.5) together with Proposition 2.1 imply that

$$\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = \gamma(\frac{1}{2}, \pi \times \pi'[r-t]; \psi). \tag{6.17}$$

The representation $\pi'[r-t]$ of $G_r(E)$ is irreducible, unitarizable and non-degenerate and it follows from Lemma 6.4 that it is also $G_r(F)$-distinguished. Thus, from Lemma 6.1 and (6.17) we get that

$$\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = c(\pi'[r-t]).$$

But by Corollaries 6.1 and 6.2 we have

$$c(\pi'[r-t]) = c(\pi') = 1.$$

Proposition 6.1 follows.

7. Distinction and the Rankin-Selberg γ-factor at $\frac{1}{2}$

In order to deduce Theorem 0.1 from Proposition 6.1 we apply the relative subrepresentation theorem of Kato-Takano. In Section 7.1 we make the results of Kato-Takano explicit in our setting. In Section 7.2 we apply this explicit description to conclude the proof of Theorem 0.1.

7.1. **The relative subrepresentation theorem for GL_n.** The purpose of this subsection is to make the relative sub-representation theorem of Kato-Takano more explicit in our case. We show that [KT08, Theorem 7.1] implies the following.

Proposition 7.1. *Let π be an irreducible and H-distinguished representation of G. Then there exists a partition $(n_1, \dots, n_r)$ of n with $n_i = n_{r+1-i}$ and irreducible representations τ_i of $G_{n_i}(E)$, $i = 1, \dots, r$ with the following properties. We have $\tilde{\tau}_i \simeq \bar{\tau}_{r+1-i}$ for all i, τ_i is supercuspidal for all $i \neq r+1-i$, τ_i is relatively cuspidal if $i = r+1-i$ and π imbeds as a subrepresentation of $I_Q^G(\tau_1 \otimes \cdots \otimes \tau_r)$ where Q is the standard parabolic subgroup of G of type $(n_1, \dots, n_r)$.*

Let R be a reductive group and let θ be an involution on R both defined over F. For any subgroup Q of R we set

$$Q^\theta = \{q \in Q : \theta(q) = q\}.$$

A parabolic subgroup Q of R, defined over F, is called θ-split if $\theta(Q)$ is opposite to Q. In this case

$$L = Q \cap \theta(Q)$$

is a common θ-stable Levi subgroup of Q and $\theta(Q)$. A torus S of R defined over F is called θ-split if $\theta(s) = s^{-1}$, $s \in S$. The main result we apply in this subsection, [KT08, Theorem

7.1], asserts that any irreducible, admissible and $R^\theta(F)$-distinguished representation of $R(F)$ imbeds as a sub-representation of

$$I_{Q(F)}^{R(F)}(\tau)$$

for some θ-split parabolic subgroup Q of R and some irreducible, admissible and $L^\theta(F)$-relatively cuspidal representation τ of $L(F)$.

We return to our setting and define the symmetric space

$$X = \{g \in G : g\bar{g} = I_n\}.$$

The group G acts on the symmetric space X by the twisted conjugation $x \cdot g = \bar{g}^{-1}xg$, $g \in G$, $x \in X$ and H is the stabilizer of the identity element I_n. It is well known that G acts transitively on X (cf. [AC89, Lemma 1.1]), i.e. that $X \simeq H\backslash G$. Thus, for every $x \in X$ there exists $\xi \in G$ such that $x = I_n \cdot \xi$ and we then have

$$\xi^{-1}H\xi = \{g \in G : x \cdot g = g\}.$$

If (π, V) is a representation of G then $\alpha \mapsto \alpha \circ \pi(\xi)$ is an isomorphism from $(V^*)^H$ to $(V^*)^{\xi^{-1}H\xi}$. A representation is therefore H-relatively cuspidal if and only if it is $\xi^{-1}H\xi$-relatively cuspidal. Note that $w_n \in X$. Fix once and for all $\xi_0 \in G$ such that $I_n \cdot \xi_0 = w_n$ and let $H_0 = \xi_0^{-1}H\xi_0$. We now define an algebraic group R and an involution θ on R defined over F such that $R(F) = G$ and $R^\theta(F) = H_0$. It is more convenient, in this context, to work with H_0 then with H since, as we shall soon see, every θ-split parabolic is H_0-conjugate to a standard one. Let R be the restriction of scalars from E to F of GL_n regarded as an algebraic group over E and let

