COMPACT UNITARY PERIODS

EREZ LAPID AND OMER OFFEN

Abstract. Let E be a CM-field and π a cuspidal representation of $GL_n(\mathbb{A}_E)$ which admits a spherical vector (at all places) ϕ_0. We evaluate the period of ϕ_0 with respect to any compact unitary group. The result is consistent with a conjecture of Sarnak.

1. Introduction

Recently there has been remarkable progress in the study of periods of automorphic forms in the context of the relative trace formula of Jacquet. In particular, it has been proved by Jacquet that for GL_n over a quadratic extension, the non-vanishing of periods with respect to the unitary groups precisely characterizes the image of quadratic base change. So far, however, the actual value of the period integrals received little attention. In this work we will compute explicitly the absolute value of the period integral of certain automorphic forms over anisotropic unitary groups. More precisely, let F be a totally real number field of degree d and let E be a totally imaginary quadratic extension of F, with Galois conjugation $x \to \bar{x}$. Let $\mathbf{G}' = GL_n/F$ and $\mathbf{G}$ the restriction of scalars of $\mathbf{G}'$ from E to F. Set $G' = \mathbf{G}'(F) = GL_n(F)$ and $G = \mathbf{G}(F) = GL_n(E)$. Consider a unitary group

$$\mathbf{H} = \mathbf{H}^\alpha = \{g \ \in \mathbf{G} : g\alpha\,{}^t\bar{g} = \alpha\}$$

which is assumed to be anistropic at every real place of F. That is, $\alpha \in G$ is Hermitian and either positive or negative definite in any real embedding of F. (The group $\mathbf{H}^e$ pertaining to the identity matrix will be particularly handy.) Now let π be an irreducible, everywhere unramified cuspidal representation of $G_{\mathbb{A}}$. Thus, it admits a $\mathbf{K}$-invariant, L^2-normalized automorphic form ϕ_0, where $\mathbf{K}$ is the standard maximal compact subgroup of $G_{\mathbb{A}}$. If ϕ_0 is not invariant under Galois conjugation (up to a sign), that is, if $\bar{\pi} \neq \pi$, then by an argument of

---

Date: November 1, 2005.
First-named author partially supported by BSF grant no. 2004200.

Harder-Langlands-Rapoport, the period integral

$$\int_{H^\alpha\backslash H^\alpha_{\mathbb{A}}} \phi(h)dh \tag{1}$$

is zero for all ϕ in the space of π ([HLR86]). Assume that $\bar{\pi} = \pi$, and therefore that π is a base change from a cuspidal representation π' of $G'_{\mathbb{A}}$ ([AC89]). Assume further that E/F (and therefore π') is unramified at all finite places and in addition that π' is unramified at all real places. (The latter is merely for convenience.) Let $\omega = \omega_{E/F}$ be the idèle class character attached to E/F by class field theory and let $\theta = (\theta_v) \in G_{\mathbb{A}}$ be such that $\theta_v {}^t\bar{\theta}_v = \pm\alpha_v$ for every real place v of F and $\theta_v = e$ for every finite place v of F. Our main result in this case is the following.

Theorem 1. *Under the above assumptions* [1] *we have*

$$\left|\int_{H^\alpha\backslash H^\alpha_{\mathbb{A}}} \phi_0(h\theta)\ dh\right|^2 \tag{2}$$
$$= 4\cdot 2^{-2nd}\cdot \mathrm{vol}(H^e_{\mathbb{A}}\cap\mathbf{K})^2\cdot\left|\frac{\Delta_E}{\Delta_F}\right|^{\dim B'}\cdot|P_\alpha(\pi)|^2\cdot\frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{\mathrm{Res}_{s=1}L(s,\pi'\times\tilde{\pi}')}.$$

Here $P_\alpha(\pi)$ is a product of local factors which are given explicitly in (16). In particular, $P_e(\pi) = 1$.

Note that the L-functions on the right-hand side are the completed ones. The Haar measure on $H^\alpha_{\mathbb{A}}$ is the pull-back of the one on $H^e_{\mathbb{A}}$ (via an inner twist). For the normalization of measure on $G_{\mathbb{A}}$ see §2.1 below.

We may view ϕ_0 as a function on the locally symmetric space $G\backslash G_{\mathbb{A}}/\mathbf{K}$ which is an eigenfunction for the ring of invariant differential operators (as well as for the Hecke operators). The integral of $\pi(\theta)\phi_0$ over $H^\alpha\backslash H^\alpha_{\mathbb{A}}$ amounts to a finite sum of (weighted) point evaluations. It is quite remarkable that we can evaluate it in terms of L-functions. In the case of an arithmetic quotient of the upper half plane, there is a well-known and extremely important formula of Waldspurger of the form

$$\left|\sum_{z\in\Lambda_d}\phi(z)\right|^2\sim L(\frac{1}{2},\mathrm{bc}_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{d})}\pi).$$

Here, Λ_d is the set of Heegner points of discriminant $d<0$, π is the automorphic representation emanating from ϕ and bc denotes base

[1] In particular, $|\Delta_E| = |\Delta_F|^2$ but we prefer to write (2) this way with an eye toward the general case.

change. (See [Wal85], [Jac86], [Jac87], [KS93] for various interpretations and generalizations.) Our formula (2) is of a similar nature except that it involves the special value at $s = 1$ of a quotient of L-functions. This is the first formula of this kind in higher rank. As an application we study its connection with some recent conjectures of Sarnak about the L^∞-norm of automorphic forms (see [Sar04], and §5 below).

The point of departure for the computation of the period is a global identity of Bessel distributions that follows from the relative trace formula identity obtained by Jacquet in [Jac] and in particular from the comparison of the discrete spectrum based on [Lap]. The Bessel distribution that we consider on G' is factorizable and computing the period requires an explicit computation of the local factors. This is carried out using a local identity of the Bessel and relative Bessel distributions obtained in [Off] – see §3. The explicit computations of Y. Hironaka in [Hir99] of the spherical functions for the space of Hermitian matrices are essential. Unfortunately, the latter are written only in the case where the extension is unramified – hence the restriction on E. It should be possible to carry this out in the ramified case as well in order to lift the assumption on the ramification of E/F, and in particular, to allow the case $F = \mathbb{Q}$. This was worked out in [Hir89] for the case $n = 2$ and partially in [LR00, Remark 2] for the case $n = 3$. We hope to address the general case in the future.

1.1. **Acknowledgement.** We are grateful to Hervé Jacquet for his help and inspiration. We also thank Peter Sarnak and Akshay Venkatesh for useful discussions.

The second-named author is supported by the Sir Charles Clore Postdoctoral Fellowship at the Weizmann Institute.

2. Bessel distributions for GL_n

2.1. **Notation and Preliminaries.** Let F denote either a number field or a local field of characteristic 0. In the global case we write $\mathbb{A} = \mathbb{A}_F$ for the ring of adèles of F and $\mathbb{I}_F$ for the group of idèles. We denote algebraic sets defined over F by bold letters such as $\mathbf{X}$ and the respective sets of F-rational points by plain letters, thus $X = \mathbf{X}(F)$. In the global setting we also denote $X_v = \mathbf{X}(F_v)$ for every place v of F and $X_{\mathbb{A}} = \mathbf{X}(\mathbb{A})$.

In this section $\mathbf{G} = \mathbf{G}_n$ is the group GL_n defined over a number field F and $\mathbf{Z}$ is its center. We denote by $\mathbf{B} = \mathbf{B}_n$ the standard Borel subgroup of $\mathbf{G}$, by $\mathbf{T} = \mathbf{T}_n$ the group of diagonal matrices and by $\mathbf{U} = \mathbf{U}_n$ the group of upper triangular unipotent matrices. Given a non-trivial additive character ψ of $F\backslash\mathbb{A}$ in the global setting and of

F in the local setting we associate to it a character ψ_U of $U\backslash U_{\mathbb{A}}$ or U respectively by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

We also denote by $\mathbf{K}$ the standard maximal compact of $G_{\mathbb{A}}$ in the global setting, and by K the standard maximal compact of G in the local setting. We denote by W the Weyl group of G. Let $\mathfrak{a}_0^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X^*(T)$ is the lattice of rational characters of T and denote the dual space by $\mathfrak{a}_0$. We identify $\mathfrak{a}_0^*$ and its dual space with $\mathbb{R}^n$. The W-invariant pairing $\langle \cdot, \cdot \rangle : \mathfrak{a}_0^* \times \mathfrak{a}_0 \to \mathbb{R}$ is then the standard inner product on $\mathbb{R}^n$. The height map $H : G_{\mathbb{A}} \to \mathfrak{a}_0$ is characterized by the condition $e^{\langle \alpha, H(utk) \rangle} = |\alpha(t)|$ for all $\alpha \in X^*(T)$, $u \in U_{\mathbb{A}}$, $t \in T_{\mathbb{A}}$ and $k \in \mathbf{K}$. Here $|\cdot|$ denotes the standard norm on $\mathbb{A}$.

