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In the article being corrected, I provided explicit formulas for the spherical functions and the spherical Plancherel measure for 3 cases of $\wp$-adic symmetric spaces. In Case 2, an elementary mistake in a rank one computation (Lemma 5.3) led to providing the wrong formulas and the wrong support for the spherical Plancherel measure. Here I correct the rank one computation and its consequences.

In [Offen 2004], I claimed to obtain explicit formulas for all spherical functions in three cases of a $\wp$-adic symmetric space. In this note I correct a sloppy mistake in my computations of the relative spherical functions in Case 2. The symmetric space in this case is $\mathrm{GL}_m(E)/\mathrm{GL}_m(F)$, where E/F is an unramified quadratic extension of nonarchimedean local fields of odd residual characteristic. Unless otherwise specified, from now on the discussion is focused only on this case. The mistake was observed while comparing my computations with related computations performed by B. Feigon for her PhD thesis [2006]. It is a result of carelessness in decomposing measures in a simple rank one computation and may have not been worth an announcement for its own sake. However, this computational mistake led to a wrong formula for the spherical functions Theorem 1.2, and, more importantly, it invalidates the more conceptual statement in Theorem 1.3 that the spherical Plancherel measure for the symmetric space has residues in its support. This corrigendum corrects this misleading statement and provides the correct spherical Plancherel formula. The harmonic analysis of $\wp$-adic symmetric spaces [Blanc and Delorme 2007; Delorme and Secherre 2006; Lagier 2007; Kato and Takano 2007] and more generally of $\wp$-adic spherical varieties [Sakellaridis 2008; $\geq$ 2008; Hironaka 2005; 2006] is a subject of growing interest in recent years.

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I will freely use the notation of [Offen 2004] and unless otherwise specified consider only Case 2. The correction of Theorem 1.3 is as follows. Let $d_\mu(z)$ be the measure supported on X_0 and defined by

$$d_\mu(z) = \frac{1}{2^n n!} V_0 \Delta(z)\, dz,$$

where dz is the standard Lebesgue measure on X_0 and

$$\Delta(z) = \prod_{\alpha \in R_1} \frac{1 - t_{2\alpha}^{1/2}\, e^\alpha}{1 - t_{2\alpha}^{1/2}\, t_\alpha\, e^\alpha},$$

where R_1 is the root system Σ if m is even and the root system R if m is odd. The Macdonald polynomials $P_z(\lambda)$ are given by

$$P_z(\lambda) = V_\lambda^{-1} \sum_{\sigma \in \Gamma} \sigma\left(e^\lambda \prod_{\alpha \in R_1^+} \frac{1 - t_{2\alpha}^{1/2}\, t_\alpha\, e^{-\alpha}}{1 - t_{2\alpha}^{1/2}\, e^{-\alpha}} \right),$$

and the parameters t_α and $t_{2\alpha}^{1/2}$ are defined as

$$t_\alpha = \begin{cases} q^{-2} & \text{if } \alpha \text{ is a long root of } \Sigma, \\ -q^{-1} & \text{if } \alpha \text{ is a short root of } \Sigma, \end{cases} \tag{100}$$

$$t_{2\alpha}^{1/2} = \begin{cases} -1 & \text{if } m \text{ is even}, \\ -q^{-1} & \text{if } m \text{ is odd}, \end{cases} \quad \text{whenever } \alpha \text{ is a short root of } \Sigma, \tag{101}$$

$$t_\alpha^{1/2} = 1 \quad \text{if } \alpha \text{ is not a root of } R. \tag{102}$$

This is a correction for the parameters defined on page 101. (Page numbers refer to [Offen 2004].) Note also that $\Delta(z)$ and $P_z(\lambda)$ are associated to the root system R_1; thus when m is even they associate to Σ and not to R. The main point is that now these parameters satisfy

$$t_\alpha t_{2\alpha}^{1/2} < 1 \tag{103}$$

for every $\alpha \in R_1$.

Theorem 1.3 (correction to Case 2). *For any $\phi \in \mathcal{S}(K \backslash S)$,*

$$\phi(s) = \int_{X_0} \hat{\phi}(z) \Omega_z(s)\, d\mu(z). \tag{104}$$

The source for the mistake is in the computation of Lemma 5.3. In particular, the integration taken there is not over an H-invariant measure on $H_\xi \backslash H$. Let us now redo it correctly.

