Distinguished Representations of GL(n) and Local Converse Theorems

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Abstract

These notes establish a local converse theorem for irreducible, distinguished, supercuspidal representations of GL(n) relative to GL($n-2$) twists. Our methods may also be used to give an entirely new proof of the local converse theorem of Chen, Cogdell and Piatetski-Shapiro.

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Contents

1 A local converse theorem for distinguished supercuspidal representations

1.1 Statement of the problem

Consider an irreducible, supercuspidal representation π of the group $G = \mathrm{GL}_n(E)$, where E/F is a quadratic extension of nonarchimedean local fields and $n \geq 3$. Assume that E and F do not have characteristic two. We provide a criterion that implies π is *distinguished* in the sense that the space $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \mathbb{C})$ is nonzero. The criterion is in terms of the Rankin-Selberg gamma factors $\gamma(s, \pi \times \pi', \psi)$ introduced by Jacquet, Piatetski-Shapiro and Shalika [JPSS79a, JPSS79b, JPSS83]. More precisely, our Theorem 1.5.1 says the following:

> Assume ψ is a nontrivial character of E whose restriction to F is trivial. If the central character ω_π of π is trivial on $F^\times$ and if $\gamma(1/2, \pi \times \pi', \psi) = 1$ for all distinguished, irreducible, unitary, generic representations π' of $\mathrm{GL}_{n-2}(E)$ then π is distinguished.

The converse of this result follows from the main theorem of [Off11]. We also note that a similar result to ours with GL_{n-2} replaced by GL_{n-1} was proven in [Ok97], generalizing the $n = 2$ case from [Hak91].

Theorem 1.5.1 and its proof generalize in a straightforward way to the setting in which E is replaced by $F \oplus F$ (with no restriction on the characteristic of F). In this case, the result is equivalent to the following:

> Assume ψ_F is a nontrivial character of F. Let π_1 and π_2 be irreducible, supercuspidal representations of $\mathrm{GL}_n(F)$. If the central characters of π_1 and π_2 are identical and if
>
> $$\gamma(s, \pi_1 \times \tau, \psi_F) = \gamma(s, \pi_2 \times \tau, \psi_F)$$
>
> for all irreducible, unitary, generic representations τ of $\mathrm{GL}_{n-2}(F)$ then π_1 and π_2 are equivalent.

The latter result is not new and it is in fact known to be true when π_1 and π_2 are arbitrary smooth, irreducible, generic representations of $\mathrm{GL}_n(F)$. The formulation of this result and the existing local and global proofs have their origins with Piatetski-Shapiro. The first local proof appears in the Ph.D. thesis of Piatetski-Shapiro's student Jiang-Ping Chen. (See [Che96, Theorem 4.1] and [Che06, Theorem 1.1].) The first complete global proof was given by Cogdell and Piatetski-Shapiro [CPS99, Corollary of Theorem in Section 7]. The idea of the global proof first appeared in two early papers by Piatetski-Shapiro that were informally published by the University of Maryland. (See [PS75, PS76].) The non-supercuspidal case also follows easily from the supercuspidal case, as is explained in [JNS].

The latter result for $E = F \oplus F$ and variants of it are generally referred to as *local converse theorems* and they are intimately related to the (global) converse theorems in the theory of automorphic forms and automorphic representations. Establishing stronger converse theorems, locally and globally, is important and difficult. As far as we are aware, none of the existing methods used to prove local converse theorems generalize in any straightforward way to yield results for distinguished representations with respect to quadratic extensions E/F. The fact that our methods work both for $E = F \oplus F$ and for E/F quadratic is an encouraging sign that these methods might be more powerful than existing methods.

We now roughly outline the contents of the paper. In §1.2, we establish some notations and recall basic facts about Rankin-Selberg integrals and gamma factors. In §1.3, we introduce appropriate variants of the Rankin-Selberg integrals for studying distinguished representations. For the proof of Theorem 1.5.1, we need to establish certain relations between the latter integrals and this is done in §1.4. Theorem 1.5.1 and its proof appear in §1.5. Ok's local converse theorem for distinguished supercuspidal representations, mentioned above, is recalled in §1.6 and its proof is contrasted with our proof of Theorem 1.5.1. In §1.7, we give an alternate, but related, proof of Theorem 1.5.1 in the case of $n = 3$ and we compare the two proofs. For $n = 3$, we obtain the converse theorem more generally for irreducible, unitary, generic representations. In §2, we sketch how one adapts our proofs to the case $E = F \oplus F$ to obtain the local converse theorem of Chen, Cogdell and Piatetski-Shapiro. In §3, we discuss stronger forms of Theorem 1.5.1 that may be obtained by either broadening the class of π considered, or restricting the class of π' allowed. Finally, in §4, we give an expanded presentation of Ok's proof of Lemma 1.3.2.

1.2 Preliminaries

1.2.1 Notation

Let F be a non-archimedean local field of characteristic different than two and let E be a quadratic extension of F. Given a positive integer m, we let G_m be the F-group GL_m and consider the following F-subgroups:

- the center Z_m,
- the subgroup A_m of diagonal matrices,
- the subgroup N_m of upper triangular unipotent matrices,
- the mirabolic subgroup P_m of matrices with last row $(0, \ldots, 0, 1)$,
- the standard parabolic subgroup $P_{m-1,1} = P_m Z_m$,
- the unipotent radical $N_{m-1,1}$ of $P_{m-1,1}$.

(Note that with our notation N_m is *not* the unipotent radical of P_m.)

Fix, until the completion of the proof of Theorem 1.5.1,

- a nontrivial character ψ of E that is trivial on F, and
- an irreducible, supercuspidal representation π of $G_n(E)$ whose central character ω_π is trivial on $F^\times$.

Throughout the paper, representations and characters are assumed to be smooth.

We call a representation π' of $G_m(E)$ distinguished if $\mathrm{Hom}_{G_m(F)}(\pi', \mathbb{C})$ is nonzero. The condition $\omega_\pi | F^\times = 1$ is an obvious necessary condition for the distinction of π. Note that this condition implies that ω_π is unitary and hence π is also unitary (or, rather, unitarizable). In other words, every distinguished, irreducible, supercuspidal representation of $G_n(E)$ is necessarily unitary.

Regard ψ as a character of each group $N_m(E)$ by setting

$$\psi(u) = \psi(u_{1,2} + \cdots + u_{m-1,m}).$$

1.2.2 $P_{n-1,1}\backslash G_n/P_{n-1,1}$

Our main proof uses the fact that the group $G_n(F)$ is generated by $P_{n-1,1}(F)$ and any element of $G_n(F) - P_{n-1,1}(F)$. This fact is a consequence of:

Lemma 1.2.1. *$P_{n-1,1}\backslash G_n/P_{n-1,1}$ has cardinality two.*

Proof. Let S_n be the group of permutation matrices in G_n and let $S_{n-1} = S_n \cap P_{n-1,1}$. By the generalized Bruhat decomposition (see, for example [Spr79, §3.7]) $P_{n-1,1}\backslash G_n/P_{n-1,1}$ is in bijection with $S_{n-1}\backslash S_n/S_{n-1}$. Suppose σ and τ are two elements of $S_n - S_{n-1}$. Let us view them as permutations of $\{1, \ldots, n\}$. Since $\sigma(n)$ and $\tau(n)$ are not equal to n, there exists $\kappa \in S_{n-1}$ such that $\kappa(\sigma(n)) = \tau(n)$. It follows that the element $\lambda = \sigma^{-1}\kappa^{-1}\tau$ lies in S_{n-1}. We have $\tau = \kappa\sigma\lambda$, with $\kappa, \lambda \in S_{n-1}$. This shows that $S_{n-1}\backslash S_n/S_{n-1}$ has two elements. It follows that $P_{n-1,1}\backslash G_n/P_{n-1,1}$ has two elements. □

1.2.3 Whittaker and Kirillov models

We observe that our representation π is necessarily generic (by [GK75, Theorem B] or [Jac77, Theorem 2.1]) and we let $\mathcal{W}(\pi, \psi)$ denote the ψ-Whittaker model of π.

According to [GK75, Theorem 5], $\mathcal{W}(\pi, \psi)$ is a subrepresentation of the induced representation $\mathrm{ind}_{Z_n(E)N_n(E)}^{G_n(E)}(\omega_\pi \otimes \psi)$. Thus the elements of $\mathcal{W}(\pi, \psi)$ are smooth functions f on $G_n(E)$ such that

$$f(zug) = \omega_\pi(z)\,\psi(u)\,f(g),$$

for $z \in Z_n(E)$, $u \in N_n(E)$ and $g \in G_n(E)$, and such that the support of f has image in a compact subset of $Z_n(E)N_n(E)\backslash G_n(E)$.

Restriction of functions from $G_n(E)$ to $P_n(E)$ defines a $P_n(E)$-equivariant linear isomorphism of $\mathcal{W}(\pi, \psi)$ with the induced representation $\mathrm{ind}_{N_n(E)}^{P_n(E)}(\psi)$. (See [BZ76, 5.19 and 5.20], [BZ77, 4.10], [JS83, Proposition 3.2] and [Ber84, Corollary 6.5].) This allows one to identify the space of $\mathrm{ind}_{N_n(E)}^{P_n(E)}(\psi)$ with the representation space for π. When this is done, one obtains the *Kirillov model* of π (with respect to ψ).

Define a nonzero linear form $\mu : \mathcal{W}(\pi, \psi) \to \mathbb{C}$ by

$$\mu(W) = \int_{N_{n-1}(F)\backslash G_{n-1}(F)} W\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh.$$

Both convergence and the fact that μ is nonzero follow immediately from our remarks in the previous two paragraphs. (According to [Fli88, page 306], these facts also hold when π is an arbitrary irreducible, unitary, generic representation of $G_n(E)$.)