For an algebraic group $\mathbf{Q}$ defined over F, we denote by δ_Q the modulus function of $Q_{\mathbb{A}}$ in the global setting and of Q in the local setting. Denote by $\rho \in \mathfrak{a}_0^*$ half the sum of the positive roots in $X^*(T)$ with respect to B, thus

$$\delta_B = e^{\langle 2\rho, H(\cdot) \rangle}.$$

Measures. Our conventions for Haar measures will be the following. Discrete groups will be endowed with the counting measure. The measures on the local groups will be determined by a non-trivial character ψ of F as follows. On F we put the measure dx which is self-dual with respect to ψ. If we change ψ to $\psi_a = \psi(a\cdot)$, $a \in F^*$ then the measure is changed by a factor of $|a|^{\frac{1}{2}}$. Set

$$\mathfrak{d}_F = \mathfrak{d}_F^{\psi} = \begin{cases} \operatorname{vol}(\mathcal{O}_F) & F \text{ non-archimedean}, \\ \operatorname{vol}([0,1]) & F \text{ real}, \\ \frac{1}{2} \operatorname{vol}(\{x + iy : 0 \le x, y \le 1\}) & F \text{ complex}. \end{cases}$$

If F is non-archimedean and ψ has conductor $\mathcal{O}_F$ then $\mathfrak{d}_F^{\psi} = 1$. The same is true if F is archimedean and $\psi(x) = e^{2\pi i \operatorname{Tr}_{F/\mathbb{R}} x}$. We have $\mathfrak{d}_F^{\psi_a} = |a|^{\frac{1}{2}} \mathfrak{d}_F^{\psi}$. Next, we put on U the measure $\otimes_{i<j} dx_{i,j}$. On F^* we take the measure $L(1, \mathbf{1}_{F^*}) \frac{dx}{|x|}$ where $L(1, \mathbf{1}_{F^*})$ is the local L-factor of Tate. The measure on T will be determined by the isomorphism $T = (F^*)^n$. On G we take the measure $dt\, du\, dk$ with respect to the Iwasawa decomposition where dk is the measure on K with total mass 1. If ψ has conductor $\mathcal{O}_F$ then the measure on G gives $\operatorname{vol}(K) = 1$.

Globally, we fix a non-trivial character ψ of $F\backslash\mathbb{A}$. On $\mathbb{A}$ we take the self-dual measure with respect to ψ. It is also given by $\otimes_v dx_v$ where dx_v are defined with respect to ψ_v. This does not depend on the choice of ψ, and we have $\operatorname{vol}(F\backslash\mathbb{A}) = 1$. Similarly, $\mathfrak{d}_F := \prod_v \mathfrak{d}_{F_v}(\psi_v)$ does not depend on ψ and in fact $\mathfrak{d}_F = |\Delta_F|^{-\frac{1}{2}}$ where Δ_F is the discriminant of F. On $\mathbb{I}_F$ we put the measure $\otimes_v\ dt_v$. On $\mathbb{I}_F^1$, the kernel of the norm map, we take the measure so that the measure induced on $\mathbb{I}_F^1\backslash\mathbb{I}_F$ is the pull back of $\frac{dt}{t}$ under the isomorphism $|\cdot| : \mathbb{I}_F^1\backslash\mathbb{I}_F \to \mathbb{R}_+$. Then $\mathrm{vol}(F^*\backslash\mathbb{I}_F^1) = \lambda_{-1} = \mathrm{Res}_{s=1} L(s, \mathbf{1}_{F^*})$ where $L(s, \mathbf{1}_{F^*})$ is the completed Dedekind ζ function for F. Similarly, on $G_{\mathbb{A}}$ we take $dg = \otimes_v\ dg_v$, which is also the measure determined by the Iwasawa decomposition. We induce a measure on $G_{\mathbb{A}}^1$ by identifying $G_{\mathbb{A}}/G_{\mathbb{A}}^1$ with $\mathbb{R}_+$ via $|\det|$.

Let (π_i, V_i), $i = 1, 2$ be a pair of admissible smooth representations of G with a G-invariant pairing $(\cdot,\cdot)$ which is linear in the first variable and conjugate linear in the second. For any continuous linear forms l_i on V_i, $i = 1, 2$ the Bessel distribution is defined by

$$\mathfrak{B}_{V_1,V_2}^{l_1,l_2,(\cdot,\cdot)}(f) = \mathfrak{B}^{l_1,l_2,(\cdot,\cdot)}(f) = \overline{l_2}[l_1 \circ \pi_1(f)]$$

for any $f \in C_c^\infty(G)$. Here we view $l_1 \circ \pi_1(f)$ as an element of $V_1^\vee$ and $\overline{l_2}$ as a linear form on $V_1^\vee$ through the pairing $(\cdot,\cdot)$ (cf. [JLR04, §4.1]). In particular, if π is unitary with an invariant inner product $(\cdot,\cdot)$ then

$$\mathfrak{B}_{V,V}^{l_1,l_2,(\cdot,\cdot)}(f) = \sum_{\varphi\in \mathrm{ob}(\pi)} l_1(\pi(f)\varphi)\overline{l_2(\varphi)}$$

for any continuous linear forms l_i on V where $\mathrm{ob}(\pi)$ is any choice of an orthonormal basis for V.

2.2. **Bessel distributions and factorization.** For any automorphic form ϕ on $G\backslash G_{\mathbb{A}}$ denote by $W^\psi(\phi)$ its ψ-th Fourier coefficient given by

$$W^\psi(\phi, g) = \int_{U\backslash U_{\mathbb{A}}} \phi(ug)\overline{\psi_U(u)}du.$$

We also denote by

$$\mathcal{W}^\psi(\phi) = W^\psi(\phi, e)$$

the Whittaker functional and by $\overline{\mathcal{W}}^\psi(\phi)$ its complex conjugate.

Let π be an irreducible, cuspidal representation of $G_{\mathbb{A}}$. The Bessel distribution attached to π is defined by

$$B_\pi^\psi(f) = \mathfrak{B}^{\mathcal{W}^\psi,\mathcal{W}^\psi,(\cdot,\cdot)_{G\backslash G_{\mathbb{A}}^1}}(f).$$

It is explained in [Jac01] how to decompose the Bessel distribution into local Bessel distributions, up to an explicit global factor. This is based on the factorization of the inner product. To recall how this is done we now turn to the local setting. Let π be an irreducible, generic, unitary representation of G. We denote by $\mathcal{W}^\psi(\pi)$ the ψ-th Whittaker model of π, on which π acts by right translation. An invariant inner product on $\mathcal{W}^\psi(\pi)$ is given by

$$[W_1, W_2] = \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^*}) \cdot \int_{U_{n-1}\backslash G_{n-1}} W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \overline{W}_2 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] \, dg$$

(cf. [Bar03]). Note the normalization by a local Tate factor and discriminant which appears for convenience. The integral is absolutely convergent. We define the local Bessel distribution

$$B_\pi^\psi(f) = \mathfrak{B}^{\delta_e,\delta_e,[\cdot,\cdot]}_{\mathcal{W}^\psi(\pi),\mathcal{W}^\psi(\pi)}(f)$$

where δ_e is the evaluation at the identity.