$$\theta(x) = w_n\bar{x}w_n^{-1},\ x \in R.$$

Thus, indeed $R(F) = G$ and $R^\theta(F) = H_0$. Let A be the standard, maximal F-split torus of R, i.e. A is the torus in R such that $A(F)$ consists of all diagonal matrices in G with entries in F, and let

$$S = \{a \in A : \theta(a) = a^{-1}\}.$$

Then S is a maximal F-split and θ-split torus of R. Let $M_0 = Z_R(S)$. Then M_0 is the θ-stable Levi subgroup of some minimal θ-split parabolic subgroup of R (cf. [HW93, Section 4.7]).

Lemma 7.1. *We have*

$$(M_0R^\theta)(F) = M_0(F)R^\theta(F).$$

Proof. Identify each algebraic group over F with its points over the algebraic closure $\overline{F}$ of F. We realize R as a subgroup of GL_{2n} as follows. Write any matrix $g \in GL_{2n}(\overline{F})$ in 2×2-block form as $g = (g_{i,j})$, $1 \le i, j \le n$. Let $\kappa = \iota^2 \in F^\times$ and for a, $b \in \overline{F}$ let

$$g(a, b) = \begin{pmatrix} a & b \\ \kappa b & a \end{pmatrix} \text{ and } \bar{g}(a, b) = \begin{pmatrix} a & -b \\ -\kappa b & a \end{pmatrix} = g(a, -b).$$

If $m = g(a, b)$ we write $\bar{m}$ for $\bar{g}(a, b)$. We have

$$R = \{g \in GL_{2n}(\overline{F}) : g_{i,j} = g(a_{i,j}, b_{i,j}) \text{ for some } a_{i,j},\ b_{i,j} \in \overline{F}\}.$$

Note then that

$$M_0 = \{g \in R : g_{i,j} = 0,\, i \neq j\}$$

and

$$R^\theta = \{g \in R : g_{w_n(i),w_n(j)} = \bar{g}_{i,j},\, 1 \leq i, j \leq n\}.$$

Let $m = \operatorname{diag}(m_1, \dots, m_n) \in M_0$ and $g = (g_{i,j}) \in R^\theta$ be such that $mg \in R(F)$. To prove the lemma we show that there exists $m' \in M_0(F)$ such that

$$m'mg \in R^\theta(F). \tag{7.1}$$

Note that for every i there exist a_i, $b_i \in F$ with $a_i + \iota b_i \in E^\times$ such that

$$\bar{m}_i m_{w_n(i)}^{-1} = g(a_i, b_i). \tag{7.2}$$

Indeed, there exists $j = j_i$ such that $(mg)_{i,j} = m_i g_{i,j} \neq 0$ and is therefore invertible with entries in F. Since $g \in R^\theta$ we have

$$\bar{m}_i m_{w_n(i)}^{-1} = \bar{m}_i \bar{g}_{i,j} g_{w_n(i),w(j)}^{-1} m_{w_n(i)}^{-1} = \overline{(mg)_{i,j}}((mg)_{w_n(i),w_n(j)})^{-1}$$