It is easy to verify that μ is $P_{n-1,1}(F)$-invariant. To establish our main theorem, we show that the stated gamma factor conditions imply that μ is $G_n(F)$-invariant and hence π is distinguished.

1.2.4 Rankin-Selberg integrals and gamma factors

Fix $m \in \{1, \ldots, n-1\}$. Let π' be an irreducible, generic representation of $G_m(E)$. Let W' be a vector in $\mathcal{W}(\pi', \psi^{-1})$. For integers j and k

in $\{0,\ldots,n-m-1\}$ with $n=j+k+m+1$, we define, following [JPSS83, page 387]:

$$\Psi(s,W,W';j)$$
$$= \int_{N_m(E)\backslash G_m(E)} \int_{M_{j\times m}(E)} W\left[\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{k+1} \end{pmatrix}\right] W'(g)\,|\det g|_E^{s-\frac{n-m}{2}}\,dx\,dg.$$

Given our remarks in §1.2.3, it follows that the integrals are absolutely convergent and that the resulting functions of s are entire. (These integrals converge absolutely on a right half plane and admit meromorphic continuation even if we replace π by any generic irreducible representation of $G_n(E)$.)

The functional equation and gamma factors for these Rankin-Selberg integrals are defined as follows. First, we define some permutation matrices. For each positive integer m, let $w_m \in G_m$ be the permutation matrix with ones on the anti-diagonal, i.e., $w_m = (\delta_{i,m+1-j})$. If $m \le n$, let

$$w_{n,m} = \begin{pmatrix} I_m & 0 \\ 0 & w_{n-m} \end{pmatrix}, \quad \text{and} \quad \alpha^m = \begin{pmatrix} 0 & I_{n-m} \\ I_m & 0 \end{pmatrix}.$$

Note that (as the notation suggests) α^m is indeed the mth power of $\alpha = \alpha^1$. Given a Whittaker function $W' \in \mathcal{W}(\pi',\psi^{-1})$, define $\widetilde{W}' \in \mathcal{W}(\tilde{\pi}',\psi)$ by
$$\widetilde{W}'(g) = W'(w_m\,{}^t g^{-1}),$$
where $\tilde{\pi}'$ is the contragredient of π'.

Given a Whittaker function $W \in \mathcal{W}(\pi,\psi)$, define $W^\bullet \in \mathcal{W}(\tilde{\pi},\psi^{-1})$ by

$$W^\bullet(g) = W(w_n\,{}^t g^{-1}\,w_{n,m}).$$

Note that $\widetilde{\widetilde{W}}' = W'$ and $W^{\bullet\bullet} = W$.

The Rankin-Selberg gamma factors $\gamma(s,\pi\times\pi',\psi)$ are defined by the functional equation

$$\Psi(1-s,W^\bullet,\widetilde{W}';k) = \omega_{\pi'}(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\,\Psi(s,W,W';j) \quad (1)$$

for any pair of non-negative integers (j,k) for which $n=j+k+m+1$. (See [JPSS83, page 391].)

1.3 Rankin-Selberg integrals for distinguished representations

Fix $m \in \{1, \ldots, n-1\}$. For the purpose of studying distinguished representations, it is useful to introduce the following variants of the Rankin-Selberg integrals:

$$\Psi_{m,j}(W) = \int_{N_m(F)\backslash G_m(F)} \int_{M_{j\times m}(E)} W\left[\begin{pmatrix} h & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{k+1} \end{pmatrix}\right] \frac{dx\, dh}{|\det h|_F^{n-m-1}},$$

$$\widetilde{\Psi}_{m,j}(W) = \int_{N_m(F)\backslash G_m(F)} \int_{M_{m\times k}(E)} W\left[\alpha^m \begin{pmatrix} h & 0 & x \\ 0 & I_{j+1} & 0 \\ 0 & 0 & I_k \end{pmatrix}\right] \frac{dx\, dh}{|\det h|_F^{k-j}}.$$

(As before, (j,k) is a pair of non-negative integers such that $n = j+k+m+1$.) Note that these integrals are absolutely convergent, according to the remarks in §1.2.3.

The latter integrals satisfy a variant of the Rankin-Selberg functional equation when π satisfies the following condition:

Condition C(m). $\gamma(1/2, \pi \times \pi', \psi) = 1$ *for every irreducible, unitary, generic distinguished representation* π' *of* $G_m(E)$.

We show in this section that when Condition C(m) is satisfied then the integrals $\Psi_{m,j}$ and $\widetilde{\Psi}_{m,j}$ are related as follows:

Proposition 1.3.1. *If Condition* C(m) *holds then*

$$\Psi_{m,j}(W) = \widetilde{\Psi}_{m,j}(W),$$

for all $j \in \{0, \ldots, n-m-1\}$ *and all* $W \in \mathcal{W}(\pi, \psi)$.

The proof of Proposition 1.3.1 requires one technical ingredient from [Ok97] that we now recall. Let $C_c^\infty(N_m(E)\backslash G_m(E), \psi)$ be the space of smooth functions Φ on $G_m(E)$ such that

$$\Phi(ug) = \psi(u)\Phi(g)$$

for all $u \in N_m(E)$ and $g \in G_m(E)$ whose support has compact image in $N_m(E)\backslash G_m(E)$. (In other words, this is the space of the induced representation $\mathrm{ind}_{N_m(E)}^{G_m(E)}(\psi)$, where we are using smooth induction with compact support.)

The following is [Ok97, Lemma 11.1.2]:

Lemma 1.3.2. *Suppose* $\Phi \in C_c^\infty(N_m(E)\backslash G_m(E), \psi)$. *If*

$$\int_{N_m(E)\backslash G_m(E)} \Phi(g)\, W'(g)\, dg = 0,$$

for every Whittaker function W' *in the Whittaker model* $\mathcal{W}(\pi', \psi^{-1})$ *of every irreducible, unitary, generic distinguished representation* π' *of* $G_m(E)$ *then*

$$\int_{N_m(F)\backslash G_m(F)} \Phi(h)\, dh = 0.$$

Since the proof of Lemma 1.3.2 in [Ok97] is lacking details, we have included an appendix (§4 below) to further clarify things. Note that our assumption that our fields do not have characteristic two is based on a similar assumption in [Ok97].

Proof of Proposition 1.3.1. We begin by expressing the integrals $\Psi(1-s, W^\bullet, \widetilde{W'}; k)$ in a more convenient form. After applying the automorphism $g \mapsto w_m\, {}^t g^{-1}\, w_m$ of $N_m(E)\backslash G_m(E)$ and then $g \mapsto g w_m$, we see that $\Psi(1-s, W^\bullet, \widetilde{W'}; k)$ equals

$$\int_{N_m(E)\backslash G_m(E)} \int_{M_{k\times m}(E)} W\left[\alpha^m \begin{pmatrix} g & 0 & -g\,{}^t x\, w_k \\ 0 & I_{j+1} & 0 \\ 0 & 0 & I_k \end{pmatrix}\right] W'(g)$$
$$|\det g|_E^{s-1+\frac{n-m}{2}}\, dx\, dg$$

or, equivalently,

$$\int_{N_m(E)\backslash G_m(E)} \int_{M_{m\times k}(E)} W\left[\alpha^m \begin{pmatrix} g & 0 & x \\ 0 & I_{j+1} & 0 \\ 0 & 0 & I_k \end{pmatrix}\right] W'(g)$$
$$|\det g|_E^{s-1-k+\frac{n-m}{2}}\, dx\, dg.$$

Now assume Condition C(m) and apply the functional equation (1) at $s = 1/2$ and Lemma 1.3.2 with

$$\Phi(g) = \int_{M_{j\times m}(E)} W\left[\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{k+1} \end{pmatrix}\right] dx\ |\det g|_E^{\frac{-(n-m-1)}{2}}$$
$$- \int_{M_{m\times k}(E)} W\left[\alpha^m \begin{pmatrix} g & 0 & x \\ 0 & I_{j+1} & 0 \\ 0 & 0 & I_k \end{pmatrix}\right] dx \cdot |\det g|_E^{-\frac{k-j}{2}}.$$

□

1.4 Elementary relations between the integrals

The purpose of this section is to develop those relations between the $\Psi_{m,j}$'s and $\widetilde{\Psi}_{m,j}$'s that are required for the proof of our main result.

1.4.1 A relation between the $\Psi_{m,0}$ and $\Psi_{m-1,0}$

Lemma 1.4.1. *For $m \in \{2, \ldots, n-1\}$ and $W \in \mathcal{W}(\pi, \psi)$, one has*

$$\Psi_{m,0}(W) = \int_{F^\times} \int_{F^{m-1}} \Psi_{m-1,0} \left[\begin{pmatrix} I_{m-1} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix} W \right] |b|_F^{1+m-n} dc\, d^\times b.$$

Proof. Let $k = n - m - 1$. Then

$$\begin{aligned}
\Psi_{m,0}(W) &= \int_{A_m(F)} \int_{{}^tN_m(F)} W \left[\begin{pmatrix} \ell a & 0 \\ 0 & I_{n-m} \end{pmatrix} \right] |\det a|_F^{-k}\, d\ell\, da \\
&= \int_{F^\times} \int_{A_{m-1}(F)} \int_{F^{m-1}} \int_{{}^tN_{m-1}(F)} W \left[\begin{pmatrix} \ell & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix} \right] \\
&\qquad |b \det a|_F^{-k} d\ell\, dc\, da\, d^\times b \\
&= \int_{F^\times} \int_{A_{m-1}(F)} \int_{F^{m-1}} \int_{{}^tN_{m-1}(F)} W \left[\begin{pmatrix} \ell a & 0 \\ 0 & I_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix} \right] \\
&\qquad |\det a|_F^{-k-1}\, d\ell\, dc\, da\, |b|_F^{-k} d^\times b \\
&= \int_{F^\times} \int_{F^{m-1}} \int_{N_{m-1}(F)\backslash G_{m-1}(F)} W \left[\begin{pmatrix} h & 0 \\ 0 & 1_{n-m+1} \end{pmatrix} \begin{pmatrix} I_{m-1} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix} \right] \\
&\qquad |\det h|_F^{-k-1}\, dh\, dc\, |b|_F^{-k} d^\times b \\
&= \int_{F^\times} \int_{F^{m-1}} \Psi_{m-1,0} \left[\begin{pmatrix} I_{m-1} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix} W \right] dc\, |b|_F^{1+m-n} d^\times b.
\end{aligned}$$