To decompose the global Bessel distribution we first write the inner product in terms of the Whittaker function using a Rankin-Selberg integral ([JS81]). Namely, for a vector ϕ in the space of $\pi = \otimes_v \pi_v$ which is a pure tensor we may write $W^\psi(\phi, g) = \prod_v W_v(g_v)$ with $W_v \in \mathcal{W}^{\psi_v}(\pi_v)$ and $W_v(e) = 1$ almost everywhere. Let S be a finite set of places containing the archimedean places, so that for $v \notin S$, π_v is unramified, ψ_v has conductor $\mathcal{O}_v$, W_v is spherical and $W_v(e) = 1$. Then

$$(\phi, \phi)_{G\backslash G^1_{\mathbb{A}}} = \operatorname{Res}_{s=1} L^S(s, \pi \times \tilde{\pi}) \prod_{v \in S} [W_v, W_v] \tag{3}$$

where

$$L^S(s, \pi \times \tilde{\pi}) = \prod_{v \notin S} L(s, \pi_v \times \tilde{\pi}_v)$$

is the partial Rankin-Selberg L-function.

To obtain (3) we recall the Eisenstein series

$$\mathcal{E}_\Phi(g, s) = \int_{Z\backslash Z_{\mathbb{A}}} \sum_{v \in \mathbb{A}^n \backslash \{\underline{0}\}} \Phi(vzg) \left|\det(zg)\right|^{s+\frac{1}{2}} \, dz$$

for any Schwartz-Bruhat function $\Phi \in \mathcal{S}(\mathbb{A}^n)$. The integral-sum converges absolutely for $\operatorname{Re}(s) > \frac{1}{2}$ and admits meromorphic continuation as a Tate integral. Its residue at $s = \frac{1}{2}$ is $\hat{\Phi}(0)$ provided that the measure on $Z_{\mathbb{A}}$ is defined by taking the measure on $Z^1_{\mathbb{A}}$ such that $\operatorname{vol}(Z\backslash Z^1_{\mathbb{A}}) = 1$ and the measure on $Z_A/Z^1_{\mathbb{A}}$ determined by the isomorphism $|\det| : Z_{\mathbb{A}}/Z^1_{\mathbb{A}} \to \mathbb{R}_+$.

The unfolding gives

$$\int_{G\backslash G^1_{\mathbb{A}}} \phi_1(g)\overline{\phi_2(g)}\mathcal{E}_\Phi(g, s) \, dg$$
$$= \int_{U_{\mathbb{A}}\backslash G^1_{\mathbb{A}}} W^\psi(\phi_1, g)\overline{W}^\psi(\phi_2, g) \int_{Z_{\mathbb{A}}} \Phi(v_0 zg) \left|\det(zg|\right)^{s+\frac{1}{2}} \, dz \, dg$$

where $v_0 = (0, \dots, 0, 1)$. This can be written as

$$\int_{U_{\mathbb{A}}\backslash G_{\mathbb{A}}} W^{\psi}(\phi_1, g)\overline{W}^{\psi}(\phi_2, g)\Phi(v_0 g)\left|\det(g)\right|^{s+\frac{1}{2}}\ dg.$$

We write this as

$$\int_{P_{\mathbb{A}}\backslash G_{\mathbb{A}}}\int_{U_{\mathbb{A}}\backslash P_{\mathbb{A}}} W^{\psi}(\phi_1, pg)\overline{W}^{\psi}(\phi_2, pg)\Phi(v_0 g)\left|\det(pg)\right|^{s+\frac{1}{2}}\left|\det(p)\right|^{-1}\ dp\, dg$$

where $\mathbf{P} = \mathbf{P}_n$ is the mirabolic subgroup (the stabilizer of v_0). (The measure on P is given through the isomorphism $P \simeq G_{n-1} \ltimes U_n/U_{n-1}$.) By a local unramified computation it is

$$\prod_{v\in S}\int_{P_v\backslash G_v}\int_{U_v\backslash P_v} W_v^1(pg)\overline{W}_v^2(pg)\Phi_v(v_0 g)\left|\det(pg)\right|^{s+\frac{1}{2}}$$
$$\left|\det(p)\right|^{-1}\ dp\ dg \times L^S(s+\frac{1}{2}, \pi\times\tilde{\pi}).$$

The residue at $s = \frac{1}{2}$ is therefore given by

$$\prod_{v\in S}\int_{P_v\backslash G_v}\int_{U_v\backslash P_v} W_v^1(pg)\overline{W}_v^2(pg)\ dp\ \Phi_v(v_0 g)\left|\det(g)\right|\ dg$$
$$\times \operatorname{Res}_{s=1} L^S(s, \pi\times\tilde{\pi}) = \prod_{v\in S}[W_v^1, W_v^2]\cdot\hat{\Phi}(0)\cdot\operatorname{Res}_{s=1} L^S(s, \pi\times\tilde{\pi})$$

since the pairing $[\cdot,\cdot]$ is G-invariant and

$$\int_{P_v\backslash G_v}\Phi_v(v_0 g)\left|\det g\right|\ dg = \mathfrak{d}_v^{1-n} L(n, \mathbf{1}_{F_v^*})\hat{\Phi}_v(0)$$

by polar coordinates.

The factorization (3) gives rise to the decomposition

$$B_{\pi}^{\psi}(\otimes_{v\in S} f_v \otimes_{v\notin S}\mathbf{1}_{K_v}) = \frac{1}{\operatorname{Res}_{s=1} L^S(s, \pi\times\tilde{\pi})}\prod_{v\in S} B_{\pi_v}^{\psi_v}(f_v).$$

We now go back to a local setting. As we have already mentioned in the introduction, if π is spherical we will evaluate the local Bessel distribution $B_{\pi}^{\psi}(f)$ using the local identity of Bessel distributions obtained in [Off]. We first need to compare our normalization of the Bessel distribution for principal series with the slightly different one of [loc. cit.]. For a unitary character ν of T and $\lambda \in \mathbb{C}^n$ we denote by $I(\nu, \lambda)$ the principal series representation induced from the character $\nu e^{\langle\lambda, H(\cdot)\rangle}$ of B to G. We identify the spaces of $I(\nu, \lambda)$ with the space $I(\nu)$ of smooth sections $\varphi : G \to \mathbb{C}$ such that

$$\varphi(bg) = \nu(b)e^{\langle\rho, H(b)\rangle}\varphi(g),\ b \in B,\ g \in G.$$

The identification is through $\varphi \mapsto \varphi_\lambda = e^{\langle \lambda, H(\cdot)\rangle} \cdot \varphi$. The action is given by

$$I(g,\nu,\lambda)\varphi = (\varphi_\lambda(\cdot g))_{-\lambda} = e^{\langle \lambda, H(\cdot g) - H(\cdot)\rangle}\varphi(\cdot g).$$

When $\nu = 1$ (i.e. for unramified principal series) we often suppress ν from the notation. We consider the standard inner product on $I(\nu)$ given by

$$(\varphi_1, \varphi_2) = \int_{B\backslash G} \varphi_1(g)\overline{\varphi_2(g)}dg = \int_K \varphi_1(k)\overline{\varphi_2(k)}dk.$$

Note that $(\ ,\ ) : I(\nu,\lambda) \times I(\nu, -\bar{\lambda}) \to \mathbb{C}$ is G-invariant. Also we remark that

$$(\varphi_1, \varphi_2) = \frac{\prod_{i=1}^n L(i, \mathbf{1}_{F^*})}{L(1, \mathbf{1}_{F^*})^n} \mathfrak{d}_F^{-\dim U} \int_U \varphi_1(wu)\overline{\varphi_2(wu)}\ du \tag{4}$$

(cf. [Lan66]). Here $w = w_n$ is the permutation matrix with unit anti-diagonal. We will only consider λ so that $|\mathrm{Re}(\lambda_i)| < \frac{1}{2}$ for all i, in which case $I(\nu, \lambda)$ is irreducible. All unramified unitarizable representations are of this type. For a principal series representation $\pi = I(\nu, \lambda)$ it will be convenient to set $\mathcal{W}^\psi(\nu,\lambda) = \mathcal{W}^\psi(\pi)$. The Jacquet integral

$$W^\psi(\varphi, \lambda, g) = \int_U \varphi_\lambda(wug)\overline{\psi_U(u)}du$$

converges for $\mathrm{Re}\,\lambda$ in the Weyl chamber, admits an analytic continuation and defines an isomorphism $\varphi \mapsto W^\psi(\varphi, \lambda)$ between $I(\nu, \lambda)$ and $\mathcal{W}^\psi(\nu, \lambda)$. We also set

$$\mathcal{W}^\psi(\varphi, \lambda) = W^\psi(\varphi, \lambda, e).$$

The local Bessel distribution considered in [Off] was

$$B_\nu^\psi(f, \lambda) = \mathfrak{B}_{I(\nu,\lambda), I(\nu, -\bar{\lambda})}^{\mathcal{W}^\psi(\cdot,\lambda), \mathcal{W}^\psi(\cdot, -\bar{\lambda}), (\cdot,\cdot)}(f).$$

At first site this depends on λ itself and not only on the equivalence class of the representation $I(\nu, \lambda)$. However, we shall soon see that this is not the case.