Lemma 5.3 (correction to Case 2). *Let* $\chi = (\chi_1, \chi_1^{-1})$, *where* $\chi_1 = |\cdot|^z$. *If* $\operatorname{Re} z > 0$, *then the integral*

$$\int_{H_\xi \backslash H} \varphi(\xi h)\, dh$$

is convergent for all $\varphi \in I(\chi)$. *Normalizing the* H*-invariant measure so that* $\int_{H_\xi \backslash H_\xi (H \cap K_2)} dh = 1$, *we have*

$$\int_{H_\xi \backslash H} \varphi_{K_2,\chi}(\xi h)\, dh = \frac{1+q^{-1-2z}}{1-q^{-2z}}. \tag{105}$$

Proof. Since every $\varphi \in I(\chi)$ can be bounded by a constant multiple of the absolute value of $\varphi_{K_2,\chi}$, it is enough to compute the integral

$$I_\chi = \int_{H_\xi \backslash H} \varphi_{K_2,\chi}(\xi h)\, dh$$

for $\operatorname{Re} z > 0$. Note that

$$H_\xi = \left\{ \begin{pmatrix} a & b \\ \tau b & a \end{pmatrix} \in \mathrm{GL}_2(F) : a,\ b \in F \right\} \simeq E^\times$$

and that if $Z_H \simeq F^\times$ is the center of H, then $Z_H \backslash H_\xi \simeq F^\times \backslash E^\times$ is compact. We therefore have

$$I_\chi = \int_{Z_H \backslash H} \varphi_{K_2,\chi}(\xi h)\, dh,$$

where the H-invariant measure is normalized so that $Z_H \backslash Z_H (H \cap K_2)$ has volume one. Using the Iwasawa decomposition

$$h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \quad \text{for } a, b \in F^\times,\ x \in F, \text{ and } k \in K_2,$$

we get that $dh = |a|_F^{-1} dx\, d^\times a\, dk$ is the H-invariant measure on $Z_H \backslash H$, where dx is the Haar measure on F such that $\int_{\mathcal{O}_F} dx = 1$ and

$$d^\times x = (1-q^{-1})^{-1} \frac{dx}{|x|_F}. \tag{106}$$

We then have

$$I_\chi = \int_{F^\times} \int_F |a|_F^{-1} \varphi_{K_2,\chi}\left(\begin{pmatrix} \iota a & 1+\iota x \\ -\iota a & 1-\iota x \end{pmatrix} \right) dx\, d^\times a.$$

Since $|a + \iota b|_E = \max(|a|_E, |b|_E) = \max(|a|_F^2, |b|_F^2)$, we have

$$|a|_F^{-1} \varphi_{K_2,\chi}\left(\begin{pmatrix} \iota a & 1+\iota x \\ -\iota a & 1-\iota x \end{pmatrix} \right) = \frac{|a|_F^{2z}}{\max(|a|_F, |x|_F, 1)^{4z+2}}.$$

Splitting the domain of integration to $|x|_F \le 1$ and $|x|_F > 1$, we then have

$$I_\chi = J(1) + \int_{|x|_F>1} J(x)\, dx, \tag{107}$$

where, for every $x \in F$, $J(x)$ is the integral computed by

$$\begin{aligned}
J(x) &= \int_{F^\times} \frac{|a|_F^{2z}}{\max(|a|_F, |x|_F)^{4z+2}}\, d^\times a \\
&= |x|_F^{-4z-2} \int_{|a|_F \le |x|_F} |a|_F^{2z}\, d^\times a + \int_{|a|_F>|x|_F} |a|_F^{-2z-2}\, d^\times a \\
&= |x|_F^{-2z-2} \Big(\int_{|a|_F \le 1} |a|_F^{2z}\, d^\times a + \int_{|a|_F>1} |a|_F^{-2z-2}\, d^\times a \Big) \\
&= |x|_F^{-2z-2} \Big(\sum_{n=0}^{\infty} q^{-2nz} + \sum_{n=1}^{\infty} q^{-2n(z+1)} \Big) \\
&= |x|_F^{-2z-2} \Big(\frac{1}{1-q^{-2z}} + \frac{q^{-2-2z}}{1-q^{-2-2z}} \Big) \\
&= |x|_F^{-2z-2} \frac{1-q^{-2-4z}}{(1-q^{-2z})(1-q^{-2-2z})}.
\end{aligned}$$

Plugging this into (107), we have

$$I_\chi = \frac{1-q^{-2-4z}}{(1-q^{-2z})(1-q^{-2-2z})} \Big(1 + \int_{|x|_F>1} |x|_F^{-2z-2}\, dx \Big).$$

Applying (106), we get that

$$\begin{aligned}
\int_{|x|_F>1} |x|_F^{-2z-2}\, dx &= (1-q^{-1}) \int_{|x|_F>1} |x|_F^{-2z-1}\, d^\times x \\
&= (1-q^{-1}) \sum_{n=1}^{\infty} q^{-n(2z+1)} = (1-q^{-1}) \frac{q^{-1-2z}}{1-q^{-1-2z}}.
\end{aligned}$$

Thus

$$\begin{aligned}
I_\chi &= \frac{1-q^{-2-4z}}{(1-q^{-2z})(1-q^{-2-2z})} \Big(1 + (1-q^{-1}) \frac{q^{-1-2z}}{1-q^{-1-2z}} \Big) \\
&= \frac{1-q^{-2-4z}}{(1-q^{-2z})(1-q^{-2-2z})} \cdot \frac{1-q^{-2-2z}}{1-q^{-1-2z}}.
\end{aligned}$$

Simplifying the last expression, the lemma follows. □

For any expression y, we shall now adopt the convention that

$$[[y]] = \begin{cases} 1 & \text{if } m \text{ is even,} \\ y & \text{if } m \text{ is odd.} \end{cases}$$