□

For our main result, we only require the following special case (corresponding to $m = n-1$) of Lemma 1.4.1:

Corollary 1.4.2. *For $W \in \mathcal{W}(\pi, \psi)$, one has*

$$\mu(W) = \int_{F^\times} \int_{F^{n-2}} \Psi_{n-2,0} \left[\begin{pmatrix} I_{n-2} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{pmatrix} W \right] dc\, d^\times b.$$

1.4.2 A relation between $\widetilde{\Psi}_{n-1,0}$ and $\widetilde{\Psi}_{n-2,0}$

Lemma 1.4.3. *For $W \in \mathcal{W}(\pi, \psi)$, one has*

$$\widetilde{\Psi}_{n-1,0}(W) = \int_{F^\times} \int_{F^{n-2}} \widetilde{\Psi}_{n-2,0} \left[\begin{pmatrix} I_{n-2} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha W \right] dc\, d^\times b.$$

Proof. Let

$$\lambda(W) = \int_{F^\times} \int_{F^{n-2}} \widetilde{\Psi}_{n-2,0} \left[\begin{pmatrix} I_{n-2} & 0 & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha W \right] dc\, d^\times b.$$

Using the definitions and some matrix multiplication, one obtains:

$$\lambda(W) = \int_{F^\times} \int_{F^{n-2}} \int_{N_{n-2}(F) \backslash G_{n-2}(F)} \int_{E^{n-2}} W \left[\alpha^{n-1} \begin{pmatrix} 1 & 0 & 0 \\ x & h & 0 \\ 0 & c & b \end{pmatrix} \right] |\det h|_F^{-1}\, dx\, dh\, dc\, d^\times b.$$

Sending b to b^{-1} and using the assumption $\omega_\pi | F^\times = 1$ gives:

$$\begin{aligned} \lambda(W) &= \int_{F^\times} \int_{F^{n-2}} \int_{N_{n-2}(F) \backslash G_{n-2}(F)} \int_{E^{n-2}} \\ &\qquad W \left[\alpha^{n-1} \begin{pmatrix} 1 & 0 & 0 \\ x & h & 0 \\ 0 & c & b^{-1} \end{pmatrix} \right] |\det h|_F^{-1}\, dx\, dh\, dc\, d^\times b \\ &= \int_{F^\times} \int_{F^{n-2}} \int_{N_{n-2}(F) \backslash G_{n-2}(F)} \int_{E^{n-2}} \\ &\qquad W \left[\alpha^{n-1} \begin{pmatrix} b & 0 & 0 \\ xb & hb & 0 \\ 0 & cb & 1 \end{pmatrix} \right] |\det h|_F^{-1}\, dx\, dh\, dc\, d^\times b. \end{aligned}$$

Some obvious changes of variables yield:

$$\begin{aligned}\lambda(W) &= \int_{F^\times}\int_{F^{n-2}}\int_{N_{n-2}(F)\backslash G_{n-2}(F)}\int_{E^{n-2}} \\ &\qquad W\left[\alpha^{n-1}\begin{pmatrix} b & 0 & 0 \\ xb & h & 0 \\ 0 & cb & 1\end{pmatrix}\right] |b|_F^{n-2}|\det h|_F^{-1}\, dx\, dh\, dc\, d^\times b \\ &= \int_{F^\times}\int_{F^{n-2}}\int_{N_{n-2}(F)\backslash G_{n-2}(F)}\int_{E^{n-2}} \\ &\qquad W\left[\alpha^{n-1}\begin{pmatrix} b & 0 & 0 \\ xb & h & 0 \\ 0 & c & 1\end{pmatrix}\right] |\det h|_F^{-1}\, dx\, dh\, dc\, d^\times b \\ &= \int_{F^\times}\int_{F^{n-2}}\int_{N_{n-2}(F)\backslash G_{n-2}(F)}\int_{E^{n-2}} \\ &\qquad W\left[\alpha^{n-1}\begin{pmatrix} b & 0 & 0 \\ xb & h & 0 \\ 0 & ch & 1\end{pmatrix}\right] dx\, dh\, dc\, d^\times b \\ &= \int_{F^\times}\int_{F^{n-2}}\int_{N_{n-2}(F)\backslash G_{n-2}(F)}\int_{E^{n-2}} \\ &\qquad W\left[\alpha^{n-1}\begin{pmatrix} 1 & 0 & 0 \\ x & I_{n-2} & 0 \\ 0 & c & 1\end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1\end{pmatrix}\right] dx\, dh\, dc\, d^\times b.\end{aligned}$$

The matrix identity

$$\alpha^{n-1}\begin{pmatrix} 1 & 0 & 0 \\ x & I_{n-2} & 0 \\ 0 & c & 1\end{pmatrix} = \begin{pmatrix} 1 & -cx & c \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\alpha^{n-1}\begin{pmatrix} 1 & 0 & 0 \\ x & I_{n-2} & 0 \\ 0 & 0 & 1\end{pmatrix}$$

implies

$$\begin{aligned}\lambda(W) &= \int_{F^\times}\int_{F^{n-2}}\int_{N_{n-2}(F)\backslash G_{n-2}(F)}\int_{E^{n-2}} \\ &\qquad W\left[\alpha^{n-1}\begin{pmatrix} 1 & 0 & 0 \\ x & I_{n-2} & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1\end{pmatrix}\right]\psi(-cx)\, dx\, dh\, dc\, d^\times b.\end{aligned}$$

Next we apply the identity (see Lemma 2.2 [Off11])

$$\int_{F^{n-2}}\int_{E^{n-2}} f(x)\psi(-cx)\, dx\, dc = \int_{F^{n-2}} f(x)\, dx$$

for $f \in C_c^\infty(E^{n-2})$ to get

$$\lambda(W) = \int_{F^\times} \int_{N_{n-2}(F)\backslash G_{n-2}(F)} \int_{F^{n-2}} W\left[\alpha^{n-1}\begin{pmatrix}1 & 0 & 0\\ x & I_{n-2} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}b & 0 & 0\\ 0 & h & 0\\ 0 & 0 & 1\end{pmatrix}\right] dx\, dh\, d^\times b.$$

Our assertion follows from the identity

$$\int_{N_{n-1}(F)\backslash G_{n-1}(F)} f(h)\ dh = \int_{F^\times} \int_{N_{n-2}(F)\backslash G_{n-2}(F)} \int_{F^{n-2}} f\left[\begin{pmatrix}1 & 0\\ x & I_{n-2}\end{pmatrix}\begin{pmatrix}b & 0\\ 0 & h\end{pmatrix}\right] dx\, dh\, d^\times b$$

for $f \in C_c^\infty(N_{n-1}(F)\backslash G_{n-1}(F), \psi)$. □

1.5 Our main theorem

We now come to our main theorem:

Theorem 1.5.1. *If π is an irreducible, supercuspidal representation of $G_n(E)$ that satisfies $C(n-2)$ and if $\omega_\pi | F^\times = 1$ then π is distinguished.*

Proof. Corollary 1.4.2 implies that

$$\mu(W) = \Psi_{n-1,0}(W) = \int_{F^\times} \int_{F^{n-2}} \Psi_{n-2,0}\left[\begin{pmatrix}I_{n-2} & 0 & 0\\ c & b & 0\\ 0 & 0 & 1\end{pmatrix} W\right] dc\, d^\times b,$$

whereas Lemma 1.4.3 says

$$\widetilde{\Psi}_{n-1,0}(\alpha^{-1}W) = \int_{F^\times} \int_{F^{n-2}} \widetilde{\Psi}_{n-2,0}\left[\begin{pmatrix}I_{n-2} & 0 & 0\\ c & b & 0\\ 0 & 0 & 1\end{pmatrix} W\right] dc\, d^\times b.$$

But, according to Proposition 1.3.1, Condition C$(n-2)$ implies $\Psi_{n-2,0} = \widetilde{\Psi}_{n-2,0}$. Therefore, Condition C$(n-2)$ implies

$$\mu(W) = \widetilde{\Psi}_{n-1,0}(\alpha^{-1}W).$$

Since the linear form $W \mapsto \widetilde{\Psi}_{n-1,0}(W)$ is invariant under ${}^tP_{n-1,1}(F)$, it follows that μ is invariant under $\alpha \cdot {}^tP_{n-1,1}(F) \cdot \alpha^{-1}$. But the latter group consists of the matrices in $G_n(F)$ of the form

$$\begin{pmatrix} h & 0 & x \\ b & v & y \\ c & 0 & z \end{pmatrix},$$

where h is an $(n-2)\times(n-2)$ matrix. Since this contains matrices not in $P_{n-1,1}(F)$, it follows from Lemma 1.2.1 that μ is invariant under $G_n(F)$. □

1.6 Ok's local converse theorem

In his thesis [Ok97], Ok proves the following local converse theorem for distinguished representations [Ok97, Main Theorem III]:

> Let π be an irreducible, supercuspidal representation of $G_n(E)$, where $n \geq 2$. If $\omega_\pi|F^\times = 1$ and if $\gamma(1/2, \pi\times\pi', \psi) = 1$ for all distinguished, irreducible, unitary, generic representations π' of $G_{n-1}(E)$ then π is distinguished.