Proposition 1. *For $\lambda \in i\mathfrak{a}_0^*$ we have*

$$(\varphi_1, \varphi_2) = \frac{[W^\psi(\varphi_1, \lambda), W^\psi(\varphi_2, -\bar{\lambda})]}{L(1, \mathbf{1}_{F^*})^n}.$$

Proof. We prove this by induction on n, the case $n = 1$ being trivial. We can assume of course that $\varphi_2 = \varphi_1 = \varphi$. For the induction step we identify $\pi = I(\nu, \lambda)$ with $I_Q^G(\pi')$ where Q is the parabolic of type $(1, n-1)$ and $\pi' = \mathrm{Ind}_B^Q(\nu, \lambda)$. Explicitly, for $\varphi \in I(\nu, \lambda)$ we write

$$F_\varphi(g)(q) = \delta_Q(q)^{-\frac{1}{2}}\varphi(qg), \quad g \in G,\ q \in Q$$

so that $F_\varphi(g)(\cdot) \in \pi'$. We will assume that φ has the property that F_φ is compact supported in Qw_nU' where U' is the unipotent radical of the parabolic subgroup of type $(n-1,1)$. These sections are dense in π. Realizing π' in its Whittaker model using the Jacquet integral (in GL_{n-1}) we also write

$$W_\varphi(g) = W^Q(F_\varphi(g), \lambda, \cdot) \in \mathcal{W}(\pi') \quad g \in G$$

where the superscript signifies that we work in the (Levi subgroup of the) group Q. Thus,

$$W_\varphi(g)(q) = \delta_Q(q)^{-\frac{1}{2}} \int_{U_{n-1}} \varphi_\lambda(j(w_{n-1}u)qg)\psi_{U_{n-1}}(u)\, du$$

(in the sense of analytic continuation) where we set $j(x) = \begin{pmatrix} 1 & \\ & x \end{pmatrix}$ for $x \in GL_{n-1}$. Using Fubini and the relation (4) we write

$$(\varphi, \varphi) = \frac{L(n, \mathbf{1}_{F^*})}{L(1, \mathbf{1}_{F^*})} \mathfrak{d}_F^{1-n} \int_{U'} (F_\varphi(w'u'), F_\varphi(w'u'))_{\pi'}\, du'$$

where w' is such that $j(w_{n-1})w' = w_n$. By induction hypothesis we get

$$\frac{L(n, \mathbf{1}_{F^*})}{L(1, \mathbf{1}_{F^*})^n} \mathfrak{d}_F^{1-n} \int_{U'} [W_\varphi(w'u'), W_\varphi(w'u')]_{n-1}\, du'.$$

Using Parseval identity (for vector-valued functions) the integral is equal to the L^2-norm of the Fourier transform of $W_\varphi(w'\cdot)$. The value of this Fourier transform at the character $u' \mapsto \psi(pu'p^{-1})$ of U' ($p \in GL_{n-1}$ imbedded as $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ in GL_n) is

$$\begin{aligned} \int_{U'} W_\varphi(w'u')\psi(pu'p^{-1})\, du' &= \int_{U'} W^Q(F_\varphi(w'u'))\psi(pu'p^{-1})\, du' \\ &= |\det p|^{-1} \int_{U'} W^Q(F_\varphi(w'p^{-1}u'p))\psi(u')\, du' \\ &= |\det p|^{-\frac{1}{2}}\, \pi'(j(p^{-1})) \int_{U'} W^Q(F_\varphi(w'u'p))\psi(u')\, du'. \end{aligned}$$

Integrating over the characters of U' amounts to integrating over $p \in P_{n-1}\backslash GL_{n-1}$ against $|\det p|$ times the factor $\mathfrak{d}_F^{n-2}/L(n-1, \mathbf{1}_{F^*})$. Therefore, since $[\cdot,\cdot]_{n-1}$ is GL_{n-1}-invariant we obtain $\frac{1}{L(1,\mathbf{1}_{F^*})^n}$ times

$$L(n, \mathbf{1}_{F^*}) \frac{\int [\int W^Q(F_\varphi(w'u'p))\psi(u')\ du', \int W^Q(F_\varphi(w'u'p))\psi(u')\ du']_{n-1}\ dp}{\mathfrak{d}_F L(n-1, \mathbf{1}_{F^*})}$$
$$= \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^*}) \iint \left| \int W^Q(F_\varphi(w'u'p), j(p'))\psi(u')\ du' \right|^2\ dp'\ dp$$
$$= \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^*}) \iint \left| \int W^Q(F_\varphi(j(p')w'u'p), e)\psi(u')\ du' \right|^2 |\det p'|\ dp'\ dp$$
$$= \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^*}) \iint \left| \int W^Q(F_\varphi(w'u'p'p), e)\psi(u')\ du' \right|^2 |\det p'|^{-1}\ dp'\ dp$$
$$= \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^*}) \iint |W(\varphi, p'p)|^2 |\det p'|^{-1}\ dp'\ dp = [W(\varphi), W(\varphi)]$$

as required. In the last series of equalities p, p' and u' are integrated over $P_{n-1}\backslash GL_{n-1}$, $U_{n-1}\backslash P_{n-1}$ and U' respectively. The justification for all the steps above follows directly from the convergence of $[W(\varphi), W(\varphi)]$. ☐

The statement of the proposition extends by analytic continuation to all $\lambda \in \mathbb{C}^n$ such that $|\mathrm{Re}(\lambda_i)| < \frac{1}{2}$ (in which case, the inner product $[\cdot,\cdot]$ converges). We conclude that at least for such λ

$$B_\nu^\psi(f, \lambda) = L(1, \mathbf{1}_{F^*})^n \cdot \mathfrak{B}_{\mathcal{W}(\pi),\mathcal{W}(\pi^*)}^{\delta_e,\delta_e,[\cdot,\cdot]}(f)$$

where π^* denotes the conjugate contragredient of π. In particular, if $I(\nu, \lambda)$ is unitary then

$$B_\nu^\psi(f, \lambda) = L(1, \mathbf{1}_{F^*})^n B_{I(\nu,\lambda)}^\psi(f). \tag{5}$$

We also note that in the unramified case

$$\left| W^\psi(\varphi_0, -\bar{\lambda}, g) W^\psi(\varphi_0, \lambda, g) \right| = L(1, \mathbf{1}_{F^*})^n \left| W_1^\psi(g) \right|^2 \tag{6}$$

where W_1^ψ is a spherical Whittaker function of π normalized so that $[W_1^\psi, W_1^\psi] = 1$ and φ_0 is the spherical section normalized so that $\varphi_0(e) = 1$. Indeed, $W^\psi(\varphi_0, \lambda, \cdot)$ and $W^\psi(\varphi_0, -\overline{\lambda}, \cdot)$ are both proportional to W_1^ψ. If the proportionality constants are c_1 and c_2 respectively then $c_1\overline{c_2} = L(1, \mathbf{1}_{F^*})^n$ by Proposition 1.

3. Local identities of distributions

For the rest of the paper, we switch the notation from the previous section as follows. We will have a quadratic extension E/F of either local or global fields of characteristic zero. In the global case we will assume that F is totally real and E is totally complex. That is, E is a CM-field and F is its maximal real subfield. In the local setting we will also consider the split case where $E = F \oplus F$. We denote by $\mathrm{Nm}(x) = x\bar{x}$ the norm map from E^* to F^*, by E^1 – its kernel and by ω the quadratic character of F^* attached to E/F by class field theory.