and the right hand side has entries in F. Hence the existence of a_i, $b_i \in F$ satisfying (7.2). Note that if n is odd then $\bar{m}_{\frac{n+1}{2}} m_{\frac{n+1}{2}}^{-1} = g(a_{\frac{n+1}{2}}, b_{\frac{n+1}{2}})$ satisfies $(a_{\frac{n+1}{2}} + \iota b_{\frac{n+1}{2}})(a_{\frac{n+1}{2}} - \iota b_{\frac{n+1}{2}}) = 1$. It follows from Hilbert 90 that there exist c, $d \in F$ such that $a_{\frac{n+1}{2}} + \iota b_{\frac{n+1}{2}} = (c + \iota d)(c - \iota d)^{-1}$, i.e. such that $g(a_{\frac{n+1}{2}}, b_{\frac{n+1}{2}}) = g(c, d)\bar{g}(c, d)^{-1}$. We then set $m'_{\frac{n+1}{2}} = \bar{g}(c, d)^{-1}$. For every $i < \frac{n+1}{2}$ let

$$m'_i = I_2 \text{ and } m'_{w_n(i)} = g(a_i, b_i).$$

With this construction $m' = \operatorname{diag}(m'_1, \dots, m'_n) \in M_0(F)$ satisfies (7.1) and the lemma follows. ☐

A standard parabolic subgroup Q of R is such that $Q(F)$ is a standard parabolic subgroup of G.

Corollary 7.1. *Let Q be a θ-split parabolic subgroup of R then there exists a partition $(n_1, \dots, n_r)$ of n with $n_i = n_{r+1-i}$ and $h \in R^\theta(F) = H_0$ such that hQh^{-1} is the standard parabolic subgroup of G of type $(n_1, \dots, n_r)$.*

Proof. It is easily verified that a standard parabolic subgroup of R of type $(n_1, \dots, n_r)$ is θ-split if and only if $n_{r+1-i} = n_i$, $i = 1, \dots, r$. Taking Lemma 7.1 into consideration, the corollary follows from [KT08, Lemma 2.5 (2)]. ☐

We are now ready to prove Proposition 7.1. Let π be an irreducible and H-distinguished representation of G. As we already remarked, π is then also $R^\theta(F)$-distinguished. It follows from [KT08, Theorem 7.1] and Corollary 7.1 that there exists a standard parabolic subgroup Q of R of type $(n_1, \dots, n_r)$ with $n_{r+1-i} = n_i$, $i = 1, \dots, r$ and an irreducible and $L^\theta(F)$-relatively cuspidal representation τ of $L(F)$ where $L = Q \cap \theta(Q)$ such that π imbeds as a sub-representation of $I_{Q(F)}^G(\tau)$. Note that $L(F) \simeq G_{n_1}(E) \times \cdots \times G_{n_r}(E)$ and that

$$L^\theta(F) = \{\operatorname{diag}(g_1, \dots, g_r) : g_i \in G_{n_i}(E),\, g_{r+1-i} = w_{n_i} \bar{g}_i w_{n_i},\, i = 1, \dots, r\}.$$

Write, $\tau = \tau_1 \otimes \cdots \otimes \tau_r$ where τ_i is an irreducible representation of $G_{n_i}(E)$. We make explicit the fact that τ is $L^\theta(F)$-relatively cuspidal. Assume first that $i \neq r+1-i$. The existence of a non-zero $L^\theta(F)$-invariant linear form on the space of τ implies that there is a non-zero linear form α on $\tau_i \otimes \tau_{r+1-i}$ such that

$$\alpha(\tau_i(g)v_1 \otimes \tau_{r+1-i}(w_{n_i}\bar{g}w_{n_i})v_2) = \alpha(v_1 \otimes v_2)$$

for every v_1 in the space of τ_i, v_2 in the space of τ_{r+1-i} and $g \in G_{n_i}(E)$. This identifies $\bar{\tau}_{r+1-i}$ with $\tilde{\tau}_i$. The relative cuspidality of τ implies further that τ_i is supercuspidal (as explained for example in [KT08, Section 1.5]). Assume now that $i = r+1-i$. The existence of a non-zero $L^\theta(F)$-invariant linear form on the space of τ implies that there is a non-zero linear form α on τ_i such that

$$\alpha(\tau_i(g)v) = \alpha(v)$$

whenever $g = w_{n_i}\bar{g}w_{n_i} \in G_{n_i}(E)$ and v is in the space of τ_i. As already explained, this is equivalent to the statement that τ_i is $G_{r_i}(F)$-distinguished. The relative cuspidality of τ implies that τ_i is also relatively cuspidal. The proof of Proposition 7.1 is now complete.