Whereas, our proof of Theorem 1.5.1 reduces to showing that C($n-$2) implies

$$\mu(W) = \widetilde{\Psi}_{n-1,0}(\alpha^{-1}W),$$

Ok's proof boils down to showing that C($n-1$) implies

$$\mu(W) = \widetilde{\Psi}_{n-1,0}(W).$$

Indeed, once Ok establishes [Ok97, Lemma 11.1.2] (stated as Lemma 1.3.2 above), it immediately follows that

$$\mu(W) = \Psi_{n-1,0}(W) = \widetilde{\Psi}_{n-1,0}(W).$$

But since $\widetilde{\Psi}_{n-1,0}$ is obviously ${}^tP_{n-1,1}(F)$-invariant and since $G_n(F)$ is generated by $P_{n-1,1}(F)$ and ${}^tP_{n-1,1}(F)$, one deduces that μ is $G_n(F)$-invariant and thus π is H-distinguished.

1.7 An alternate GL_3 proof

For the $n = 3$ case of Theorem 1.5.1, the authors presented a different proof at a January 12, 2013 talk "Local converse theorems for distinguished representations" at the San Diego meeting of the American Mathematical Society. Instead of using Lemma 1.3.2, we use the elementary identities

$$\int_F f(x)\ dx = \int_F \hat{f}(x)\ dx = \int_F \int_E f(u)\,\psi(xu)\,du\,dx, \tag{2}$$

for $f \in C_c^\infty(E)$, and

$$\sum_{\chi \in (E^\times/F^\times)^\wedge} \int_{E^\times} f(y)\,\chi(y)\ d^\times y = \int_{F^\times} f(y)\ d^\times y \tag{3}$$

for $f \in C_c^\infty(E^\times)$. Here, $(E^\times/F^\times)^\wedge$ is the unitary dual of $E^\times/F^\times$ and the formula is a consequence of the Fourier inversion formula for the compact abelian group $E^\times/F^\times$. It is therefore valid whenever $f : E^\times \to \mathbb{C}$ is a continuous function that satisfies

$$f \in L^1(E^\times), \quad \forall a \in E^\times \quad \int_{F^\times} |f(ax)|\ d^\times x < \infty. \tag{4}$$

Applying estimates on Whittaker functions due to Lapid-Mao [LM14], we obtain our local converse theorem for any irreducible, unitary, generic representation of $\mathrm{GL}_3(E)$.

Theorem 1.7.1. *If π is an irreducible, unitary, generic representation of $G_3(E)$ that satisfies $C(1)$ and if $\omega_\pi|F^\times = 1$ then π is distinguished.*

1.7.1 The proof

We remark that the absolute convergence of the integrals follows from the estimates on Whittaker functions given in [LM14, Corollary 2.2].

Let $W \in \mathcal{W}(\pi, \psi)$. Using the Bruhat decomposition of $G_3(F)$, we can rewrite the integral defining μ as follows:

$$\mu(W) = \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} y & 0 & 0 \\ x & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \frac{d^\times y}{|y|_F} d^\times b\,dx.$$

Now, using Equation (4) above, we get the following expression for $\mu(W)$

$$\int_F \int_{F^\times} \sum_{\chi \in (E^\times/F^\times)^\wedge} \int_{E^\times} W\left[\begin{pmatrix} y & 0 & 0 \\ x & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \chi(y) \frac{d^\times y}{|y|_E^{1/2}} d^\times b\,dx.$$

The latter integral is the same as

$$\int_F \int_{F^\times} \sum_{\chi \in (E^\times/F^\times)^\wedge} \Psi\left(\frac{1}{2}, \begin{pmatrix} 1 & 0 & 0 \\ x & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot W, \chi, 0\right) d^\times b\, dx.$$

Now if $\gamma(1/2, \pi \times \chi, \psi) = 1$ then the functional equation (1) and the most obvious changes of variables yield

$$\Psi\left(\frac{1}{2}, \begin{pmatrix} 1 & 0 & 0 \\ x & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot W, \chi, 0\right) = \int_{E^\times} \int_E W\left[\begin{pmatrix} x & b & 0 \\ 0 & 0 & 1 \\ a & 0 & u \end{pmatrix}\right] du\, \chi(a) \frac{d^\times a}{|a|_E^{1/2}}.$$

We obtain the following expression for $\mu(W)$:

$$\int_F \int_{F^\times} \sum_{\chi \in (E^\times/F^\times)^\wedge} \int_{E^\times} \int_E W\left[\begin{pmatrix} x & b & 0 \\ 0 & 0 & 1 \\ a & 0 & u \end{pmatrix}\right] du\ \chi(a) \frac{d^\times a}{|a|_E^{1/2}}\, d^\times b\, dx.$$

Applying (4) again this simplifies to:

$$\int_F \int_{F^\times} \int_{F^\times} \int_E W\left[\begin{pmatrix} x & b & 0 \\ 0 & 0 & 1 \\ a & 0 & u \end{pmatrix}\right] du\ \frac{d^\times a}{|a|_F}\, d^\times b\, dx.$$

Send x to ax and use the matrix identity:

$$\begin{pmatrix} ax & b & 0 \\ 0 & 0 & 1 \\ a & 0 & u \end{pmatrix} = \begin{pmatrix} 1 & -xu & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 1 \\ a & 0 & u \end{pmatrix}$$

to deduce

$$\mu(W) = \int_F \int_{F^\times} \int_{F^\times} \int_E W\left[\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 1 \\ a & 0 & u \end{pmatrix}\right] \psi(-xu)\, du\, d^\times a\, d^\times b\, dx.$$

Using Equation (2), we get

$$\mu(W) = \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 1 \\ a & 0 & x \end{pmatrix}\right] d^\times a\, d^\times b\, dx.$$

Now sequentially apply the changes of variables $b \mapsto bx$, $a \mapsto -ax^2$, $x \mapsto -a^{-1}x$. The matrix in the integral becomes

$$\begin{pmatrix} 0 & -a^{-1}bx & 0 \\ 0 & 0 & 1 \\ -a^{-1}x^2 & 0 & -a^{-1}x \end{pmatrix}$$

and it is equal to

$$\begin{pmatrix} -a^{-1}x & 0 & 0 \\ 0 & -a^{-1}x & 1 \\ 0 & 0 & -a^{-1}x \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}.$$

We obtain:

$$\mu(W) = \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix} \right] \frac{d^\times a}{|a|_F} d^\times b \, dx.$$

This shows that μ is invariant under matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$, with $x \in F$. To complete the proof that μ is $G_3(F)$-invariant, one observes that we have shown that μ is invariant under a set of generators of $G_3(F)$. Indeed, μ is invariant under $P_{2,1}(F)$ and the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$, with $x \in F$. The invariance under $G_3(F)$ follows from Lemma 1.2.1.

1.7.2 Connection with the proof of Theorem 1.5.1

We have

$$\begin{aligned}
\mu(W) &= \widetilde{\Psi}_{2,0}(\alpha^{-1}W) \\
&= \int_{N_2(F)\backslash G_2(F)} W\left[\alpha^{-1}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1}\right] dh \\
&= \int_F \int_{F^\times} \int_{F^\times} W\left[\alpha^{-1}\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\alpha^{-1}\right] d^\times a\, d^\times b\, dx \\
&= \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & x \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}\right] d^\times a\, d^\times b\, dx \\
&= \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & x \end{pmatrix}\begin{pmatrix} a^{-1}b & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] d^\times a\, d^\times b\, dx \\
&= \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & x \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] d^\times a\, d^\times b\, dx.
\end{aligned}$$

Now we use the following matrix identity

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & x \end{pmatrix} = x\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x^{-1} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{pmatrix}\begin{pmatrix} -x^{-2} & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to get the formula

$$\mu(W) = \int_F \int_{F^\times} \int_{F^\times} W\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] d^\times a\, d^\times b\, dx.$$

This is essentially the formula in our GL_3 proof. It directly exhibits the invariance of μ under matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2 A new proof of a local converse theorem

The statement and proof of Theorem 1.5.1 assume that E is a quadratic extension of F, however, everything carries over in a straightforward way to the case of $E = F \oplus F$. In other words, our approach treats both cases in a uniform way. In writing this paper, the authors considered discussing both cases simultaneously, but that would have required continual remarks on how things should be interpreted in the case $E = F \oplus F$. We have chosen to collect these various remarks in this section. We emphasize though that, in every case, the $E = F \oplus F$ interpretation is quite straightforward, if not immediately obvious.

The key points are as follows:

- The analogue of the nontrivial Galois automorphism of E/F is the automorphism $(x, y) \mapsto (y, x)$ of E.
- $G_m(E) = \mathrm{GL}_m(F) \times \mathrm{GL}_m(F)$ and $G_m(F) \cong \mathrm{GL}_m(F)$ embeds in $G_m(E)$ diagonally. The same thing applies to the various subgroups defined in §1.2.1.
- The character ψ of E has the form $\psi_F \times \psi_F^{-1}$, where ψ_F is an arbitrary nontrivial character of F.
- An irreducible representation π of $G_m(E)$ is a tensor product of two irreducible representations of $G_m(F)$. It is convenient to write π as $\pi_1 \times \tilde{\pi}_2$, where π_1 and π_2 are irreducible representations of $G_m(F)$ and $\tilde{\pi}_2$ is the contragredient of π_2. The reason for this is that π is then distinguished precisely when π_1 and π_2 are equivalent. In other words, the question of whether π is distinguished translates into a question of whether two representations of $G_m(F)$ are equivalent.
- Note that $\omega_\pi = \omega_{\pi_1} \times \omega_{\pi_2}^{-1}$. Therefore, the condition $\omega_\pi | F^\times = 1$ simply says that π_1 and π_2 have the same central character.
- We have
$$\mathcal{W}(\pi, \psi) = \mathcal{W}(\pi_1, \psi_F) \otimes \mathcal{W}(\tilde{\pi}_2, \psi_F^{-1})$$
and we focus on those Whittaker functions W that are elementary tensors $W_1 \otimes W_2^\bullet$, where $W_2 \mapsto W_2^\bullet$ is defined as in §1.2.4.
- Similarly, when $\pi' = \tau \times \tilde{\tau}$ is a distinguished, generic representation of $G_m(E)$, we have
$$\mathcal{W}(\pi', \psi^{-1}) = \mathcal{W}(\tau, \psi_F^{-1}) \otimes \mathcal{W}(\tilde{\tau}, \psi_F)$$

and we use elementary tensors $W' = W'_1 \otimes \widetilde{W}'_2$.