Let $\mathbf{G}' = \mathbf{G}'_n$ denote the group GL_n regarded as an algebraic group defined over F and let $\mathbf{G} = R_{E/F}(\mathbf{G}')$ be the restriction of scalars of $\mathbf{G}'$ from E to F. All the notation and conventions of the previous section will apply to $\mathbf{G}$ and E, using the character $\psi \circ \mathrm{Tr}_{E/F}$. Notation pertaining to $\mathbf{G}'$ will be appended by a prime. The measure on E^1 is defined by the relation

$$\int_{E^*} f(z)\, dz = \int_{\mathrm{Nm}(E^*)\subset F^*} F(x)\, dx \quad \text{where} \quad F(\mathrm{Nm}\, t) = \int_{E^1} f(yt)\, dy.$$

Finally, note that $H(g) = 2H'(g)$, $g \in G'_{\mathbb{A}}$.

Let $\mathbf{H} = \mathbf{H}^\alpha$ be the unitary group defined by the Hermitian form α. It will be assumed to be anisotropic at the real places.

3.1. **Relative Bessel distributions.** We start with the global setting. Let

$$P^H(\phi) = \int_{H\backslash H_{\mathbb{A}}} \phi(h) dh$$

denote the period over H of a cusp form ϕ. Let π be a cuspidal automorphic representation of $G_{\mathbb{A}}$. The relative Bessel distribution is defined for a function $f \in C_c^\infty(G_{\mathbb{A}})$ by

$$\tilde{B}_\pi^\psi(f) = \sum_{\phi \in \mathrm{ob}(\pi)} P^H(\pi(f)(\phi)) \overline{\mathcal{W}}^\psi(\phi).$$

We turn to the local setting. For simplicity we consider only unramified principal series representations $I(\lambda)$ of G since this is the case needed for Theorem 1. For any character ν of T' such that $\nu \circ \mathrm{Nm} \equiv 1$ define the stable intertwining period of $\varphi \in I(\lambda)$ by

$$J_\nu^{st,\alpha}(\varphi, \lambda) = {\sum_{a\in A}}' (\nu\nu_\omega)^{-1}(a) e^{-\langle \rho+\lambda, H'(t)\rangle} \int_{H_\eta\backslash H} \varphi_\lambda(\eta h) dh$$

(cf. [Off]). Here $A = T'/\mathrm{Nm}(T) \simeq (F^*/\mathrm{Nm}(E^*))^n$, and we sum over $a \in A$ which are in the G-orbit of α. For each such a we choose η such that $\eta\alpha\, {}^t\bar{\eta} = t \in a$ and set $H_\eta = H \cap \eta^{-1} B \eta$ which is isomorphic to

$(E^1)^n$ (with the measure inherited from the one on E^1). Finally, ν_ω is the character $(\omega, \omega^2, \dots, \omega^n)$ of T'. The integral extends meromorphically and the expression does not depend on the choice of η. The functionals $J_\nu^{st,\alpha}$ constitute a basis of H-invariant functionals on $I(\lambda)$. We will suppress ν from the notation of J if $\nu = 1$.

In the case where E/F is p-adic, unramified or split and $\varphi_0 \in I(\lambda)$ is the K-invariant section with $\varphi_0(e) = 1$, $J^{st,\alpha}(\varphi_0, \lambda)$ can be interpreted as Hironaka's spherical function evaluated at α ([Off, Lemma 8.5]) in the inert case, and the zonal spherical function at α, multiplied by a suitable c-function in the split case. These values are computed explicitly in [Hir99, Theorem 1] and [Mac95, p. 299] respectively. On the other hand, in the archimedean case we have

$$J^{st,\alpha}(I(\theta,\lambda)\varphi_0,\lambda) = \nu_\omega(\pm e) \int_{H_{\theta^{-1}}\backslash H} e^{\left\langle \lambda+\rho, H(\theta^{-1}h\theta)\right\rangle}\, dh$$
$$= \nu_\omega(\pm e) \int_{H_e^e\backslash H^e} e^{\langle \lambda+\rho, H(h)\rangle}\, dh = \nu_\omega(\pm e)\,\mathrm{vol}(H_e^e\backslash H^e)$$

where $\theta\, {}^t\bar{\theta} = \pm\alpha$. (Note that $H^e = K = \theta^{-1}H\theta$ in this case.) The upshot is that in both cases we have

$$J^{st,\alpha}(I(\theta,\lambda)\varphi_0,\lambda) = \mathrm{vol}(((H_e^e)\cap K)\backslash(H^e\cap K))P_\alpha(\lambda)\prod_{i<j}\frac{L(\lambda_i-\lambda_j,\omega)}{L(\lambda_i-\lambda_j+1,\mathbf{1}_{F^*})} \tag{7}$$

where in the p-adic case we set $\theta = e$ and where $P_\alpha(\lambda)$ is defined as follows. If E/F is p-adic, unramified or split

$$P_\alpha(\lambda) = \nu_\omega(\varpi^m)\frac{\prod_{i=1}^n L(i,\omega^i)}{L(1,\omega)^n}\sum_{\sigma\in W}\sigma\left(e^{\langle\lambda-\rho,\varpi_\alpha\rangle}\prod_{i<j}\frac{L(\lambda_i-\lambda_j,\mathbf{1}_{F^*})}{L(\lambda_i-\lambda_j+1,\omega)}\right)$$

where in the sum σ acts on λ and where ϖ_α is the dominant co-weight of α, i.e. it is $\log q(m_1,\dots,m_n)$ if there exists $k \in K$ such that

$$k\alpha\, {}^t\bar{k} = \varpi^m = \mathrm{diag}(\varpi^{m_1},\dots,\varpi^{m_n})$$

with $m_1 \geq \cdots \geq m_n$ for a uniformizer ϖ of F. Up to a constant depending on α, $P_\alpha(\lambda)$ is the ϖ_α-th Hall-Littlewood polynomial evaluated at q^λ and $t = \omega(\varpi)q$. In the case $F = \mathbb{R}$ and $E = \mathbb{C}$ set $P_\alpha(\lambda) = \nu_\omega(\pm e)$. Note that in the latter case the quotient of L-functions in (7) is 1 because ω is the signum character!

The stable local relative Bessel distribution is defined by

$$\tilde{B}_\nu^{st,\psi}(f,\lambda) = \sum_{\varphi\in\mathrm{ob}(I(\lambda))} J_\nu^{st,\alpha}(I(f,\lambda)\varphi,\lambda)\overline{\mathcal{W}}^\psi(\varphi,-\bar{\lambda}).$$

As before we suppress ν from the notation if $\nu = 1$. In the case where E/F is unramified, split, or archimedean we obtain from the previous computation

$$(8)\quad \tilde{B}^{st,\psi}(f_\theta,\lambda) = \hat{f}(\lambda)P_\alpha(\lambda)J^{st,e}(\varphi_0,\lambda)\overline{\mathcal{W}}^\psi(\varphi_0,-\bar{\lambda}) = \hat{f}(\lambda)P_\alpha(\lambda)\left(\prod_{i<j}\frac{L(\lambda_i-\lambda_j,\omega)}{L(\lambda_i-\lambda_j+1,\mathbf{1}_{F^*})}\right)\overline{\mathcal{W}}^\psi(\varphi_0,-\bar{\lambda})\upsilon$$

for any bi-K-invariant f, where we write $f_\theta = f(\theta^{-1}\cdot)$, $\upsilon = \mathrm{vol}((H_e^e \cap K)\backslash(H^e \cap K))$ and where $\hat{f}$ is the spherical transform of f. Note that $I(f_\theta,\lambda)\varphi = I(\theta,\lambda)I(f,\lambda)\varphi$ for $\varphi \in I(\lambda)$.

3.2. **Matching functions.** We recall the notion of matching of functions on G' and on G in our setting. Fix α as before. Locally, we say that $f' \in C_c^\infty(G')$ and $f \in C_c^\infty(G)$ match with respect to ψ and write $f' \overset{\psi}{\leftrightarrow} f$ if for any diagonal matrix $a = \mathrm{diag}(a_1,\dots,a_n) \in T'$

$$\int_{U'}\int_{U'} f'(u_1wau_2)\psi_{U'}(u_1u_2)\,du_1\,du_2 = \begin{cases}\nu_\omega(a)\int_U\int_{H^\alpha} f(h\eta u)\psi_U(u)\,dh\,du & \text{if } a = {}^t\bar{\eta}\alpha^{-1}\eta,\\ 0 & \text{if } a \notin \{{}^t\bar{g}\alpha^{-1}g : g \in G\}.\end{cases}$$

Globally, by definition $f' = \prod_v f'_v \in C_c^\infty(G'_{\mathbb{A}})$ and $f = \prod_v f_v \in C_c^\infty(G_{\mathbb{A}})$ match with respect to ψ if $f'_v \overset{\psi_v}{\leftrightarrow} f_v$ for all places v of F.