7.2. **Proof of Theorem 0.1.** Let $r \geq t$ and let π (resp. π') be an irreducible $G_r(F)$-distinguished (resp. $G_t(F)$-distinguished) representation of $G_r(E)$ (resp. $G_t(E)$). Let $\tau_1, \dots, \tau_k$ be given by Proposition 7.1 and similarly, let $\tau'_1, \dots, \tau'_{k'}$ be given by applying Proposition 7.1 to π'. If either $i \neq k+1-i$ or $j \neq k'+1-j$ then since $\bar{\tau}_{k+1-i} = \tilde{\tau}_i$ and $\bar{\tau}'_{k'+1-j} = \tilde{\tau}'_j$ it follows from (6.6) that

$$\frac{L(\frac{1}{2}, \tau_i \times \tau'_j)}{L(\frac{1}{2}, \tilde{\tau}_i \times \tilde{\tau}'_j)} \frac{L(\frac{1}{2}, \tau_{k+1-i} \times \tau'_{k'+1-j})}{L(\frac{1}{2}, \tilde{\tau}_{k+1-i} \times \tilde{\tau}'_{k'+1-j})} = 1$$

and

$$\epsilon(\frac{1}{2}, \tau_{k+1-i} \times \tau'_{k'+1-j}; \psi) = \epsilon(\frac{1}{2}, \tilde{\tau}_i \times \tilde{\tau}'_j; \bar{\psi}).$$

Since $\bar{\psi} = \psi^{-1}$ it follows further from (0.2) that

$$\epsilon(\frac{1}{2}, \tau_i \times \tau'_j; \psi)\, \epsilon(\frac{1}{2}, \tau_{k+1-i} \times \tau'_{k'+1-j}; \psi) = 1.$$

We therefore see that if either $i \neq k+1-i$ or $j \neq k'+1-j$ then

$$\gamma(\frac{1}{2}, \tau_i \times \tau'_j; \psi)\, \gamma(\frac{1}{2}, \tau_{k+1-i} \times \tau'_{k'+1-j}; \psi) = 1. \tag{7.3}$$

If both $i = k+1-i$ and $j = k'+1-j$ then τ_i and τ'_j are both relatively cuspidal and by Proposition 6.1 we have

$$\gamma(\frac{1}{2}, \tau_i \times \tau'_j; \psi) = 1. \tag{7.4}$$

It follows from (6.5) that

$$\gamma(s, \pi \times \pi'; \psi) = \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k'}} \gamma(s, \tau_i \times \tau'_j; \psi).$$

Plugging in (7.3) and (7.4) shows that $\gamma(s, \pi \times \pi'; \psi) = 1$. This completes the proof of Theorem 0.1.

We conclude with an interesting integral identity for Whittaker functions of distinguished representations.

Corollary 7.2. *Let π be an irreducible, unitarizable, non-degenerate and $G_r(F)$-distinguished representation of $G_r(E)$. Then $c(\pi) = 1$, i.e. for every $W \in \mathcal{W}(\pi, \psi)$ we have*

$$\int_{U_{r-1}(F)\backslash G_{r-1}(F)} W\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh = \int_{U_{r-1}(F)\backslash G_{r-1}(F)} W\left[\begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix}\right] dh. \tag{7.5}$$

Proof. Let π' be an irreducible and relatively cuspidal representation of $G_r(E)$. It follows from Theorem 0.1 that $\gamma(\frac{1}{2}, \pi \times \pi'; \psi) = 1$ and therefore from Lemma 6.1 that $c(\pi) = 1$. □

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