- Given $g = (g_1, g_2) \in G_m(E)$, we take $|\det g|_E = |\det(g_1 g_2)|_F$.
- Define $W^\bullet = W_1^\bullet \otimes W_2$ and $\widetilde{W}' = \widetilde{W}'_1 \otimes W'_2$.
- One can now define $\Psi(s, W, W'; j)$ using the integrals in 1.2.4 together with the remarks above. This gives

$$\Psi(s, W, W'; j) = \Psi(s, W_1, W'_1; j)\Psi(s, W_2^\bullet, \widetilde{W}'_2; j)$$

and

$$\Psi(1-s, W^\bullet, \widetilde{W}'; k) = \Psi(1-s, W_1^\bullet, \widetilde{W}'_1; k)\Psi(1-s, W_2, W'_2; k).$$

In order to get the same functional equation as before, we need to take

$$\gamma(s, \pi\times\pi', \psi) = \frac{\gamma(s, \pi_1 \times \tau, \psi_F)}{\gamma(1-s, \pi_2 \times \tau, \psi_F)} (= \gamma(s, \pi_1\times\tau, \psi_F)\gamma(s, \tilde{\pi}_2\times\tilde{\tau}, \psi_F^{-1}))).$$

- Note that the condition $\gamma(1/2, \pi \times \pi', \psi) = 1$ becomes

$$\gamma(1/2, \pi_1 \times \tau, \psi_F) = \gamma(1/2, \pi_2 \times \tau, \psi_F).$$

If the latter condition holds when τ is replaced by arbitrary twists by unramified characters of $F^\times$ then we have

$$\gamma(s, \pi_1 \times \tau, \psi_F) = \gamma(s, \pi_2 \times \tau, \psi_F)$$

for all $s \in \mathbb{C}$.

- We now give some background for the desired analogue of Lemma 1.3.2. Let $L^2(N_m(F)\backslash G_m(F), \psi_F)$ be the unitary representation of $G_m(F)$ induced from the character ψ_F of $N_m(F)$. Then we have a direct integral decomposition

$$L^2(N_m(F)\backslash G_m(F), \psi_F) = \int \pi \, d\mu(\pi),$$

where the representations π are (topologically) irreducible and unitary (in fact tempered). The corresponding inner product formula is

$$\begin{aligned}(\Phi_1, \Phi_2) &= \int_{N_m(F)\backslash G_m(F)} \Phi_1(h)\, \overline{\Phi_2(h)}\, dh \\ &= \int (\Phi_1(\pi), \Phi_2(\pi))_\pi \, d\mu(\pi).\end{aligned}$$

We want to apply this when Φ_1 and Φ_2 lie in the corresponding space $C_c^\infty(N_m(F)\backslash G_m(F), \psi_F)$ of smooth functions supported in a compact subset of $N_m(F)\backslash G_m(F)$. Suppose (Φ_1, Φ_2) is nonzero. Then we can choose π such that $(\Phi_1(\pi), \Phi_2(\pi))_\pi$ is nonzero. Let π^∞ be the representation obtained by restricting π to the smooth vectors in the space of π. This is a smooth, irreducible, unitary, admissible, generic representation of $G_m(F)$. (See [BZ76, Theorem 4.21].) We can take the space of π^∞ to be its ψ_F-Whittaker model. Choose a compact open subgroup K of $G_m(F)$ that fixes both Φ_1 and Φ_2. If $\{W\}$ is an orthonormal basis for K-fixed vectors for π^∞ then, up to a nonzero constant, we have

$$(\Phi_1(\pi), \Phi_2(\pi))_\pi = \sum_W (\Phi_1, W)\ (W, \Phi_2)$$

and, moreover, the latter sum is finite. For some W, the integrals

$$(\Phi_1, W) = \int_{N_m(F)\backslash G_m(F)} \Phi_1(h)\,\overline{W(h)}\,dh$$

and

$$(W, \Phi_2) = \int_{N_m(F)\backslash G_m(F)} W(h)\,\overline{\Phi_2(h)}\,dh$$

must be nonzero. Note that the fact that W lies in $\mathcal{W}(\pi^\infty, \psi_F)$ implies that $\overline{W}$ lies in $\mathcal{W}(\widetilde{\pi^\infty}, \psi_F^{-1})$.

- Now we discuss the application of the previous discussion (See also [JS85, Lemma 3.2] and [JPSS81, Lemma 3.5]). Take $\Phi = \Phi_1 \otimes \overline{\Phi}_2$. This lies in

$$C_c^\infty(N_m(E)\backslash G_m(E), \psi) \\ = C_c^\infty(N_m(F)\backslash G_m(F), \psi_F) \otimes C_c^\infty(N_m(F)\backslash G_m(F), \psi_F^{-1}).$$

Now, in the previous discussion, take $\tau = \widetilde{\pi^\infty}$ and let $W' = W_1' \otimes \widetilde{W_2'} = W \otimes \overline{W}$. Then we have just outlined the proof of (the contrapositive of) the natural analogue of Lemma 1.3.2. The condition

$$\int_{N_m(F)\backslash G_m(F)} \Phi(h)\,dh = 0$$

translates to

$$(\Phi_1, \Phi_2) = 0.$$

So we have assumed the latter condition does not hold and then shown that

$$\int_{N_m(E)\backslash G_m(E)} \Phi(g)\, W'(g)\, dg = (\Phi_1, W)(W, \Phi_2) \neq 0.$$

- Lemma 1.3.2 is applied in the proof of Proposition 1.3.1. There we take

$$\begin{aligned}\Phi(g) \quad &= \quad \int_{M_{j\times m}(E)} W\left[\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{k+1}\end{pmatrix}\right] dx\ |\det g|_E^{\frac{-(n-m-1)}{2}} \\ &\quad - \int_{M_{m\times k}(E)} W\left[\alpha^m \begin{pmatrix} g & 0 & x \\ 0 & I_{j+1} & 0 \\ 0 & 0 & I_k\end{pmatrix}\right] dx \cdot |\det g|_E^{-\frac{k-j}{2}}.\end{aligned}$$

  In the present context, we use the same formula to define Φ, but now $W = W_1 \otimes W_2^{\bullet}$. Therefore, our Φ is not an elementary tensor but rather a difference of two elementary tensors.

- With $W = W_1 \otimes \widetilde{W_2^{\bullet}}$, we have

$$\begin{aligned}\Psi_{m,j}(W) = &\int_{N_m(F)\backslash G_m(F)} \partial W_1\left[\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right] \\ &\qquad\qquad \partial W_2^{\bullet}\left[\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right] \frac{dh}{|\det h|_F^{n-m-1}}, \\ \widetilde{\Psi}_{m,j}(W) = &\int_{N_m(F)\backslash G_m(F)} \partial W_1\left[\alpha^m \begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right] \\ &\qquad\qquad \partial W_2^{\bullet}\left[\alpha^m \begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right] |\det h|_F^{n-m-1}\, dh.\end{aligned}$$

  where $\partial = \partial_{m,j}$ and $\tilde{\partial} = \tilde{\partial}_{m,j}$ are given by

$$\partial W_1 = \int_{M_{j\times m}(F)} \begin{pmatrix} I_m & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{k+1}\end{pmatrix} \cdot W_1\ dx,$$

$$\tilde{\partial} W_1 = \int_{M_{m\times k}(F)} \begin{pmatrix} I_m & 0 & x \\ 0 & I_{j+1} & 0 \\ 0 & 0 & I_k\end{pmatrix} \cdot W_1\ dx.$$

- We remark that the Rankin-Selberg gamma factors for $G_n(F) \times G_m(F)$ satisfy

$$\gamma(s, \tau \times \tau', \psi_F) = \gamma(1/2, \tau \times (\tau' \otimes |\ |_F^{s-(1/2)}), \psi_F),$$

where $(\tau' \otimes |\ |_F^s)(h) = \tau'(h)\,|\det h|_F^s$, and thus Condition C($m$) can be restated as:

$$\gamma(s, \pi_1 \times \tau, \psi_F) = \gamma(s, \pi_2 \times \tau, \psi_F)$$

for all irreducible, unitary, generic representations τ of $G_m(F)$. This shows that our Condition C(m) is equivalent to the condition used in the statements of traditional local converse theorems, where the gamma factors are considered for all values of s, not just $s = 1/2$.

- The rest of the proof follows the same calculations as in the case in which E/F is a quadratic extension.

3 Extensions and variants of the main result

Above, we have proved the following, both when E is a quadratic extension of F and when it is $F \oplus F$:

> Suppose π is an irreducible, supercuspidal representation of $G_n(E)$ with $\omega_\pi|F^\times = 1$. Fix ψ with $\psi|F = 1$ and assume $\gamma(1/2, \pi \times \pi', \psi) = 1$ for all distinguished, irreducible, unitary, generic representations π' of $G_{n-2}(E)$. Then π is distinguished.

There are natural, but more difficult, variants of this problem obtained by either (a) broadening the class of π considered, or (b) restricting the class of π' allowed.

Ideally, we could reduce the more difficult problems to the case that we have already treated. When $E = F \oplus F$, one can use multiplicativity of gamma factors and the Langlands/Zelevinsky decomposition to obtain the following (known) result from the above result:

> Assume ψ_F is a nontrivial character of F. Let π_1 and π_2 be irreducible, generic representations of $G_n(F)$. If the central characters of π_1 and π_2 are identical and if
>
> $$\gamma(s, \pi_1 \times \tau, \psi_F) = \gamma(s, \pi_2 \times \tau, \psi_F)$$

for all irreducible, unitary, supercuspidal representations τ of $G_{n-2}(F)$ then π_1 and π_2 are equivalent.