3.3. **Local Bessel identities.** We recall the main result of [Off]. Set

$$\gamma(\nu,\lambda,\psi) = \prod_{i<j}\gamma(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j,\psi)$$

where for a character μ of F^* and $s \in \mathbb{C}$, $\gamma(\mu,s,\psi)$ is the Tate gamma factor

$$\gamma(\mu,s,\psi) = \frac{L(s,\mu)}{\varepsilon(s,\mu,\psi)L(1-s,\mu^{-1})}.$$

There exists a root of unity $\kappa_{E/F} = \kappa_{E/F}(\psi)$ for which we do not need to pay much attention, such that for any pair of matching functions $f' \overset{\psi}{\leftrightarrow} f$ we have the following equality of meromorphic functions

$$\tilde{B}_\nu^{st,\psi}(f,\lambda) = \kappa_{E/F}\gamma(\nu,\lambda,\psi)B_\nu^\psi(f',\lambda).$$

It follows from (5) that if $I'(\nu,\lambda)$ is unitary then

$$(9)\quad \tilde{B}_\nu^{st,\psi}(f,\lambda) = \kappa_{E/F}L(1,\mathbf{1}_{F^*})^n\gamma(\nu,\lambda,\psi)B^\psi_{I'(\nu,\lambda)}(f').$$

In particular, if $\nu = 1$, E/F is either unramified or archimedean and $f' \overset{\psi}{\leftrightarrow} f_\theta$ with f bi-K-invariant and θ as in §3.1 then by (8) and (9)

$$B^{\psi}_{I'(\lambda)}(f') = \kappa_{E/F}\gamma(\lambda,\psi)(L(1,\mathbf{1}_{F^*}))^n)^{-1}\upsilon\hat{f}(\lambda)P_\alpha(\lambda) \left(\prod_{i<j}\frac{L(\lambda_i-\lambda_j,\omega)}{L(\lambda_i-\lambda_j+1,\mathbf{1}_{F^*})}\right)\overline{\mathcal{W}}^{\psi}(\varphi_0,-\bar{\lambda}) = \kappa^{-1}_{E/F}(L(1,\mathbf{1}_{F^*}))^{-n}\upsilon\hat{f}(\lambda) P_\alpha(\lambda)\left(\prod_{i<j}\frac{L(\lambda_j-\lambda_i+1,\omega)\varepsilon(\lambda_i-\lambda_j,\omega,\psi)}{L(\lambda_i-\lambda_j+1,\mathbf{1}_{F^*})}\right)\overline{\mathcal{W}}^{\psi}(\varphi_0,-\bar{\lambda}). \tag{10}$$

Since $I'(\lambda)$ is assumed to be unitarizable, $I'(\lambda) \simeq I'(-\overline{\lambda})$ and therefore the right-hand side must be invariant under $\lambda \mapsto -\overline{\lambda}$. Thus,

$$\overline{B^{\psi}_{I'(\lambda)}(f')} = \kappa_{E/F}(L(1,\mathbf{1}_{F^*}))^{-n}\upsilon\overline{\hat{f}(\lambda)}\,\overline{P_\alpha(\lambda)} \prod_{i>j}\frac{L(\lambda_j-\lambda_i+1,\omega)\varepsilon(\lambda_i-\lambda_j,\omega,\overline{\psi})}{L(\lambda_i-\lambda_j+1,\mathbf{1}_{F^*})}\mathcal{W}^{\psi}(\varphi_0,\lambda). \tag{11}$$

Using (10), (11) and the equality

$$\varepsilon(\omega,s,\psi)\varepsilon(\omega,-s,\overline{\psi}) = (\frac{\mathfrak{d}_F}{\mathfrak{d}_E})^2$$

we get

$$\left|B^{\psi}_{I'(\lambda)}(f')\right|^2 = \left|\left(\frac{\mathfrak{d}_F}{\mathfrak{d}_E}\right)^{\dim U'}\hat{f}(\lambda)P_\alpha(\lambda)\right|^2\frac{1}{L(1,\mathbf{1}_{E^*})^n}\upsilon^2 \frac{L(1,\pi'\times\tilde{\pi}'\times\omega)}{L(1,\pi'\times\tilde{\pi}')}\left|\mathcal{W}^{\psi}(\varphi_0,-\bar{\lambda})\overline{\mathcal{W}}^{\psi}(\varphi_0,\lambda)\right|.$$

Finally, using (6) and the equality

$$L(s,\pi\times\tilde{\pi}) = L(s,\pi'\times\tilde{\pi}')L(s,\pi'\times\tilde{\pi}'\times\omega)$$

we obtain

$$\left|B^{\psi}_{I'(\lambda)}(f')\right|^2 = \left|\hat{f}(\lambda)(\frac{\mathfrak{d}_F}{\mathfrak{d}_E})^{\dim U'}\upsilon P_\alpha(\lambda)\right|^2\frac{L(1,\pi\times\tilde{\pi})}{L(1,\pi'\times\tilde{\pi}')^2}\left|W^{\psi}_1(e)\right|^2$$

where W^{ψ}_1 is as in §2.2. We stress that for this equality to hold we do not need to assume that f' is bi-K'-invariant.

Note that if $f^g_\theta = f(\theta^{-1}\cdot g)$ then by a simple change of the orthonormal basis we have

$$\tilde{B}^{st,\psi}_{\nu}(f^g_\theta,\lambda) = \sum_{\varphi\in\mathrm{ob}(I(\chi,\lambda))} J^{st,\alpha}(I(f_\theta,\lambda)\varphi,\lambda)\overline{W}^{\psi}(\varphi,-\bar{\lambda},g).$$

Therefore, in the unramified case, if $f' \overset{\psi}{\leftrightarrow} f^g_\theta$ then by the same reasoning as before

$$(12)\quad \left|B^\psi_{I'(\lambda)}(f')\right|^2 = \left|\hat{f}(\lambda)(\frac{\mathfrak{d}_F}{\mathfrak{d}_E})^{\dim U'} \upsilon P_\alpha(\lambda)\right|^2 \frac{L(1,\pi\times\tilde{\pi})}{L(1,\pi'\times\tilde{\pi}')^2}\left|W^\psi_1(g)\right|^2.$$

4. The Computation of the period

We now turn to the setting of Theorem 1. We assume that E/F is unramified at all finite places and consider an irreducible, cuspidal everywhere unramified automorphic representation π' of $G'_{\mathbb{A}}$ such that $\pi'\otimes\omega \not\simeq \pi'$. Thus, $\pi = \mathrm{bc}(\pi') = \mathrm{bc}(\pi'\otimes\omega)$ is a cuspidal, everywhere unramified automorphic representation of $G_{\mathbb{A}}$. We write $\pi'_v = I'(\lambda_v)$ for all places v of F. Let ϕ_0 be the $\mathbf{K}$-invariant cusp form in the space of π which is L^2-normalized and let $\theta \in G_{\mathbb{A}}$ be as in Theorem 1. Fix $g \in G_{\mathbb{A}}$ such that $W^\psi(\phi_0, g) \neq 0$.