The upshot of Lemma 5.3 is that on page 140 the scalar $\zeta(\chi)$ should be corrected to

$$\zeta(\chi)=\Big(\prod_{\alpha\in\Sigma^{+L}}\frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}}\Big)\Big(\prod_{\alpha\in\Sigma^{+S}}\frac{1+q^{-1}e^{-\alpha}}{1-e^{-\alpha}}\Big[\!\!\Big[\frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}}\Big]\!\!\Big]\Big).$$

With this correction Proposition 5.14 is now valid also in Case 2. To complete the computation of the spherical functions as in Section 5.6, we point out another misprint on page 145. The formula for $c_{\sigma_l}(\chi)$ should be corrected to

$$c_{\sigma_l}(\chi)=\Big(\prod_{\alpha\in\Sigma^{+L}}\frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}}\Big)^2\Big(\prod_{\alpha\in\Sigma^{+S}}\frac{1-q^{-2}e^{-2\alpha}}{1-e^{-2\alpha}}\Big[\!\!\Big[\Big(\frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}}\Big)^2\Big]\!\!\Big]\Big).$$

It follows that the scalar $\epsilon(\chi)$ defined on page 144 satisfies

$$\epsilon(\chi)=\frac{c_{\sigma_l}(\chi)}{\zeta(\chi)}=\prod_{\alpha\in R_1^+}\frac{1-t_\alpha t_{2\alpha}^{1/2}e^{-\alpha}}{1-t_{2\alpha}^{1/2}e^{-\alpha}}.$$

Now Theorem 1.2 holds in Case 2 with $P_z(\lambda)$ defined in terms of the parameters t_α, $t_{2\alpha}^{1/2}$ given by (100), (101) and (102) and the root system R_1 and *not* as defined on page 101.

Remark. In fact, in all 3 cases, we obtained (but did not claim earlier) a formula for the nonnormalized spherical functions ω_χ defined by the meromorphic continuation of the integral

$$\omega_\chi(\theta(g))=\int_{H_\xi\backslash H}\varphi_{K,\chi}(\xi h g^{-1})\,dh,$$

where the H-invariant measure is normalized by $\int_{H_\xi\backslash H_\xi(H\cap K)}dh=1$. The formula for this integral is then given by

$$\omega_{\chi_{\nu(z)}}(d_\lambda)=q_1^{-(\lambda\cdot\rho)}\frac{V_\lambda}{V_0}\,\zeta(\chi_{\nu(z)})\,P_z(\lambda).$$

It follows from (103) that we now fall into what Macdonald refers to as the standard case, (as opposed to what I claimed on page 148). The techniques of [Macdonald 1971, Section 5.1] can therefore be applied in order to obtain the spherical Plancherel formula — Theorem 1.3. There is *no* residual support to the Plancherel measure.

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References

[Blanc and Delorme 2007] P. Blanc and P. Delorme, "Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H", Preprint, 2007. arXiv math/0412435v2

[Delorme and Secherre 2006] P. Delorme and V. Secherre, "An analogue of the Cartan decomposition for p-adic reductive symmetric spaces", Preprint, 2006. arXiv math/0612545v1

[Feigon 2006] B. Feigon, *Periods and relative trace formulas for $GL(2)$ in the local setting*, PhD thesis, University of California at Los Angeles, 2006.

[Hironaka 2005] Y. Hironaka, "Functional equations of spherical functions on p-adic homogeneous spaces", *Abh. Math. Sem. Univ. Hamburg* **75** (2005), 285–311. MR 2006m:11079 Zbl 02245680

[Hironaka 2006] Y. Hironaka, "Spherical functions on p-adic homogeneous spaces", pp. 81–95 in *Number theory* (Xi'an, 2004), edited by W. Zhang and Y. Tanigawa, Dev. Math. **15**, Springer, New York, 2006. MR 2007a:22009 Zbl 05064856

[Kato and Takano 2007] S.-i. Kato and K. Takano, "Subrepresentation theorem for p-adic symmetric spaces", Preprint, 2007. arXiv 0706.0567v2

[Lagier 2007] N. Lagier, "Asymptotiques de fonctions sur un espace symétrique réductif p-adique", *C. R. Math. Acad. Sci. Paris* **344**:7 (2007), 421–424. MR 2008d:22012 Zbl 1114.43005

[Macdonald 1971] I. G. Macdonald, *Spherical functions on a group of p-adic type*, Publications of the Ramanujan Institute **2**, Madras, 1971. MR 55 #8261 Zbl 0302.43018

[Offen 2004] O. Offen, "Relative spherical functions on $\wp$-adic symmetric spaces (three cases)", *Pacific J. Math.* **215**:1 (2004), 97–149. MR 2005f:11103 Zbl 1059.22019

[Sakellaridis 2008] Y. Sakellaridis, "On the unramified spectrum of spherical varieties over p-adic fields", *Compositio Math.* (2008). To appear. arXiv math.RT/0606130v1

[Sakellaridis $\geq$ 2008] Y. Sakellaridis, "Spherical functions on spherical varieties", In preparation.

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