To obtain the latter result from the previous result, one can use the same argument that appears in [JNS, Section 2.4] in a different, but similar, context.

In the case in which E/F is quadratic, we know of no satisfactory reduction arguments similar to those given in [JNS]. We spend the remainder of this section discussing preliminary results in this direction. We recall from [Fli91, Proposition 12] that if π is an irreducible, distinguished representation of $G_n(E)$ then $\tilde{\pi} \simeq \bar{\pi}$. We further recall that for any generic representations π of $G_n(E)$ and τ of $G_m(E)$ the functional equations of Rankin-Selberg integrals, applied twice, give

$$\epsilon(s, \pi \times \tau, \psi)\epsilon(1-s, \tilde{\pi} \times \tilde{\tau}, \psi^{-1}) = \gamma(s, \pi \times \tau, \psi)\gamma(1-s, \tilde{\pi} \times \tilde{\tau}, \psi^{-1}) = 1. \tag{5}$$

Recall further that

$$L(s, \bar{\pi} \times \bar{\tau}) = L(s, \pi, \tau), \quad \gamma(s, \bar{\pi} \times \bar{\tau}, \bar{\psi}) = \gamma(s, \pi \times \tau, \psi) \tag{6}$$

and $\bar{\psi} = \psi^{-1}$.

Lemma 3.0.2. *Let π be an irreducible representation of $G_n(E)$ satisfying $\tilde{\pi} \simeq \bar{\pi}$ but not satisfying Condition* C(m) *for some $m \leq n-1$. Then there exists a distinguished, discrete series representation π' of $G_r(E)$ for some $r \leq m$ such that $\gamma(1/2, \pi \times \pi', \psi) = -1$.*

Proof. By assumption, there exists a distinguished, irreducible, unitary, generic representation π' of $G_m(E)$ such that $\gamma(1/2, \pi \times \pi', \psi) \neq 1$. By (5) and (6) we have $\gamma(1/2, \pi \times \pi', \psi)^2 = 1$ and therefore $\gamma(1/2, \pi \times \pi', \psi) = -1$. It follows from [Mat11, Theorem 5.2] that π is obtained by normalized parabolic induction from

$$\bar{\tau}_1 \otimes \tilde{\tau}_1 \otimes \cdots \otimes \bar{\tau}_k \otimes \tilde{\tau}_k \otimes \delta_1 \otimes \cdots \otimes \delta_\ell,$$

where each τ_i is an essentially discrete series representation of $G_{n_i}(E)$ and each δ_j is a distinguished, discrete series representation of $G_{m_j}(E)$ and $n = 2\sum_i n_i + \sum_j m_j$.

We have

$$\gamma(s, \pi \times \pi', \psi) = \left[\prod_{i=1}^{k} \gamma(s, \pi \times \bar{\tau}_i, \psi)\gamma(s, \pi \times \tilde{\tau}_i, \psi)\right] \prod_{i=1}^{\ell} \gamma(s, \pi \times \delta_i, \psi).$$

By (5) and (6) we have

$$\gamma\left(1/2, \pi \times \bar{\tau}_i, \psi\right) \gamma\left(1/2, \pi \times \tilde{\tau}_i, \psi\right) = 1$$

and $\gamma(1/2, \pi \times \delta_i, \psi) \in \{\pm 1\}$ for all i. The lemma follows. □

We now turn our attention to the case of discrete series representations π of $G_n(E)$ such that $\tilde{\pi} \simeq \bar{\pi}$ and the question of detecting whether or not π is distinguished from its gamma factors. We know from the main result of [Off11] that if π is distinguished then $\gamma(1/2, \pi \times \pi', \psi) = 1$ for all distinguished, irreducible, generic, unitary representations of $G_m(E)$ for $m \leq n$. So we are interested in those π in the discrete series that are not distinguished and are not supercuspidal (but satisfy $\tilde{\pi} \simeq \bar{\pi}$).

The discrete series representations may be described as follows. Suppose m and k are positive integers such that $mk = n$. If ρ is a supercuspidal representation of $G_m(E)$, we define the "generalized Steinberg representation" $\mathrm{St}(k, \rho)$ to be the unique irreducible quotient of the representation of $G_n(E)$ obtained from

$$|\ |_E^{(1-k)/2} \rho \otimes \cdots \otimes |\ |_E^{(k-1)/2} \rho$$

by normalized parabolic induction.

Proposition 3.0.3. *Suppose $\pi = \mathrm{St}(k, \rho)$ is a (non-supercuspidal) generalized Steinberg representation of $G_n(E)$ such that $\tilde{\pi} \simeq \bar{\pi}$ and $\omega_\pi | F^\times = 1$. If π is not distinguished then $\gamma(1/2, \pi \times \pi', \psi) = -1$ for some $m \leq \lfloor n/2 \rfloor$ and some irreducible, discrete series representation π' of $G_m(E)$.*

The first step in proving Proposition 3.0.3 is to recall the relation between the distinction of $\mathrm{St}(k, \rho)$ and the distinction of ρ. Let $\eta_{E/F}$ be the nontrivial character of $F^\times / N_{E/F}(E^\times)$. We say π is $\eta_{E/F}$-distinguished if $\mathrm{Hom}_{G_n(F)}(\pi, \eta_{E/F})$ is nonzero.

Lemma 3.0.4.

1. *If π is a discrete series representation of $G_n(E)$ such that $\tilde{\pi} \simeq \bar{\pi}$ then π is either distinguished or $\eta_{E/F}$-distinguished but not both. If $\omega_\pi | F^\times = 1$ then this means that when n is odd π is distinguished, but $\pi \otimes \eta_{E/F}$ cannot be.* ([Mat09, Proposition 2.18], [Kab04, Main Theorem])
2. *Suppose that $\bar{\rho} \simeq \tilde{\rho}$. The generalized Steinberg representation $\mathrm{St}(k, \rho)$ is distinguished if and only if ρ is $\eta_{E/F}^{k-1}$-distinguished if and only if ρ is not $\eta_{E/F}^{k}$-distinguished.* ([Mat09, Corollary 4.2])

The next step is to assemble the required facts about Rankin-Selberg L-functions.

Lemma 3.0.5. *Let π is an irreducible representation of $G_n(E)$.*

1. *Then*
$$\{q_E^{-u} \,:\, \pi \otimes |\cdot|_E^u \simeq \pi\}$$
is a (cyclic) subgroup of the group of complex n-th roots of unity and hence it is equal to the group of k_π-th roots of unity for some unique positive divisor k_π of n.

2. *If π is also supercuspidal then*
$$L(s, \pi \times \tilde{\pi}) = \frac{1}{1 - q_E^{-k_\pi s}}.$$

3. *If $\tau = \tilde{\pi} \otimes |\cdot|_E^u$ is an unramified twist of $\tilde{\pi}$ then*
$$L(s, \pi \times \tau) = L(s + u, \pi \times \tilde{\pi}).$$

4. *If $\tau = \tilde{\pi} \otimes |\cdot|_E^u$ is supercuspidal,*
$$\frac{L(s, \pi \times \tau)}{L(-s, \pi \times \tau)} = -q_E^{k_\pi (s+u)}$$
and thus
$$\lim_{s \to -u} \frac{L(s, \pi \times \tau)}{L(-s, \pi \times \tau)} = -1.$$

5. *If τ is a supercuspidal representation of $G_m(E)$ for some m and if τ is not an unramified twist of $\tilde{\pi}$ then $L(s, \pi \times \tau) = 1$.*

Proof. If π is an irreducible representation of $G_n(E)$ then the set
$$\{q_E^{-u} \,:\, \pi \otimes |\cdot|_E^u \simeq \pi\}$$
is clearly a group. Looking at the central characters, we see that the latter group is a subgroup of the group of complex n-th roots of unity. So this group must be cyclic or, in other words, there must be a positive divisor k_π of n such that this group is just the group of k_π-th roots of unity. So we have
$$x^{k_\pi} - 1 = \prod_{\{q_E^{-u} : \pi \otimes |\cdot|_E^u \simeq \pi\}} (x - q_E^{-u}).$$

Now plug in q_E^s for x and multiply each factor by q_E^{-s} to get

$$1 - q_E^{-k_\pi s} = \prod_{\{q_E^{-u}: \pi\otimes|\cdot|_E^u \simeq \pi\}} (1 - q_E^{-u-s}).$$

Taking reciprocals gives

$$\frac{1}{1 - q_E^{-k_\pi s}} = \prod_{\{q_E^{-u}:\ \pi\otimes|\cdot|_E^u \simeq \pi\}} \frac{1}{1 - q_E^{-u-s}}.$$

Now assume π is supercuspidal. Then the right hand side of the latter identity is $L(s, \pi \times \tilde{\pi})$, according to [JPSS83, Proposition 8.1]. So in the supercuspidal case, we have

$$L(s, \pi \times \tilde{\pi}) = \frac{1}{1 - q_E^{-k_\pi s}}.$$

It follows that

$$\frac{L(s, \pi \times \tilde{\pi})}{L(-s, \pi \times \tilde{\pi})} = -q_E^{k_\pi s}.$$

Hence

$$\lim_{s\to 0} \frac{L(s, \pi \times \tilde{\pi})}{L(-s, \pi \times \tilde{\pi})} = -1.$$