Let S be a finite set of places of F containing all the archimedean and all the even places, so that for $v \notin S$ the character ψ_v is unramified and $g_v, \alpha_v \in K_v$. We consider a function f on $G_{\mathbb{A}}$ of the form

$$f = \prod_{v\in S} f_v \prod_{v\notin S} \mathbf{1}_{K_v}$$

where f_v is a bi-K_v-invariant function for all $v \in S$. Let $f^g_\theta(x) = f(\theta^{-1}xg)$, $x \in G_{\mathbb{A}}$. For f^g_θ there is a matching function f' (with respect to ψ) of the form

$$f' = \prod_{v\in S} f'_v \prod_{v\notin S} \mathbf{1}_{K'_v}$$

on $G'_{\mathbb{A}}$ with f'_v supported on $\pm U'_v w T'^{+}_v U'_v$ for $v|\infty$ and f'_v is supported on the set of $g' \in G'_v$ such that $\det g' \in \det(w\alpha_v^{-1})\,\mathrm{Nm}(E^*_v)$ for $v < \infty$. Here $T'^{+}_v = \{\mathrm{diag}(a_1,\dots,a_n) : a_i > 0\}$. For the non-archimedean places this follows from [Jac03] and [Jac04]. For the real places note that $f^{g_v}_{\theta_v}$ is left-H_v-invariant, since $K_v = H^e_v = \theta_v^{-1}H_v\theta_v$, and that its restriction to B is of compact support. Therefore the function

$$\Omega(a) = \begin{cases} \mathrm{vol}(H_v)\int_{U_v} f(\theta_v^{-1}\eta u g_v)\psi_{U_v}(u)du & \text{if } a = {}^t\bar{\eta}\alpha_v^{-1}\eta \\ 0 & \text{if } a \notin \pm T'^{+}_v \end{cases}$$

is smooth and of compact support on $\pm T'^{+}_v$. We can now take $f'_v(u_1 w a u_2) = \Omega(a)\varphi(u_1)\varphi(u_2)$ where $\varphi \in C^\infty_c(U')$ is chosen such that $\int_{U'}\varphi(u)\psi_{U'}(u)\,du = 1$. From the relative trace formula identity of Jacquet obtained in [Jac] it follows that

$$\tilde{B}^\psi_\pi(f^g_\theta) = B^\psi_{\pi'}(f') + B^\psi_{\pi'\otimes\omega}(f').$$

For f' as above we have

$$B^{\psi}_{\pi'}(f') = B^{\psi}_{\pi'\otimes\omega}(f')$$

since globally $\omega(\det(w\alpha^{-1})) = 1$ and therefore the support of f' is contained in the kernel of $\omega \circ \det$. Thus, we obtain

$$\tilde{B}^{\psi}_{\pi}(f^g_{\theta}) = 2B^{\psi}_{\pi'}(f'). \tag{13}$$

By considering an orthonormal basis containing $\pi(g)\phi_0$ and using that f is bi-$\mathbf{K}$-invariant we have

$$\tilde{B}^{\psi}_{\pi}(f^g_{\theta}) = \hat{f}_S(\pi_S)P^H(\pi(\theta)\phi_0)\overline{W^{\psi}(\phi_0, g)}$$

where

$$\hat{f}_S(\pi_S) = \prod_{v\in S}\hat{f}_v(\pi_v)$$

is the spherical Fourier transform of f. By (3) we have

$$\left|W^{\psi}(\phi_0, g)\right|^2 = \frac{1}{\operatorname{Res}_{s=1} L^S(s, \pi\times\tilde{\pi})}\prod_{v\in S}\left|W^{\psi_v}_{1,v}(g_v)\right|^2.$$

Thus,

$$\left|\tilde{B}^{\psi}_{\pi}(f^g_{\theta})\right|^2 = \frac{\left|\hat{f}_S(\pi_S)P^H(\pi(\theta)\phi_0)\right|^2}{\operatorname{Res}_{s=1} L^S(s, \pi\times\tilde{\pi})}\prod_{v\in S}\left|W^{\psi_v}_{1,v}(g_v)\right|^2. \tag{14}$$

On the other hand we can write

$$B^{\psi}_{\pi'}(f') = \frac{1}{\operatorname{Res}_{s=1} L^S(s, \pi'\times\tilde{\pi}')}\prod_{v\in S} B^{\psi_v}_{\pi'_v}(f'_v).$$

Combining this with (12) we get

$$\left|B^{\psi}_{\pi'}(f')\right|^2 = \upsilon^2\left|\frac{\Delta_E}{\Delta_F}\right|^{\dim U'}\left(\frac{\left|\hat{f}_S(\pi_S)\right|}{\operatorname{Res}_{s=1} L(s, \pi'\times\tilde{\pi}')}\right)^2 \prod_{v\in S} L(1, \pi_v\times\tilde{\pi}_v)\left|W^{\psi_v}_{1,v}(g_v)P_{\alpha_v}(\lambda_v)\right|^2 \tag{15}$$

where $\upsilon = \operatorname{vol}(((H^e_e)_{\mathbb{A}}\cap\mathbf{K})\backslash(H^e_{\mathbb{A}}\cap\mathbf{K}))$. Comparing (14) and (15) via (13) and taking into account the equality

$$L(s, \pi\times\tilde{\pi}) = L(s, \pi'\times\tilde{\pi}')L(s, \pi'\times\tilde{\pi}'\times\omega)$$

and the fact that $\operatorname{vol}((H^e_e)_{\mathbb{A}}\cap\mathbf{K}) = 2^{dn}\left|\frac{\Delta_F}{\Delta_E}\right|^{n/2}$ we get Theorem 1 with

$$P_{\alpha}(\pi) = \prod_{v} P_{\alpha_v}(\lambda_v). \tag{16}$$

Recall that $P_{\alpha_v} \equiv 1$ if $v \notin S$.

General CM-fields. We now drop the assumption that E/F is unramified at all finite places and denote by S_r the set of finite places where E/F ramifies. The representation π and the cusp form ϕ_0 remain as in Theorem 1 and $\pi' = \otimes_v \pi'_v$ is a cuspidal representation of $G'_{\mathbb{A}}$ so that $\pi = bc(\pi')$. Thus, for each v π'_v is one of the 2^n (not necessarily unramified) principal series representations of G'_v that base-changes to π_v.

The same argument used to prove Theorem 1 yields the formula (17)

$$\left|P^{H^\alpha}(\pi(\theta)\phi_0)\right|^2 = \frac{L^S(1,\pi'\times\tilde{\pi}'\otimes\omega)}{\operatorname{Res}_{s=1} L^S(s,\pi'\times\tilde{\pi}')} \times \prod_{v\in S} \frac{L(1,\mathbf{1}_{E_v^*})^n}{L(1,\mathbf{1}_{F_v^*})^{2n}} \left|\frac{J_{\nu_v}^{st,\alpha_v}(\varphi_{0,v},\lambda_v)}{\gamma(\nu_v,\lambda_v,\psi_v)}\right|^2.$$

As before, we can interpret $J_{\nu_v}^{st,\alpha_v}(\varphi_{0,v},\lambda_v)$ as Hironaka's spherical function evaluated at α_v at all finite places (cf. [Off, Lemma 8.5]). For $v \notin S_r$ their value is known. Otherwise, this is not the case except for $n = 2$ where the spherical function is given by [Hir89, Theorem 1, p. 28] if the residual characteristic is odd. It follows for instance that in the odd ramified case

$$J^{st,e}_{\nu=(\nu_1,\nu_2)}(\varphi_0,\lambda) = \begin{cases} 0 & \text{if } \nu_1 = \nu_2, \\ \operatorname{vol}(H_e^e \backslash H_e \cap K) \frac{L(\lambda_1-\lambda_2,\mathbf{1}_{F^*})}{L(\lambda_1-\lambda_2,(\cdot,-\epsilon))} & \text{otherwise,} \end{cases}$$

where $(\cdot,\cdot)$ is the Hilbert symbol and $\epsilon \in \mathcal{O}_F^* \setminus (\mathcal{O}_F^*)^2$.