Now consider the general formula

$$L(s, \pi \times \tau) = \prod_{\{q_E^{-u}:\ \tilde{\pi}\otimes|\cdot|_E^u \simeq \tau\}} \frac{1}{1 - q_E^{-u-s}}$$

that holds for supercuspidal π and τ. Suppose there exists u_0 such that $\tilde{\pi} \otimes |\cdot|_E^{u_0} \simeq \tau$. Now suppose u satisfies $\tau \otimes |\cdot|_E^u \simeq \tau$. Then $\tilde{\pi} \otimes |\cdot|_E^{u_0+u} \simeq \tau$. We see that

$$L(s, \pi \times (\tilde{\pi}|\cdot|_E^{u_0})) = L(s + u_0, \pi \times \tilde{\pi}).$$

It follows that

$$\frac{L(s, \pi \times (\tilde{\pi}|\cdot|_E^{u_0}))}{L(-s, \pi \times (\tilde{\pi}|\cdot|_E^{u_0}))} = -q_E^{k_\pi(s+u_0)}.$$

Hence

$$\lim_{s\to -u_0} \frac{L(s, \pi \times (\tilde{\pi}|\cdot|_E^{u_0}))}{L(-s, \pi \times (\tilde{\pi}|\cdot|_E^{u_0}))} = -1.$$

The last part is part of [JPSS83, Proposition 8.1]. □

Proof of Proposition 3.0.3. Let $\pi = \mathrm{St}(k, \rho)$ be the representation in the statement of the proposition. We consider first the case in which $k = 2\ell$ is even. In this case we assume ρ is distinguished since that is equivalent to assuming that π is not distinguished. We claim $\gamma(1/2, \pi \times \tilde{\rho}, \psi) = -1$. The multiplicativity of gamma factors gives

$$\gamma(s, \pi \times \tilde{\rho}, \psi) = \prod_{j=1}^{k} \gamma(s + j - \frac{1}{2}, \rho \times \tilde{\rho}, \psi)\gamma(s - j + \frac{1}{2}, \rho \times \tilde{\rho}, \psi).$$

We observe that

$$\begin{aligned} &\gamma(s + j - \frac{1}{2}, \rho \times \tilde{\rho}, \psi)\gamma(s - j + \frac{1}{2}, \rho \times \tilde{\rho}, \psi) \\ &= \epsilon(s + j - \frac{1}{2}, \rho \times \tilde{\rho}, \psi)\epsilon(s - j + \frac{1}{2}, \rho \times \tilde{\rho}, \psi) \\ &\qquad \frac{L(\frac{3}{2} - s - j, \rho \times \tilde{\rho})}{L(s + j - \frac{1}{2}, \rho \times \tilde{\rho})} \frac{L(\frac{1}{2} - s + j, \rho \times \tilde{\rho})}{L(s - j + \frac{1}{2}, \rho \times \tilde{\rho})}. \end{aligned}$$

Since ϵ factors are entire, it follows from (5) and (6) that

$$\epsilon(j, \rho \times \tilde{\rho}, \psi)\epsilon(1 - j, \rho \times \tilde{\rho}, \psi) = 1.$$

Note that by statement 2 of Lemma 3.0.5, for every $j = 1, \ldots, k$ we have that $L(\frac{1}{2} - s + j, \rho \times \tilde{\rho})$ and $L(s + j - \frac{1}{2}, \rho \times \tilde{\rho})$ are holomorphic at $s = \frac{1}{2}$ and the value at $s = \frac{1}{2}$ of their quotient is 1. Furthermore, for $j = 2, \ldots, k$ we have that $L(\frac{3}{2} - s - j, \rho \times \tilde{\rho})$ and $L(s - j + \frac{1}{2}, \rho \times \tilde{\rho})$ are holomorphic at $s = \frac{1}{2}$ and the value at $s = \frac{1}{2}$ of their quotient is 1. Altogether, it follows that

$$\gamma\left(\frac{1}{2}, \pi \times \tilde{\rho}, \psi\right) = \lim_{s \to \frac{1}{2}} \frac{L(\frac{1}{2} - s, \rho \times \tilde{\rho})}{L(s - \frac{1}{2}, \rho \times \tilde{\rho})} = -1.$$

This proves our claim when k is even.

Now assume $k = 2\ell + 1$ is odd. Then ρ is a representation of $G_t(E)$, with $t = n/k$, that is $\eta_{E/F}$-distinguished but it is not distinguished. It must be the case that t is even since if m were odd then $\omega_\pi | F^\times = \omega_\rho | F^\times = \eta_{E/F} \neq 1$. Furthermore $\bar{\rho} \simeq \tilde{\rho}$ since $\rho \otimes \Omega$ is distinguished for every extension Ω of $\eta_{E/F}$ to a character of $E^\times$. By Theorem 1.5.1 or [Hak91] for $t = 2$, there exists $r < t$ such that ρ does not satisfy $C(r)$. It therefore follows from Lemma 3.0.2 that there exists a distinguished discrete series representation π' of $G_r(E)$ so that $\gamma(1/2, \rho \times \pi', \psi) =$

-1. By statement 2 of Lemma 3.0.5, for every $j = 1, \ldots, k$ both $\gamma(s+j, \rho \times \pi', \psi)$ and $\gamma(s-j, \rho \times \pi', \psi)$ are holomorphic at $1/2$. By (5) and (6) the value of their product at $s = 1/2$ equals 1. Our claim follows. $\square$

4 Appendix: Ok's lemma

In this appendix, we provide supplementary details for the proof of Lemma 11.1.2 [Ok97] (stated above as Lemma 1.3.2).

For convenience, we recall the statement of Ok's lemma. Let m be a positive integer, $G = G_m(E)$ and $H = G_m(F)$. Then Lemma 11.1.2 [Ok97] says:

> Suppose $\Phi \in C_c^\infty(N_m(E)\backslash G, \psi)$. If
>
> $$\int_{N_m(E)\backslash G} \Phi(g)\, W(g)\, dg = 0,$$
>
> for every Whittaker function W in the Whittaker model $\mathcal{W}(\pi, \psi^{-1})$ of every irreducible, unitary, generic, distinguished representation π of G then
>
> $$\int_{N_m(F)\backslash H} \Phi(h)\, dh = 0.$$

Note that Ok assumes that his fields do not have characteristic two and thus we must also have this restriction in Theorem 1.5.1.

The following additional notations will be convenient throughout this appendix. Let Π be the set of (equivalence classes of) irreducible representations of G. We add the subscript gen, unit or dist for the class of generic, unitary or distinguished representations, respectively.

4.1 Sketch of the proof

Ok states without proof in §11.2 [Ok97]:

The Inversion Formula on $H\backslash G$. *There exists a [unique] measure $d\mu$ on $\Pi_{\text{unit,dist}}$ such that for all $f \in C_c^\infty(G)$*

$$\int_H f(h)\, dh = \int_{\Pi_{\text{unit,dist}}} \sum_v \lambda_\pi(\pi(f)v)\, \overline{\lambda_\pi(v)}\, d\mu(\pi),$$

where a nonzero λ_π is chosen in $\mathrm{Hom}_H(\pi, 1)$*, for each* π*, and the sum is over an orthonormal basis of the space of* π*. (The measure* μ *depends on the normalization of the linear forms* λ_π*.)*

We will discuss this inversion formula in §4.2. We refer to $d\mu$ as the *Plancherel measure on* $H\backslash G$.

Appendix 12 in [Ok97] is credited to Jacquet and it contains:

Theorem 12.1 [Ok97]. *The Plancherel measure on* $H\backslash G$ *is supported on the set of generic representations.*

We examine this in §4.3. It implies that in the above inversion formula, we only need to consider those irreducible, unitary, H-distinguished representations of G that are generic.

Next, we recall:

Ok's Main Lemma I. (page 42 [Ok97]) *Given a compact open subgroup* K_0 *of* G*, there exists an exhaustive family* $\{\Omega\}$ *of compact open subsets of* $N_m(E)/N_m(F)$ *such that for every irreducible, unitary, generic* H*-distinguished representation* π *of* G *and every nonzero* $\lambda_\pi \in \mathrm{Hom}_H(\pi, 1)$ *and every* $N_m(F)$*-invariant Radon measure* du *on* $N_m(E)/N_m(F)$ *there exists a nonzero constant* $c(\lambda_\pi, du)$ *such that for all* K_0*-fixed fixed* $W \in \mathcal{W}(\pi, \psi)$*, we have*

$$\int_\Omega \lambda_\pi(\pi(u)^{-1}W)\,\psi(u)\,du = c(\lambda_\pi, du)\,W(1).$$

To say that $\{\Omega\}$ is an "exhaustive family of compact open subsets of $N_m(E)/N_m(F)$" means that every compact subset of $N_m(E)/N_m(F)$ is contained in some Ω.

Now we combine the Inversion Formula on $H\backslash G$ with Ok's Theorem 12.1 and Main Lemma I. We first apply the Inversion Formula with f replaced by $g \mapsto \int_\Omega f(ug)\,\psi(u)\,du$ where Ω is a compact open subset of $N_m(E)/N_m(F)$. Then the left hand side of the Inversion Formula is replaced by

$$\int_\Omega \int_H f(uh)\,dh\,\psi(u)\,du.$$

On the right hand side, the expression $\pi(f)v$ is replaced by

$$\int_\Omega \int_G f(ug)\, \psi(u)\, \pi(g)v\, dg\, du$$

or, equivalently,

$$\int_\Omega \left(\pi(u)^{-1}\pi(f)v\right)\ \psi(u)\, du.$$

Since we are only interested in generic π, we can take π in Whittaker form. In other words, we assume the space V_π of π is the Whittaker model $\mathcal{W}(\pi, \psi)$ and then the vector v becomes a Whittaker function W. Applying Main Lemma I with K_0 small enough so that f is K_0-bi-invariant, we see that

$$\int_\Omega \lambda_\pi \left(\pi(u)^{-1}\pi(f)v\right)\ \psi(u)\, du = c(\lambda_\pi, du)\, (\pi(f)W)(1).$$

But

$$(\pi(f)W)(1) = \int_G f(g)\, W(g)\, dg.$$

Taking

$$\Phi(g) = \int_{N_m(E)} f(ug)\, \psi(u)\, dy,$$

we have

$$(\pi(f)W)(1) = \int_{N_m(E)\backslash G} \Phi(g)\, W(g)\, dg.$$

In terms of Φ, the left hand side of the Inversion Formula becomes

$$\int_{N_m(H)\backslash H} \Phi(h)\, dh.$$

So the Inversion Formula yields:

Ok's Proposition 11.2.1. *For all $\Phi \in C_c^\infty(N_m(E)\backslash G, \psi^{-1})$, one has*

$$\int_{N_m(F)\backslash H} \Phi(h)\, dh = \int_{\Pi_{\text{gen,unit,dist}}} \sum_W \int_{N_m(E)\backslash G} \Phi(g)\, W(g)\, dg\, \overline{\lambda_\pi(W)}\, c(\lambda_\pi, du)\, d\mu(\pi).$$

Lemma 11.1.2 [Ok97] follows immediately from this.