To illustrate the global case (for $n = 2$), we assume for simplicity that $\alpha = e$, $S_r \neq \emptyset$ consists of odd places and as before that π'_v is unramified at the archimedean places. Arguing as in §3.3 we ultimately get

Proposition 2. *Under the above assumptions, $P^{H^e}(\phi_0) = 0$ unless $\omega_{\pi'}\omega$ is unramified at all finite places, in which case,*

$$\left|P^{H^e}(\phi_0)\right|^2 = 4\operatorname{vol}(H^e_{\mathbb{A}} \cap \mathbf{K})^2 \cdot \left|\frac{\Delta_E^2}{\Delta_F}\right| 2^{-4(d+|S_r|)} \cdot \frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{\operatorname{Res}_{s=1} L(s,\pi'\times\tilde{\pi}')} \times \prod_{v\in S_r} \frac{1}{L(0,\pi'_v\times\tilde{\pi}'_v\otimes(\omega(\cdot,-\epsilon_v)))}.$$

5. Connection to a conjecture of Sarnak

Recall that for a co-compact arithmetic quotient of the upper half plane one expects to have for any $\epsilon > 0$ an estimate $\|\phi\|_\infty \ll \lambda^\epsilon$ for any L^2-normalized eigenfunction ϕ of the Laplacian with eigenvalue λ. (See [IS95] for a discussion of this problem.) The situation is rather different in higher dimension. By our assumption ϕ_0 is a cusp form on the locally symmetric space $G\backslash G_{\mathbb{A}}/\mathbf{K}$, which is an arithmetic quotient of several copies (according to the class number of E) of $G(F\otimes\mathbb{R})/H^e(F\otimes\mathbb{R}) = (GL_n(\mathbb{C})/U_n)^d$ where $d = [F : \mathbb{Q}]$ – a symmetric space of dimension n^2d. The form ϕ_0 is an eigenfunction of the ring of invariant differential operators (of rank nd), as well as of the Hecke operators. In [Sar04] it is proved that for any L^2-normalized form ϕ which is an eigenfunction of the ring of invariant differential operators, one has

$$\|\phi\|_\infty \ll \lambda_\phi^\delta \tag{18}$$

for $\delta = 1$ where

$$\lambda_\phi = \prod_{k=1}^d \prod_{i<j} \left|\lambda_i^{(k)} - \lambda_j^{(k)}\right|$$

and $(\lambda_1^{(k)}, \dots, \lambda_n^{(k)})_{k=1}^d$ parameterize the eigenvalues of ϕ (i.e., it is the infinitesimal character in Harish-Chandra's parameterization of the corresponding representation of $GL_n(\mathbb{C})^d$). In fact, more recently Sarnak and Venkatesh showed in a more general setting that it is possible to take $\delta < 1$. (The parameter λ_ϕ is related to the Harish-Chandra's c-function in the general setting of a locally symmetric space.) Assume for simplicity that $\alpha = e$, i.e. that H is H^e. Under the above interpretation of ϕ_0,

$$\int_{H\backslash H_{\mathbb{A}}} \phi_0(h)\, dh = \operatorname{vol}(\mathbf{K}\cap H_{\mathbb{A}}) \sum_i \frac{1}{\#\{x_i\mathbf{K}x_i^{-1}\cap H\}}\phi_0(x_i)$$

where $H_{\mathbb{A}} = \cup_{i=1}^n Hx_i(\mathbf{K}\cap H_{\mathbb{A}})$. (The x_i's comprise the genus of the hermitian form defined by e. The volume of $\mathbf{K}\cap H_{\mathbb{A}}$ can be evaluated explicitly for the Tamagawa measure - cf. [GHY01]). On the other hand, one has precise conjectures about the size of the L-functions appearing in the numerator and in the denominator of the right hand side of (2). Namely, their *finite* part, as well as its inverse, is expected to be majorized by λ_ϕ^ϵ for any $\epsilon > 0$. (These are the convexity bounds for these L-functions. They are known to hold for standard L-functions by Molteni ([Mol02])). The archimedean part of each L-function is easy to analyze by Stirling's formula and the quotient is roughly of the size of λ_ϕ. Therefore, under the above assumption on the finite part of the L-function Theorem 1 would give

$$\|\phi\|_\infty \gg \lambda_\phi^{\frac{1}{2}+\epsilon}. \tag{19}$$

Thus, one cannot expect to have $\delta < \frac{1}{2}$ in (18). In fact, the latter is already a consequence of the fact that the period is zero for representations which are not base change. Indeed, by the local Weyl law (which is known to hold at least for compact quotients), for any given finite set of points x_i in the locally symmetric space we have

$$\sum_{\mu_\phi < R^2} \left| \sum_i \phi(x_i) \right|^2 \sim cR^{(n^2-1)d}$$

where ϕ ranges over an orthonormal basis of eigenfunctions of Laplace eigenvalue $\mu_\phi < R^2$ with a fixed central character. Out of these, (the number of which is roughly $R^{(n^2-1)d}$) the number of forms which are base change is roughly $R^{d(\frac{n(n+1)}{2}-1)}$. Therefore, for the x_i as above, the weighted sum $\sum' \phi(x_i)$ is of size $R^{dn(n-1)/4}$ on average for those ϕ arising as base change, because it is zero whenever ϕ is not a base change. This is compatible with (19). This argument was used in [RS94] for the case $n = 2$. However, even in that case, our result is sharper since it holds for *any* form which is a base change. (In the case $n = 2$, the L-functions are described in terms of the standard L-function of the Gelbart-Jacquet lift ([GJ78]) and therefore the convexity bounds of [Mol02] apply.)

This example illustrates the connection between the large L^∞- norm and functoriality. In general, the conjecture predicts that the exceptional forms (those with large L^∞-norm) are rare. In the best possible scenario they are all accounted for by functoriality from smaller groups and their L^∞-norm is close to a rational power of λ_ϕ which depends on the group from which the form originates.

References

[AC89] James Arthur and Laurent Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[Bar03] Ehud Moshe Baruch. A proof of Kirillov's conjecture. *Ann. of Math. (2)*, 158(1):207–252, 2003.

[GHY01] Wee Teck Gan, Jonathan P. Hanke, and Jiu-Kang Yu. On an exact mass formula of Shimura. *Duke Math. J.*, 107(1):103–133, 2001.

[GJ78] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.

[Hir89] Yumiko Hironaka. Spherical functions of Hermitian and symmetric forms. II. *Japan. J. Math. (N.S.)*, 15(1):15–51, 1989.

[Hir99] Yumiko Hironaka. Spherical functions and local densities on Hermitian forms. *J. Math. Soc. Japan*, 51(3):553–581, 1999.

[HLR86] G. Harder, R. P. Langlands, and M. Rapoport. Algebraische Zyklen auf Hilbert-Blumenthal-Flächen. *J. Reine Angew. Math.*, 366:53–120, 1986.

[IS95] H. Iwaniec and P. Sarnak. L^∞ norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995.

[Jac] Hervé Jacquet. Kloosterman identities over a quadratic extension II. *preprint*.

[Jac86] Hervé Jacquet. Sur un résultat de Waldspurger. *Ann. Sci. École Norm. Sup. (4)*, 19(2):185–229, 1986.

[Jac87] Hervé Jacquet. Sur un résultat de Waldspurger. II. *Compositio Math.*, 63(3):315–389, 1987.

[Jac01] Hervé Jacquet. Factorization of period integrals. *J. Number Theory*, 87(1):109–143, 2001.

[Jac03] Hervé Jacquet. Smooth transfer of Kloosterman integrals. *Duke Math. J.*, 120(1):121–152, 2003.

[Jac04] Hervé Jacquet. Kloosterman identities over a quadratic extension. *Ann. of Math. (2)*, 160(2):755–779, 2004.

[JLR04] Hervé Jacquet, Erez Lapid, and Stephen Rallis. A spectral identity for skew symmetric matrices. In *Contributions to automorphic forms, geometry, and number theory*, pages 421–455. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[JS81] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981.

[KS93] Svetlana Katok and Peter Sarnak. Heegner points, cycles and Maass forms. *Israel J. Math.*, 84(1-2):193–227, 1993.

[Lan66] R. P. Langlands. The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 143–148. Amer. Math. Soc., Providence, R.I., 1966.

[Lap] Erez Lapid. On the fine spectral expansion of Jacquet's relative trace formula. *submitted*.

[LR00] Erez Lapid and Jonathan Rogawski. Stabilization of periods of Eisenstein series and Bessel distributions on GL(3) relative to U(3). *Doc. Math.*, 5:317–350 (electronic), 2000.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mol02] Giuseppe Molteni. Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product. *Duke Math. J.*, 111(1):133–158, 2002.

[Off] Omer Offen. Stable relative Bessel distributions on GL(n) over a quadratic extension. *submitted*.

[RS94] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.

[Sar04] P. Sarnak. A letter to Cathleen Morawetz. 2004.

[Wal85] J.-L. Waldspurger. Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie. *Compositio Math.*, 54(2):173–242, 1985.