4.2 Abstract Plancherel inversion for $H\backslash G$

In this section, we consider unitary representations of G that are not necessarily smooth. Following Bernstein [Ber88], we analyze the decomposition of the Hilbert space $L^2(H\backslash G)$ with respect to some fixed invariant measure on $H\backslash G$.

We have a direct integral decomposition

$$L^2(H\backslash G) = \int_{\Pi_{\text{unit}}} H_\pi \; d\mu(\pi).$$

(See [Ber88, §0.2].) We call $d\mu$ the *Plancherel measure on* $H\backslash G$.

According to the definition of the direct integral [Ber88, §1.2], an element $\phi \in L^2(H\backslash G)$ corresponds to a measurable, square integrable cross section

$$\pi \in \Pi_{\text{unit}} \mapsto \phi_\pi \in H_\pi,$$

where two such sections are considered equal when they agree for almost all π.

Naïvely, one might expect to have a G-morphism $L^2(H\backslash G) \to H_\pi$ given $\phi \mapsto \phi_\pi$. However, *a priori*, this makes no sense since $\pi \mapsto \phi_\pi$ is only well-defined up to measure zero subspaces of Π_{unit}. But, using the Gelfand-Kostyuchenko method, Bernstein shows in a very general setting that there is a system of G-morphisms

$$\alpha_\pi : C_c^\infty(H\backslash G) \to H_\pi$$

that are nonzero for almost all π.

With our setup, we can be quite explicit about the α_π's. To do this, we consider, for nonzero α_π, the adjoint G-morphism

$$\beta_\pi : V_\pi \to C^\infty(H\backslash G),$$

where V_π is the space of smooth vectors in H_π (a.k.a., the Gårding space). The morphism β_π is defined by the relation

$$\langle \alpha_\pi(\phi), v\rangle_{H_\pi} = \int_{H\backslash G} \phi(g)\, \overline{(\beta_\pi(v))(g)}\, dg.$$

(Note that β_π is not $\mathbb{C}$-linear, but rather conjugate linear.) Since we are assuming α_π is nonzero, β_π must also be nonzero and so, by

Frobenius reciprocity, there is a unique nonzero $\lambda_\pi \in \mathrm{Hom}_H(V_\pi, \mathbb{C})$ such that

$$(\beta_\pi(v))(g) = \overline{\lambda_\pi(\pi(g)v)}.$$

But $\mathrm{Hom}_H(V_\pi, \mathbb{C})$ must have dimension one, by [Fli91, Proposition 11]. Therefore, up to nonzero scalar multiples, λ_π, β_π and α_π are all well-defined, and, with the obvious interpretation of notations, we have

$$\langle \alpha_\pi(\phi), v\rangle_{H_\pi} = \overline{(\beta_\pi(v))(\phi)} = \lambda_\pi(\pi(\phi)v).$$

Following Bernstein [Ber88, Equation (**)], we take

$$\phi_\pi = \beta_\pi(\alpha_\pi(\phi))$$

and then we have the Plancherel inversion identity

$$\phi = \int_{\Pi_{\text{unit,dist}}} \phi_\pi \, d\mu(\pi).$$

The latter identity can be expressed more explicitly in terms of an orthonormal basis $\{e_i\}$ of V_π. Then we have

$$\alpha_\pi(\phi) = \sum_i \overline{\lambda_\pi(\pi(\phi)e_i)} \; e_i.$$

It follows that

$$\phi_\pi(g) = \sum_i \lambda_\pi(\pi(\phi)e_i) \; \overline{\lambda_\pi(\pi(g)e_i)}.$$

The inversion formula in §11.2 [Ok97] follows directly from the latter identity.

4.3 The Plancherel measure is supported on the generic spectrum

In the previous section, we discussed the Plancherel decomposition

$$L^2(H\backslash G) = \int_{\Pi_{\text{unit,dist}}} H_\pi \; d\mu(\pi).$$

Suppose $(\pi, H_\pi) \in \Pi_{\text{unit,dist}}$ and let (π^∞, V_π) be the associated smooth representation. Then π^∞ is irreducible according to [BZ76, Theorem 4.21].

Let $P = P_n(E)$ be the mirabolic group in G and let $N = N_n(E)$ be the group of upper triangular unipotent matrices in G. The unitary representation $(\xi, L^2(N\backslash P, \psi))$ of P induced from the character ψ of n is (topologically) irreducible.

By definition, π^∞ is generic precisely when $\mathrm{Hom}_N(\pi^\infty, \psi)$ is nonzero or, equivalently, by uniqueness of Whittaker models [Sha74], π^∞ is generic exactly when $\mathrm{Hom}_N(\pi^\infty, \psi) = \mathbb{C}$. Therefore, by Frobenius reciprocity, π^∞ is generic precisely when $\mathrm{Hom}_P(\pi^\infty, \mathrm{Ind}_N^P(\psi)) = \mathbb{C}$ or, equivalently, when the irreducible representations π^∞ and $\mathrm{Ind}_N^P(\psi)$ are equivalent. But, according to [BZ76, Theorem 4.21], $\pi^\infty|P \simeq \mathrm{Ind}_N^P(\psi)$ precisely when $\pi|P \simeq \xi$. Therefore, π^∞ is generic precisely when $\pi|P \simeq \xi$.

According to [Ber84, Theorem 0.2], every (topologically) irreducible, unitary representation of G on a Hilbert space remains (topologically) irreducible when restricted to P. It follows that the Plancherel decomposition of the G-module $L^2(H\backslash G)$ is the same as the Plancherel decomposition of the P-module $L^2(H\backslash G)|P$. (Of course, multiplicities change dramatically since many irreducible G-modules can have the same restriction to P.)

Therefore, to show that the Plancherel measure of $H\backslash G$ is supported on $\Pi_{\mathrm{unit,dist,gen}}$ reduces to showing that $L^2(H\backslash G)|P$ is a multiple of ξ in the sense that if $\pi \in \Pi_{\mathrm{unit,dist,gen}}$ contributes to the Plancherel decomposition then $\pi|P \simeq \xi$. This is [Ok97, Proposition 12.2].

We now sketch the proof of [Ok97, Proposition 12.2]. Let $Q = PZ = P_{n-1,1}(E)$ and let $U = N_{n-2,2}(E)$ be the unipotent radical of the parabolic of type $(n-2, 2)$ of G. It is easy to show that $H\backslash G/Q$ has two elements, the closed cell HQ, and the open, dense cell $G - HQ$.

It follows that for any $g \in G - HQ$ we have

$$L^2(H\backslash G)|P \simeq L^2(Q_g\backslash Q)|P$$

where $Q_g = Q \cap g^{-1}Hg$. Now take

$$g = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \tau & \bar{\tau} \\ 0 & 1 & 1 \end{pmatrix},$$

where τ is any fixed element of $E - F$ and let

$$T = Q_g = \left\{ \begin{pmatrix} h & x & \bar{x} \\ 0 & a & 0 \\ 0 & 0 & \bar{a} \end{pmatrix} : h \in G_{n-2}(F), x \in E^{n-2}, a \in E^{\times} \right\}.$$

It suffices to show that $L^2(T\backslash Q)|P$ is a multiple of ξ.

Let

$$T_1 = \left\{ \begin{pmatrix} h & x & y \\ 0 & a & 0 \\ 0 & 0 & \bar{a} \end{pmatrix} : h \in G_{n-2}(F), x,\, y \in E^{n-2}, a \in E^{\times} \right\}$$

and note that both T and U are subgroups of T_1. By transitivity of induction $L^2(T\backslash Q)$ is equivalent to the representation of Q induced from $L^2(T\backslash T_1)$. Ok shows that $L^2(T\backslash T_1) \simeq L^2(U\backslash T_1, \chi_0)$ for some non-trivial character χ_0 of $U \simeq E^{2(n-2)}$ that is described explicitly.

Note that T_1 normalises U and therefore acts on its characters. The T_1-orbit of χ_0 is open and dense. Transitivity of induction now gives that $L^2(T\backslash Q) \simeq L^2\text{-}\mathrm{Ind}_{T_1}^{Q}(L^2(U\backslash T_1, \chi_0)) \simeq L^2(U\backslash Q, \chi_0)$.

Abelian harmonic analysis decomposes $L^2(U)$ as a direct integral of the characters of U. Neglecting a set of measure zero, we can write the decomposition over $T_1 \cdot \chi_0$. Since $L^2(Q) = L^2\text{-}\mathrm{Ind}_U^Q(L^2(U))$ it can be decomposed as a direct integral over $L^2(U\backslash Q, \chi)$ with $\chi \in T_1 \cdot \chi_0$. Since $L^2(Q)|P$ is a multiple of ξ it follows that each $L^2(U\backslash Q, \chi)|P$ with $\chi \in T_1 \cdot \chi_0$ and in particular $L^2(U\backslash Q, \chi_0)|_P \simeq L^2(T\backslash Q)|P$ is a multiple of ξ. Proposition 12.2 [Ok97] therefore follows.

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